Ariketa 1	Ariketa 2	Ariketa 3	Ariketa 4	Guztira 1. zatia

LEHENENGO ZATIA

IZEN-ABIZENAK:

TALDEA:

1.- Kalkulatu $\lim_{n\to\infty}\sqrt[n]{\dfrac{2n(2n+1)(2n+2)\ldots(2n+n)}{(n+1)^n}}$.

(2 puntu)

$$\lim_{n\to\infty}\sqrt[n]{\frac{2n(2n+1)(2n+2)\ldots(2n+n)}{(n+1)^n}} \overset{Z-E}{=} \lim_{n\to\infty}\frac{\dfrac{2n(2n+1)(2n+2)\ldots(2n+n)}{(n+1)^n}}{\dfrac{(2n-2)(2n-1)2n\ldots(2n-2+n-1)}{n^{n-1}}} =$$

$$= \lim_{n\to\infty}\frac{2n(2n+1)(2n+2)\ldots 3n}{(n+1)^n}\,\frac{n^{n-1}}{(2n-2)(2n-1)2n\ldots(3n-3)} =$$

$$= \lim_{n\to\infty}\frac{1}{\left(\dfrac{n+1}{n}\right)^n}\,\frac{(3n-2)(3n-1)3n}{n(2n-2)(2n-1)} = \frac{27}{4e}$$

2.- Estudiatu $\sum_{n=1}^{\infty}\frac{n^n}{a^n\cdot n!}$ seriearen izaera $\forall a\in\mathbb{R}-\{0\}$.

(2 puntu)

Balio absolutuen seriearen izaera aztertzen hasiko gara, D´Alembert aplikatuz:

$$|a_n|=\frac{n^n}{|a|^n\cdot n!}\quad\Rightarrow\quad \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{\dfrac{(n+1)^{n+1}}{|a|^{n+1}\cdot(n+1)!}}{\dfrac{n^n}{|a|^n\cdot n!}}=\frac{1}{|a|}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n=\frac{e}{|a|}\quad\Rightarrow$$

$$\Rightarrow\quad \frac{e}{|a|}\begin{cases}<1\quad\Leftrightarrow\quad |a|>e\quad\Rightarrow\quad\begin{cases}\forall a>e\quad \sum\limits_{n=1}^{\infty}a_n\ \text{konbergentea}\\ \forall a<-e\quad \sum\limits_{n=1}^{\infty}a_n\ \text{absolutuki konbergentea}\end{cases}\\ >1\quad\Leftrightarrow\quad |a|<e\quad\Rightarrow\quad\begin{cases}\forall a\in(0,e)\quad \sum\limits_{n=1}^{\infty}a_n\ \text{dibergentea}\\ \forall a\in(-e,0)\quad \sum\limits_{n=1}^{\infty}a_n\ \text{ez absolutuki konbergentea}\quad (1)\end{cases}\\ =1\quad\Leftrightarrow\quad |a|=e\quad\Rightarrow\quad\begin{cases}a=e\quad\Rightarrow\quad a_n=\dfrac{n^n}{e^n\cdot n!}\sim\dfrac{n^n}{e^n\cdot\sqrt{2\pi n}\cdot n^n\cdot e^{-n}}=\dfrac{1}{\sqrt{2\pi n}}\quad (2)\\ a=-e\quad\Rightarrow\quad a_n=\dfrac{n^n}{(-e)^n\cdot n!}\quad\Rightarrow\quad |a_n|\sim\dfrac{1}{\sqrt{2\pi n}}\quad (3)\end{cases}\end{cases}$$

- (1) Kasu honetan Leibniz-en teorema aplikatuko diogu:

$$\lim_{n\to\infty}|a_n|=\lim_{n\to\infty}\frac{n^n}{|a|^n\cdot n!}\sim\lim_{n\to\infty}\frac{n^n}{|a|^n\cdot\sqrt{2\pi n}\cdot n^n\cdot e^{-n}}=\lim_{n\to\infty}\frac{\left(\dfrac{e}{|a|}\right)^n}{\sqrt{2\pi n}}=\infty\quad\Rightarrow\quad\sum_{n=1}^{\infty}a_n\ \text{ez konbergentea}$$

- (2) Kasu honetan $a_n\sim\frac{1}{\sqrt{2\pi n}}\quad\Rightarrow\quad\sum_{n=1}^{\infty}a_n$ dibergentea
- (3) Kasu honetan $|a_n|\sim\frac{1}{\sqrt{2\pi n}}\quad\Rightarrow\quad\sum_{n=1}^{\infty}a_n$ ez absolutuki konbergentea eta berriro Leibniz-en teorema aplikatuko diogu:

$$a=-e\quad\Rightarrow\quad\begin{cases}i)\quad \lim\limits_{n\to\infty}|a_n|=\lim\limits_{n\to\infty}\dfrac{n^n}{e^n\cdot n!}\sim\lim\limits_{n\to\infty}\dfrac{1}{\sqrt{2\pi n}}=0\ \Rightarrow \text{B.B betetzen da}\\ ii)\quad |a_{n+1}|=\dfrac{(n+1)^{n+1}}{e^{n+1}\cdot(n+1)!}<|a_n|=\dfrac{n^n}{e^n\cdot n!}\quad\Leftrightarrow\quad\dfrac{|a_{n+1}|}{|a_n|}=\dfrac{1}{e}\left(\dfrac{n+1}{n}\right)^n<1\end{cases}\quad\Rightarrow$$

$$\Rightarrow\quad\sum_{n=1}^{\infty}a_n\ \text{baldintzaz konbergentea}$$

3.- Aurkitu analitiko eta grafikoki hurrengo funtzioaren definizio-eremua:

$$f(x,y) = \mathrm{L}\left(x^2 - y\right) + \frac{\sqrt{e^2 - x^2 - y^2}}{\arctan\left(\mathrm{L}(x-y)-1\right)}$$

(2 puntu)

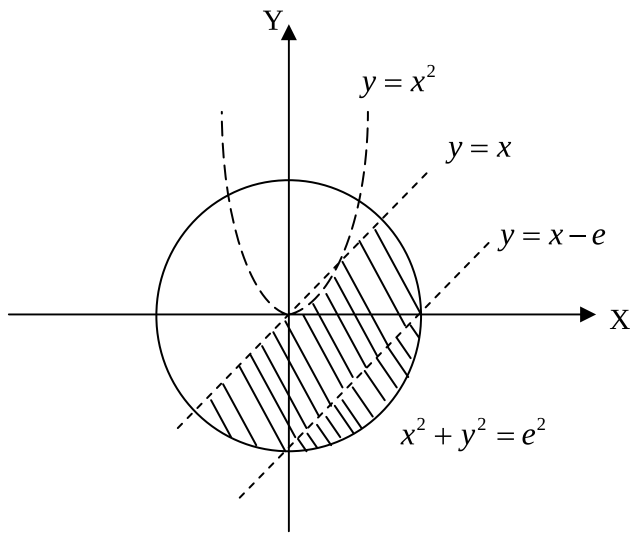
$$D = \left\{(x,y) \in \mathbb{R}^2 / \; x^2 - y > 0,\; e^2 - x^2 - y^2 \geq 0,\; \arctan\left(\mathrm{L}(x-y)-1\right) \neq 0,\; x - y > 0\right\}$$

$x^2 - y > 0 \quad \Leftrightarrow \quad y < x^2$

$e^2 - x^2 - y^2 \geq 0 \quad \Leftrightarrow \quad x^2 + y^2 \leq e^2$

$\arctan\left(\mathrm{L}(x-y)-1\right) \neq 0 \quad \Leftrightarrow \quad \mathrm{L}(x-y) - 1 \neq 0 \quad \Leftrightarrow \quad x - y \neq e \quad \Leftrightarrow \quad y \neq x - e$

$x - y > 0 \quad \Leftrightarrow \quad y < x$



4.- $f(x,y)=\begin{cases}\arctan\left(\dfrac{x^4+y^4}{x^2+y^2}\right) & \forall(x,y)\neq(0,0)\\ a & (x,y)=(0,0)\end{cases}$ **funtzioa emanik:**

a) Aurkitu $a\in\mathbb{R}$ f jarraitua izan dadin (0,0) puntuan.

Aurreko atalean lortutako *a*-ren balio horretarako:

b) Kalkulatu $f'_x(0,0)$ eta $f'_y(0,0)$.

c) Aurkitu f -ren deribatu direkzionala (1,0) puntuan, OX ardatzaren noranzko positiboarekin 60º angelua osatzen duen norabidean.

(3 puntu)

a) $$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)}\arctan\left(\frac{x^4+y^4}{x^2+y^2}\right)=\lim_{\substack{\rho\to0^+\\ \forall\theta\in[0,2\pi)}}\arctan\left(\rho^2\cdot\underbrace{\left(\cos^4\theta+\sin^4\theta\right)}_{\text{mugatua}}\right)=0$$

Beraz, f jarraitua da (0,0) puntuan $\Leftrightarrow a=0$

b) $$f'_x(0,0)=\lim_{h\to0}\frac{f(h)-f(0)}{h}=\lim_{h\to0}\frac{\arctan\left(\frac{h^4}{h^2}\right)}{h}=\lim_{h\to0}\frac{\arctan\left(h^2\right)}{h}\sim\lim_{h\to0}\frac{h^2}{h}=0$$

Eta simetriaz, $f'_y(0,0)=0$

c) Kontuan hartuta f diferentziagarria dela $\forall(x,y)\neq(0,0)$ (funtzio diferentziagarrien konposaketa baita), orduan,

$$\left.\frac{df}{d\vec{u}}\right|_{(1,0)}=f'_x(1,0)\cdot h_1+f'_y(1,0)\cdot h_2$$

non $\vec{u}=\left(h_1,h_2\right)=(\cos 60^\circ,\sin 60^\circ)=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, unitarioa dena.

$$\forall(x,y)\neq(0,0)\quad f'_x(x,y)=\frac{\dfrac{4x^3\cdot\left(x^2+y^2\right)-2x\cdot\left(x^4+y^4\right)}{\left(x^2+y^2\right)^2}}{1+\left(\dfrac{x^4+y^4}{x^2+y^2}\right)^2}=\frac{2x^5+4x^3y^2-2xy^4}{\left(x^2+y^2\right)^2+\left(x^4+y^4\right)^2}$$

Eta simetriaz, $\forall(x,y)\neq(0,0)\quad f'_y(x,y)=\dfrac{2y^5+4y^3x^2-2yx^4}{\left(x^2+y^2\right)^2+\left(x^4+y^4\right)^2}$

Orduan, $f'_x(1,0)=\dfrac{2}{1+1}=1$ eta $f'_y(1,0)=0$

Beraz, $\left.\dfrac{df}{d\vec{u}}\right|_{(1,0)}=\dfrac{1}{2}$

Ariketa 5	Ariketa 6	Ariketa 7	Guztira 2. zatia

BIGARREN ZATIA

IZEN-ABIZENAK:

TALDEA:

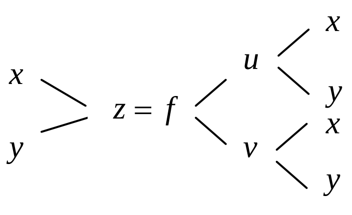
5.- Izan bedi $z = e^{uv+2u+v-2}$ **funtzioa, non** $\begin{cases} u = \dfrac{1}{e^{x+y}} \\ v = x^2 + y^2 \end{cases}$ **.**

a) Aurkitu $P(x,y) = (0,0)$ **puntuan norabidean non** z **–ren aldakuntza maximoa den.**

b) Lortu ***P*** **puntutik igarotzen den maila-kurbari dagokion zuzen ukitzailearen ekuazioa puntu horretan.**

(2 puntu)

a) z funtzio diferentziagarria da (funtzio diferentziagarrien konposaketa baita) beraz, norabidea non z –ren aldakuntza maximoa den bere gradienteak adierazten du: $\overrightarrow{\nabla z}(P) = \left(z'_x(P), z'_y(P)\right)$



Orduan, $z'_x = f'_u \cdot u'_x + f'_v \cdot v'_x$ eta $z'_y = f'_u \cdot u'_y + f'_v \cdot v'_y$

$$P(x,y) = (0,0) \quad \Rightarrow \quad (u,v) = (1,0)$$

$$f(u,v) = e^{uv+2u+v-2} \quad \Rightarrow \quad \begin{cases} f'_u = (v+2) \cdot e^{uv+2u+v-2} \\ f'_v = (u+1) \cdot e^{uv+2u+v-2} \end{cases} \quad \Rightarrow \quad \begin{cases} f'_u(1,0) = 2 \\ f'_v(1,0) = 2 \end{cases}$$

eta $\begin{cases} u = \dfrac{1}{e^{x+y}} \\ v = x^2 + y^2 \end{cases} \Rightarrow \begin{cases} u'_x = \dfrac{-1}{e^{x+y}} = u'_y \\ v'_x = 2x \quad v'_y = 2y \end{cases} \Rightarrow \begin{cases} u'_x(0,0) = -1 = u'_y(0,0) \\ v'_x(0,0) = 0 \quad v'_y(0,0) = 0 \end{cases}$

Beraz, $\overrightarrow{\nabla z}(P) = \left(z'_x(P), z'_y(P)\right) = (-2,-2)$

b) P puntutik igarotzen den maila-kurbari dagokion zuzen ukitzailearen norabide-bektorea eta gradientea elkarzutak dira beraz, $\vec{v} = (2,-2)$ bektore hori da.

Hortaz, zuzen ukitzailearen ekuazioa honako hau dugu:

$$\begin{cases} y = -x \\ z = 1 \quad \text{(maila-kurba plano horretan dago } z(P) = 1 \text{ baita)} \end{cases}$$

6.- $\begin{cases} F(x,y,z) = x^2 + y^2 + z^2 - 3 = 0 \\ G(x,y,z) = \alpha \cdot \mathrm{L}x + \beta \cdot \mathrm{L}y + \gamma \cdot \mathrm{L}z = 0 \end{cases}$ **ekuazio-sistema emanik:**

a) **Aurkitu** $\alpha, \beta, \gamma \in \mathbb{R}$ **parametroen artean egon behar duten erlazioak,** $P(x,y,z) = (1,1,1)$ **puntuaren ingurune batean aurreko sistemak** $y = y(x)$ **eta** $z = z(x)$ **funtzio diferentziagarriak defini ditzan.**

a) $\alpha = 1, \beta = 1, \gamma = 2$ **balioetarako har dezagun aurreko sistemak definituriko** $y = y(x)$ **eta** $z = z(x)$ **funtzio-bikotea. Aztertu ea** $z = z(x)$ **funtzioak mutur erlatiborik duen** $x = 1$ **puntuan eta, baiezko kasuan, zein motatakoa den.**

(2.5 puntu)

a) Funtzio inplizituaren teorema aplikatuko diegu emandako ekuazio-sistema eta puntuari:

i. $\begin{cases} F(P) = 1+1+1-3 = 0 \\ G(P) = \alpha \cdot \mathrm{L}1 + \beta \cdot \mathrm{L}1 + \gamma \cdot \mathrm{L}1 = 0 \end{cases} \quad \forall \alpha, \beta, \gamma \in \mathbb{R}$

ii. $\begin{cases} F'_x = 2x \quad F'_y = 2y \quad F'_z = 2z \\ G'_x = \dfrac{\alpha}{x} \quad G'_y = \dfrac{\beta}{y} \quad G'_z = \dfrac{\gamma}{z} \end{cases} \quad \forall \alpha, \beta, \gamma \in \mathbb{R}$ existitzen eta jarraituak dira

$P(x,y,z) = (1,1,1)$ puntuaren ingurune batean non $x \neq 0$, $y \neq 0$ eta $z \neq 0$.

iii. $\left| \dfrac{D(F,G)}{D(y,z)} \right|_P = \begin{vmatrix} 2y & 2z \\ \dfrac{\beta}{y} & \dfrac{\gamma}{z} \end{vmatrix}_P = \begin{vmatrix} 2 & 2 \\ \beta & \gamma \end{vmatrix} = 2\gamma - 2\beta \neq 0 \quad \Leftrightarrow \quad \beta \neq \gamma$

Beraz, $\forall \alpha, \beta, \gamma \in \mathbb{R}$ non $\beta \neq \gamma$, $P(x,y,z) = (1,1,1)$ puntuaren ingurune batean $\exists! \begin{cases} y = y(x) \\ z = z(x) \end{cases}$ funtzio-sistema diferentziagarria non $\begin{cases} y(1) = 1 \\ z(1) = 1 \end{cases}$.

b) $\alpha = 1, \beta = 1, \gamma = 2$ balioetarako, azter dezagun $z = z(x)$ funtzioak mutur erlatiborik ote duen $x = 1$ puntuan.

Puntu kritikoa izateko $z'(1) = 0$ bete behar da. Deribatu hori lortzeko $\begin{cases} F(x, y(x), z(x)) = 0 \\ G(x, y(x), z(x)) = 0 \end{cases}$ sisteman x-rekiko deribatuko dugu:

(1) $\begin{cases} 2x + 2y \cdot y' + 2z \cdot z' = 0 \\ \dfrac{1}{x} + \dfrac{y'}{y} + \dfrac{2z'}{z} = 0 \end{cases} \overset{P \text{ puntuan}}{\Rightarrow} \begin{cases} 2 + 2y'(1) + 2z'(1) = 0 \\ 1 + y'(1) + 2z'(1) = 0 \end{cases} \overset{\text{kenduz}}{\Rightarrow} 1 + y'(1) = 0 \quad \Leftrightarrow$

$$\Leftrightarrow \quad y'(1) = -1 \quad \Rightarrow \quad z'(1) = 0$$

Behin frogatuta $z = z(x)$ funtzioak puntu kritikoa duela $x = 1$ puntuan, sailkatuko dugu. Horretarako (1) sisteman berriro x-rekiko deribatuko dugu:

$$\begin{cases} 2+2\left(y'\right)^2+2y\cdot y''+2\left(z'\right)^2+2z\cdot z''=0 \\ -\dfrac{1}{x^2}+\dfrac{y\cdot y''-\left(y'\right)^2}{y^2}+2\dfrac{z\cdot z''-\left(z'\right)^2}{z^2}=0 \end{cases} \xRightarrow{P \text{ puntuan}} \begin{cases} 2+2+2y''(1)+2z''(1)=0 \\ -1+y''(1)-1+2z''(1)=0 \end{cases}$$

$$\begin{cases} 4+2y''(1)+2z''(1)=0 \\ -2+y''(1)+2z''(1)=0 \end{cases} \xRightarrow{\text{kenduz}} \quad 6+y''(1)=0 \quad \Leftrightarrow \quad y''(1)=-6 \quad \Rightarrow \quad z''(1)=4>0$$

Beraz, $z=z(x)$ funtzioak $x=1$ puntuan minimo erlatiboa du.

7.- Izan bedi $\begin{cases} S_1 \equiv z = 2-\sqrt{x^2+y^2} \\ S_2 \equiv z = x^2+y^2 \end{cases}$ **gainazalek osaturiko *S* gainazal itxiak mugatzen duen *V* solidoa. Izan bedi** S_1 **eta** S_2 **gainazalen arteko *C* ebakidura-kurba eta defini dezagun** $\vec{F}(x,y,z) = (x+y)\cdot\vec{i} + (2x-z)\cdot\vec{j} + (y+z)\vec{k}$ **eremu bektoriala.**

a) Kalkulatu *V* solidoaren bolumena.

b) Aurkitu *S* gainazala osatzen duen S_2 **gainazalaren zatiaren azalera.**

c) Kalkulatu *S* gainazaletik irteten den fluxua $\vec{F}$ **-ren eraginez.**

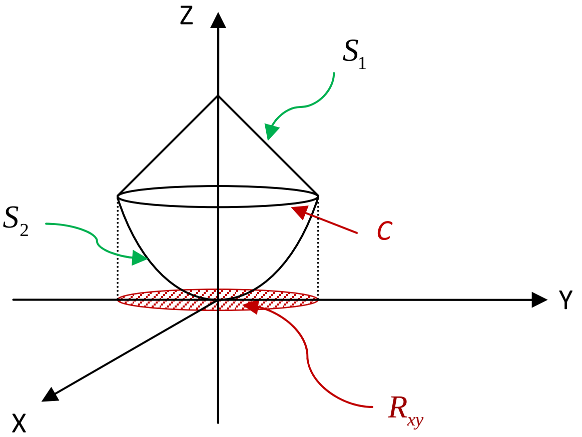
d) Kalkulatu $\vec{F}$ **-ren zirkulazioa *C* kurban zehar.**

(3.5 puntu)

a) $C = S_1 \cap S_2 \quad \Rightarrow \quad z = 2-\sqrt{z} \quad \Leftrightarrow \quad z = z^2-4z+4 \quad \Leftrightarrow \quad z^2-5z+4=0 \quad \Leftrightarrow \quad \begin{cases} z=1 \\ z=4 \end{cases}$

Baina, $S_1 \equiv z = 2-\sqrt{x^2+y^2} \quad \Rightarrow \quad z<2 \quad \Rightarrow \quad z=4$ planoak ez du balio.

Beraz, $C \equiv \begin{cases} x^2+y^2=1 \\ z=1 \end{cases}$



Eta *V* solidoaren proiekzioa $R_{xy} \equiv x^2+y^2 \le 1$ eskualdea da. Beraz, zilindrikoetan adierazten badugu:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases} \quad |J| = \rho \quad \Rightarrow \quad V \equiv \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \rho \le 1 \\ \rho^2 \le z \le 2-\rho \end{cases}$$

$$\text{Bolumena}(V) = \int_0^{2\pi}\int_0^1\int_{\rho^2}^{2-\rho} \rho\, dz\, d\rho\, d\theta = 2\pi\int_0^1 \rho\left(2-\rho-\rho^2\right)d\rho = 2\pi\left[\rho^2-\frac{\rho^3}{3}-\frac{\rho^4}{4}\right]_0^1 =$$

$$= 2\pi\left[1-\frac{1}{3}-\frac{1}{4}\right]_0^1 = \frac{5\pi}{6}$$

b) $S_2 \equiv z = x^2 + y^2 \quad \forall (x,y) \in R_{xy} \equiv x^2 + y^2 \le 1 \quad \Rightarrow \quad \begin{cases} z'_x = 2x \\ z'_y = 2y \end{cases} \quad \Rightarrow$

$$\Rightarrow \quad \text{Azalera}\left(S_2\right) = \iint_{R_{xy}} \sqrt{1+\left(z'_x\right)^2+\left(z'_y\right)^2} = \iint_{R_{xy}} \sqrt{1+4\left(x^2+y^2\right)}dxdy \overset{(*)}{=} \int_0^{2\pi}\int_0^1 \rho\sqrt{1+4\rho^2}\,d\rho\, d\theta =$$

$$= 2\pi\left[\frac{\left(1+4\rho^2\right)^{3/2}}{3/2}\cdot\frac{1}{8}\right]_0^1 = \frac{\pi}{6}\left(5^{3/2}-1\right)$$

(*) Polarretan, $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \quad |J| = \rho \quad \Rightarrow \quad R_{xy} \equiv \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \rho \le 1 \end{cases}$

c) $\Phi_S\left(\vec{F}\right) \overset{GAUSS}{=} \iiint_V \underbrace{div\left(\vec{F}\right)}_{=2} dxdydz = 2\text{Bolumena}(V) = \dfrac{5\pi}{3}$

d) $\displaystyle\oint_C \vec{F}d\vec{r} \overset{(**)}{=} \int_0^{2\pi} \left(-\sin t(\cos t+\sin t)+\cos t(2\cos t-1)\right)dt =$

$$= \int_0^{2\pi}\left(-\sin t\cdot\cos t-\sin^2 t+2\cos^2 t-\cos t\right)dt =$$

$$= \int_0^{2\pi}\left(-\sin t\cdot\cos t-\frac{1-\cos(2t)}{2}+1+\cos(2t)-\cos t\right)dt =$$

$$= \int_0^{2\pi}\left(-\sin t\cdot\cos t+\frac{3\cos(2t)}{2}+\frac{1}{2}-\cos t\right)dt = \left[\frac{\cos^2 t}{2}+\frac{3\sin(2t)}{4}+\frac{t}{2}-\sin t\right]_0^{2\pi} = \pi$$

(**) $C \equiv \begin{cases} x^2+y^2 = 1 \\ z = 1 \end{cases} \equiv \begin{cases} x = \cos t \\ y = \sin t \\ z = 1 \quad \Rightarrow \quad dz = 0 \end{cases} \qquad 0 \le t \le 2\pi$