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To Nicolae Teleman on the occasion of his 65th birthday

Cohomological Tautness for Riemannian Foliations

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Abstract. In this paper, we present new results on the tautness of Riemannian foliations in their historical context. The first part of the paper gives a short history of the problem. For a closed manifold, the tautness of a Riemannian foliation can be characterized cohomologically. We extend this cohomological characterization to a class of foliations which includes the foliated strata of any singular Riemannian foliation of a closed manifold.

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In the paper [5], using his PhD thesis, Carrière conjectured that, for Riemannian foliations of compact manifolds, the property "taut" understood as the existence of a Riemannian bundle-like metric making all leaves minimal, is equivalent to the nontriviality of the top-dimension basic cohomology group. The conjecture was based on previous results of Haefliger [16] proving that "tautness" is a transverse property and on his own research on Riemannian flows on 3-manifolds. For over a decade, the conjecture was the subject of intensive study by a group of "feuilleteurs" and was finally solved by Masa [22] and refined by Álvarez [1]. The best account of the development of the theory up to 1995 can be found in Tondeur's book [43].

The case of noncompact manifolds is much more complicated, because the tautness class of a Riemannian foliation cannot be defined in the standard way used in the case of closed manifolds [7]. However, for some noncompact manifolds, it is possible to propose a similar characterization. In a previous paper, we proved that if a foliated Riemannian manifold $(M, g, \mathcal{F})$ can be embedded as a regular stratum of a singular Riemannian foliation (SRF), then the following conditions are equivalent: (1) $\mathcal{F}$ is taut; (2) $\boldsymbol{\kappa} = 0$, where $\boldsymbol{\kappa} = [\kappa_\mu] \in H^1(M/\mathcal{F})$, and κ_μ is the mean curvature form of the bundle-like Riemannian metric μ; (3) $H^0_{\kappa_\mu}(M/\mathcal{F}) \neq 0$, where μ is a bundle-like Riemannian metric; (4) $H^n_c(M/\mathcal{F}) \neq 0$, where $n = \operatorname{codim} \mathcal{F}$ and the foliation is transversally oriented.

In this paper, we extend this characterization to a class of noncompact foliated Riemannian manifolds which include not only regular strata of SRFs but other strata as well (Theorems 3.2, 3.3, and 3.5).

Below, M and N are connected second countable Hausdorff manifolds of dimension m, without boundary and smooth (of class C^∞). All the mappings are assumed to be smooth unless otherwise stated. Consider a Riemannian foliation[1] $\mathcal{F}$ on M whose codimension is n. If V is a saturated submanifold of M, denote by $(V, \mathcal{F})$ the induced foliated manifold.

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1. HISTORICAL OVERVIEW OF THE PROBLEM

An involutive subbundle E of dimension p of TM is called a foliation of dimension p and of codimension $n = m - p$. The foliation $\mathcal{F}$ is said to be *modelled on a n-manifold* N_0 if it is defined by a cocycle $\mathcal{U} = \{U_i, f_i, g_{ij}\}_I$ modelled on N_0, i.e.,

1. $\{U_i\}$ is an open covering of M,
2. $f_i \colon U_i \longrightarrow N_0$ are submersions with connected fibres, and
3. $g_{ij} f_j = f_i$ on $U_i \cap U_j$.

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[1] For the notions related to Riemannian foliations, we refer the reader to [27, 43].

The n-manifold $T = \coprod T_i$, $T_i = f_i(U_i)$, is called the *transverse manifold* associated with the cocycle $\mathcal{U}$, and the pseudogroup $\mathcal{H}$ of local diffeomorphisms of T generated by g_{ij} is called the *holonomy pseudogroup representative* on T (associated with the cocycle $\mathcal{U}$); T is a complete transverse manifold. The equivalence class of $\mathcal{H}$ is referred to as the *holonomy pseudogroup* of $\mathcal{F}$ (or $(M, \mathcal{F})$). It can readily be seen that different cocycles defining the same foliation provide us with equivalent holonomy pseudogroups [17, 18]. In general, the converse is not true. The notion of a Riemannian foliation was introduced by Reinhart in [30, 31].

A foliation $\mathcal{F}$ on a smooth manifold M is said to be *Riemannian* if there is a Riemannian metric on T such that the local diffeomorphisms g_{ij} are isometries. Equivalently, $\mathcal{F}$ is Riemannian if there is a *bundle-like* metric μ on M for $\mathcal{F}$ (i.e., a geodesic perpendicular to a leaf of $\mathcal{F}$ at a point remains perpendicular to every leaf it meets). In a local adapted chart $(x_1, \dots, x_p, y_1, \dots, y_n)$, the bundle-like metric μ has a representation $\sum_{ij=1}^{p} \mu_{ij}(x,y) v_i \otimes v_j + \sum_{\alpha\beta=1}^{n} \mu_{\alpha\beta}(y) dy_\alpha \otimes dy_\beta$, where v_i is a 1-form annihilating the bundle $T\mathcal{F}^\perp$, and $v_i(\partial/\partial x_j) = \delta^i_j$.

Let $(M, \mathcal{F})$ be a Riemannian foliation with a bundle-like metric μ. Then it is defined by a cocycle $\mathcal{U} = \{U_i, f_i, k_{ij}\}_I$ modelled on a Riemannian manifold $(N_0, \overline{g})$ such that

(i) $f_i\colon (U_i, \mu) \to (T_0, \overline{\mu})$ is a Riemannian submersion with connected fibres;
(ii) $k_{ij}\colon f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ are local isometries of $(T_0, \overline{\mu})$;
(iii) $f_j = k_{ji} f_i$ on $f_i(U_i \cap U_j)$.

A foliation $\mathcal{F}$ on a Riemannian manifold (M, μ) is said to be *minimal* if all its leaves are minimal submanifolds of (M, μ). A foliation $\mathcal{F}$ on a manifold M is said to be *taut* if there is a Riemannian metric μ on M for which all leaves are minimal submanifolds of (M, μ).

Among other things, Reinhart introduced and studied the basic cohomology of these foliations.

In the presence of the Riemannian metric μ, the tangent bundle TM admits an orthogonal splitting $TM = T\mathcal{F} \oplus T\mathcal{F}^\perp$. A k-form α is said to be of pure type (r, s), $r + s = k$, if, for any point of M, there is an adapted chart $(x_1, \dots, x_p, y_1, \dots, y_n)$ such that $\alpha = \sum f_{IJ} v_{i_1} \wedge \dots \wedge v_{i_r} \wedge dy_{j_1} \wedge \dots \wedge dy_{j_s}$, where $1 \leqslant i_1 < \dots < i_r \leqslant p$, $1 \leqslant j_1 < \dots < j_s \leqslant n$, and $I = (i_1, \dots, i_r)$, $J = (j_1, \dots, j_s)$.

Denote by $\Omega^k(M)$ the space of k-forms M and by $\Omega^{r,s}(M)$ the space of forms of pure type (r, s). Then $\Omega^k(M) = \bigoplus_{r+s=k} \Omega^{r,s}(M)$, or, briefly, $\Omega^k = \bigoplus_{r+s=k} \Omega^{r,s}$.

The exterior differential $d\colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ can be decomposed into three components $d = d_\mathcal{F} + d_T + \delta$, where $d_\mathcal{F}$ is of bidegree $(1,0)$, d_T is of bidegree $(0,1)$, and δ is of bidegree $(-1,2)$, i.e., $d_\mathcal{F}\colon \Omega^{r,s} \to \Omega^{r+1,s}$, $d_T\colon \Omega^{r,s} \to \Omega^{r,s+1}$, and $\delta\colon \Omega^{r,s} \to \Omega^{r-1,s+2}$.

In this work, we use three types of cohomologies.

(a) The *basic cohomology* $H^*(M/\mathcal{F})$ is the cohomology of the complex $\Omega^*(M/\mathcal{F})$ of *basic forms*. A differential form ω is said to be basic if $i_X\omega = i_X d\omega = 0$ for every vector field X tangent to $\mathcal{F}$. The complex $\Omega^*(M/\mathcal{F})$ can be identified with the complex of holonomy invariant forms on the transverse manifold T, $\Omega^*_\mathcal{H}(T)$.

In particular, if the foliation $\mathcal{F}$ is developable and if $D\colon \tilde{M} \to T$ is its development with connected fibres and $h\colon \pi_1(M) \to \mathrm{Diff}(T)$ is the development representation, then the complex of basic forms $\Omega^*(M/\mathcal{F})$ can be identified with the complex of $h(\pi_1(M))$-invariant forms on T.

(b) The *compactly supported basic cohomology* $H^*_c(M/\mathcal{F})$ is the cohomology of the basic subcomplex $\Omega^*_c(M/\mathcal{F}) = \{\omega \in \Omega^*(M/\mathcal{F}) \mid \text{ the support of } \omega \text{ is compact}\}$.

(c) The *twisted basic cohomology* $H^*_\kappa(M/\mathcal{F})$ with respect to the cycle $\kappa \in \Omega^1(M/\mathcal{F})$ is the cohomology of the basic complex $\Omega^*(M/\mathcal{F})$ with respect to the differential $d_\kappa\omega = d\omega - \kappa \wedge \omega$. This cohomology does not depend on the choice of the cycle, namely, $H^*_\kappa(M/\mathcal{F}) \cong H^*_{\kappa+df}(M/\mathcal{F})$ through the isomorphism: $[\omega] \mapsto [e^f\omega]$.

1.1. Example (Ghys, [13]). Consider the unimodular matrix $A = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ inducing a diffeomorphism of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let $\mathbb{T}^3_A$ be the torus bundle over $\mathbb{S}^1$ determined by A and let $\mathcal{F}$ be the flow obtained by suspending A. Then the basic cohomology $H^2(\mathbb{T}^3_A/\mathcal{F})$ is infinite-dimensional because the basic forms correspond to A-invariant forms on $\mathbb{T}^2$; i.e., the 1-forms are written as $f(x)dx$, and thus are closed, while the 2-forms are represented as $u(x)dx \wedge dy$.

In [31], Reinhart claimed that the basic cohomology of a Riemannian foliated closed manifold is finite-dimensional and satisfies the Poincaré duality property $H^k(M/\mathcal{F}) \cong H^{n-k}(M/\mathcal{F})$.

Soon it became apparent that the proof was not rigorous and contained some gaps. For a long time, this remained an open problem. Only the beginnings of the eighties brought some striking new developments. In addition to Sullivan's characterization of taut foliations ("*a foliation is taut if and only if no foliation cycle is the limit of boundaries of tangent chains,*" [41]) and Rummler's thesis, [35], Haefliger published in 1980 [16] perhaps the most influential result in this theory claiming that the "tautness" property is a transverse property, i.e., depends only on the properties of the holonomy pseudogroup. After this, Kamber and Tondeur proved their correct version of the Poincaré duality property for taut Riemannian foliations on closed manifolds [20, 21]. However, a counterexample was not found until Carrière's thesis (1981). The thesis presented the classification of 1-dimensional tangentially orientable Riemannian foliations (flows) on closed 3-manifolds. He found some flows which are not defined by a Killing vector field, and these are characterized by the property that the 2-dimensional basic cohomology is trivial [5].

1.2. Flows. Let us begin with Carrière's example. Let A be a matrix in $SL_2(\mathbb{Z})$ with trace greater than 2 with two different eigenvalues, λ and $1/\lambda$, and with two corresponding eigenvectors v_1 and v_2, respectively. Denote by $\mathbb{T}^3_A$ the 3-manifold obtained by suspending A, i.e., a $\mathbb{T}^2$-fibre bundle over $\mathbb{S}^1$. In fact, it is obtained as the quotient space of $\mathbb{T}^2 \times \mathbb{R}$ by the equivalence relation generated by the identification of (m, t) with $(A(m), t + 1)$. The lines parallel to the eigenvectors v_1 and v_2 define A-invariant foliations (flows) Φ_1 and Φ_2, respectively, on $\mathbb{T}^2$. In turn, they induce flows on $\mathbb{T}^3_A$, which we denote by the same letters. Each flow is dense in the tori which form the fibres of our 3-manifold $\mathbb{T}^3_A$. One can show [5] that the flow Φ_2 on $\mathbb{T}^3_A$ is a transversally Lie flow modelled on the affine group GA of the real line.

As the flow is transversally Lie, there is a developing mapping $D\colon \mathbb{R}^3 \to GA$ ($\mathbb{R}^3$ is the universal covering of $\mathbb{T}^3_A$) such that the fibers of D are the leaves of the lifted flow. Moreover, there is a group homomorphism $h\colon \pi_1(\mathbb{T}^3_A) \to GA$, and its image is called the *holonomy group* Γ of the foliation. The global basic forms on $(\mathbb{T}^3_A, \Phi_2)$ correspond to Γ-invariant forms on GA, and thus to $K = \overline{\Gamma}$-invariant forms, where K is the closure of Γ in GA. Therefore, the basic cohomology of the foliated manifold $(\mathbb{T}^3_A, \Phi_2)$ is isomorphic to the cohomology of the complex of K-invariant forms on GA. If we identify the group GA with $\mathbb{R}^2$ and the product given by the formula $(t, s)(t', s') = (t + t', \lambda^t s' + s)$, then the group K can be identified with the subgroup $\{(n, s) \colon n \in \mathbb{Z}, s \in \mathbb{R}\}$.

These considerations permit us to show that $H^2(\mathbb{T}^3_A/\Phi_2) = 0$. To this end, we must prove that any K-left invariant 2-form on GA is exact. The 1-forms $\alpha = dt$ and $\beta = ds/\lambda^t$ are left-invariant. A smooth function is K-invariant if it does not depend on the variable s and if it satisfies $f(t) = f(t+1)$ for any real number t. Hence, a one-form ω is K-invariant if and only if $\omega = f\alpha + g\beta$ and both the functions f and g are K-invariant. A K-invariant 2-form Ω can be represented as $\Omega = h\alpha \wedge \beta$, where h is a K-invariant function. We must prove that, for any K-invariant function h, there are K-invariant functions f and g such that $d(f\alpha + g\beta) = h\alpha \wedge \beta$, or, equivalently, $g'(t) + g(t) \log \lambda = h(t)$ for any real number t.

If we assume that $g(t) = \lambda^{-t} g_1(t)$, then we must find a function g_1 for which $g_1'(t)\lambda^{-t} = h(t)$. However, a function of this kind is given by integration, $g_1(t) = c + \int_0^t \lambda^x h(x)\, dx$, where c is a real constant. Thus, $g(t) = \lambda^{-t}\bigl(c + \int_0^t \lambda^x h(x)\, dx\bigr)$. To obtain the invariance condition $g(t) = g(t + 1)$, we must take $c = \frac{1}{\lambda - 1} \int_0^1 \lambda^x h(x)\, dx$, which is always possible as $\lambda \neq 1$.

Moreover, it is not difficult to show that the flow Φ_2 is not isometric.

This example should be seen in the light of the following proposition [43, Prop. 6.6].

Proposition 1.2.1. *Let $\mathcal{F}$ be a flow defined by a nonsingular vector field V (with the normalized vector field $W = (1/|V|)V$) on a Riemannian manifold (M, μ). Then the following conditions are equivalent*:

(i) *all leaves of $\mathcal{F}$ are minimal submanifolds of (M, μ), i.e., the foliation is minimal*;
(ii) *the orbits of V are geodesics*;
(iii) $\theta(W)\chi_\mu = 0$;
(iv) $\nabla_W W = 0$, *where ∇ is the Levi-Civita connection of (M, μ).*

The combined effort of Gluck and Sullivan [14, 41] can be summarized in the following theorem [43, Prop. 6.7]. The equivalence of the fifth condition is due to Carrière [5].

Theorem 1.2.2. *Let $\mathcal{F}$ be a flow given by a nonsingular vector field V on an m-manifold M. Then the following conditions are equivalent*:

(i) *there exists a Riemannian metric on M making the orbits of V geodesics and V of unit length*;
(ii) *there exists a 1-form $\chi \in \Omega^1(M)$ such that $\chi(V) = 1$ and $\theta(V)\chi = 0$*;
(iii) *there exists a 1-form $\chi \in \Omega^1(M)$ such that $\chi(V) = 1$ and $i_V d\chi = 0$*;
(iv) *there exists an $(m-1)$-plane subbundle $E \subset TM$ complementary the flow and such that $[V, X]$ is a section of E for any section X of E*;
(v) *there exists a Riemannian metric on M for which V is a Killing vector field.*

1.3. Tautness and basic cohomology. In addition to all these deep results, the following one seems to indicate a close relationship between tautness and basic cohomology, cf. [43, Th. 4.32].

Theorem 1.3.1. *Let $\mathcal{F}$ be a transversally oriented Riemannian foliation of codimension n of a closed and orientable Riemannian manifold (M, μ) whose leaves are minimal. Then the basic cohomology class of the transverse volume form ν is nontrivial.*

Proof. Assume that there exists a basic form α such that $d\alpha = \nu$. Let χ_μ be the volume form along the leaves of the foliation. Then $d(\alpha \wedge \chi_\mu) = d\alpha \wedge \chi_\mu + (-1)^{n-1}\alpha \wedge d\chi_\mu$.

The minimality assumption implies that the form $d\chi_\mu$ is of degree $(p-1, 2)$ and, since the form α is of degree $(0, n-1)$, the form $\alpha \wedge d\chi_\mu$ vanishes. Therefore, the form $\nu \wedge \chi_\mu$, which is a volume form of the manifold M, is exact, a contradiction.

Moreover, Kamber and Tondeur [21] proved that the basic cohomology of a taut Riemannian foliation of a closed manifold is finite-dimensional and satisfies the PD property. The above results and Haefliger's theorem [16], which assured that the existence of a Riemannian metric making all leaves minimal is a transverse property, helped to formulate (1982) the following Carrière conjecture, first expressed for flows.

Conjecture. Let $\mathcal{F}$ be a Riemannian foliation of a closed Riemannian manifold (M, μ). Then there exists a (bundle-like) Riemannian metric making all leaves minimal (i.e., the foliation is taut) if and only if the top-dimensional basic cohomology is nontrivial.

For flows, the conjecture was solved by Molino and Sergiescu in 1985 [29] as follows.

Theorem 1.3.2. *Let $\mathcal{F}$ be a Riemannian flow on a closed oriented m-manifold M. In this case, a Riemannian metric for which $\mathcal{F}$ is an isometric flow exists if and only if the top-dimensional basic cohomology is nontrivial.*

However, at that time, the solution of the full conjecture was far away.

El Kacimi, Sergiescu, and Hector proved that the basic cohomology of a Riemannian foliation on a compact manifold is finite-dimensional [11]. For the same class of foliations, Kamber and Tondeur [20, 21] established a Hodge theorem under the extra assumption that the leaves are minimal submanifolds. The basic Hodge decomposition in the general case, without any additional assumptions, was proved by El Kacimi and Hector [10].

Theorem 1.3.3. *Let $\mathcal{F}$ be a transversally oriented Riemannian foliation on a closed oriented manifold M. Then the following two conditions are equivalent*:

(i) $H^n(M/\mathcal{F}) \neq 0$;
(ii) *the basic cohomology $H^*(M/\mathcal{F})$ satisfies the Poincaré duality property.*

This theorem, together with the Kamber–Tondeur result mentioned at the beginning of the subsection, strongly hinted that Carrière's intuition was correct. Finally, in 1991, Masa [22] showed that the tautness is equivalent to the nontriviality of the top-dimensional basic cohomology, solving the conjecture positively.

Theorem 1.3.4. *Let $\mathcal{F}$ be a transversally oriented Riemannian foliation of a closed manifold M. Then a Riemannian metric on M for which all leaves are minimal exists if and only if the top-dimensional basic cohomology $H^n(M/\mathcal{F})$ is nontrivial.*

To complete the story of basic cohomology, note that, in 1993, El Kacimi and Nicolau proved that the basic cohomology of a complete Riemannian foliated manifold is a topological invariant [12].

1.4. Mean curvature form. In the story of the proof of the tautness conjecture, a certain 1-form turned out to be of great importance.

For a foliation $\mathcal{F}$ of a Riemannian manifold (M, μ), we define the shape operator W on the leaves by using the natural splitting of the tangent bundle $TM = T\mathcal{F} \oplus T\mathcal{F}^\perp$. Namely, for any section Y of $T\mathcal{F}^\perp$ and any tangent vector field X, we have $W(Y)(X) = -\pi^\perp(\nabla_X Y)$, where $\pi^\perp \colon TM \to T\mathcal{F}^\perp$ is the orthogonal projection.

The trace of W is linear in Y, and thus it defines a section of $T\mathcal{F}^{\perp^*}$. We extend it to a global 1-form κ_μ on M by setting $\kappa_\mu(X) = \operatorname{trace} W(X)$ if $X \in T\mathcal{F}^\perp$ and $\kappa_\mu(X) = 0$ if $X \in T\mathcal{F}$. The 1-form $\kappa_\mu(X)$ is referred to as the *mean curvature 1-form* of $\mathcal{F}$ on the Riemannian manifold (M, μ).

If $f \colon (M_1, \mathcal{F}_1) \to (M, \mathcal{F})$ is a foliated embedding between two Riemannian manifolds with $f(M_1)$ saturated in M and if $\dim \mathcal{F}_1 = \dim \mathcal{F}$, then

$$f^*\mu \text{ is a bundle-like metric on } (M_1, \mathcal{F}_1) \text{ and } f^*\kappa_\mu = \kappa_{f^*\mu}. \tag{1}$$

Thus,

$$\text{if } U \text{ is an open subset of } M, \text{ then } (\kappa_\mu)|_U = \kappa_{\mu|_U}. \tag{2}$$

This form is of particular interest. In [20], the authors proved that, if the form κ_μ is basic, then it is closed. Thus, it defines a 1-basic cohomology class $[\kappa_\mu]$, which proved to be of importance in the study of taut foliations [20].

Proposition 1.4.1. *Let $\mathcal{F}$ be a Riemannian foliation on a closed manifold M with a bundle-like metric μ for which κ_μ is basic and $[\kappa_\mu] = 0$. Then the bundle-like metric μ can be modified along the leaves to obtain a bundle-like metric μ' for which all leaves of $\mathcal{F}$ are minimal.*

Proof. Since $[\kappa_\mu] = 0$, there is a smooth basic function f on $(M/\mathcal{F})$ such that $\kappa_\mu = df$. Write $\lambda = e^f$ and modify the metric μ as follows $\mu' = \lambda^{2/p}\mu_{\mathcal{F}} \oplus \mu^\perp$, where p stands for the dimension of leaves and $\mu_{\mathcal{F}}$ and $\mu^\perp$ for the Riemannian metric induced on leaves of $\mathcal{F}$ and the orthogonal subbundle, respectively. The splitting is defined by the metric μ. The mean curvature form $\kappa_{\mu'}$ is equal to $\kappa_\mu - d\log\lambda = 0$.

Let $\mathcal{F}$ be a tangentially oriented foliation. We define the characteristic form χ_μ, a p-form, by the rule that $\chi_\mu(Y_1, \dots, Y_p) = \det(\mu(Y_i, E_j)_{ij})$ for any p-tuple $(Y_1, \dots, Y_p)$ of vectors of T_xM, where $i, j = 1, \dots, p$ and $E_1, \dots, E_p$ stands for an oriented orthonormal frame of $T_x\mathcal{F}$.

There is a close relationship between the characteristic form and the mean curvature form. Namely [35], the following assertion holds.

Theorem 1.4.2. *Let $\mathcal{F}$ be a tangentially oriented foliation of a Riemannian manifold (M, μ), let χ_μ be its characteristic form, and let κ_μ be its mean curvature form. Then $\theta(Y)\chi_\mu = -\kappa_\mu(Y)\chi_\mu + \beta$ for any vector field Y orthogonal to the foliation, where β is a p-form of type $(p-1, 1)$.*

As a corollary, we obtain the following assertion.

Corollary 1.4.3. *A tangentially oriented foliation $\mathcal{F}$ is taut if and only if, for any vector field Y orthogonal to the foliation, the form $\theta(Y)\chi_\mu$ is of type $(p-1, 1)$, which is equivalent to the condition that $d\chi_\mu$ is of type $(p-1, 2)$, i.e., $d\chi_\mu(Y, Y_1, \dots, Y_p) = 0$ for any vector Y and any vectors $(Y_1, \dots, Y_p)$ tangent to the foliation.*

Research concerning the tautness conjecture has concentrated on the study of the basic cohomology and the mean curvature form.

The following theorem [20, 21] gave further evidence that tautness, the mean curvature class, and the PD property for basic cohomology are linked in some way. The result of El Kacimi and Hector suggested that the nonvanishing condition for the top-dimensional basic cohomology can be related to the tautness of the foliation, i.e., the vanishing of the mean curvature form.

Theorem 1.4.4. *Let $\mathcal{F}$ be a transversally oriented Riemannian foliation on a closed oriented manifold M. Let g be a bundle-like metric with basic mean curvature form. Then the pairing $\alpha \otimes \beta \to \int_M \alpha \wedge \beta \wedge \chi_\mu$ induces a nondegenerate pairing $H^r(M/\mathcal{F}) \otimes H^{n-r}_{\kappa_\mu}(M/\mathcal{F}) \to \mathbb{R}$ of finite-dimensional vector spaces.*

In the development of the theory, Álvarez López's paper [1] of 1992 proved to be of great interest. In the paper, it is proved that the space of smooth forms $\Omega(M)$ on a foliated closed Riemannian manifold $(M, \mu, \mathcal{F})$ can be decomposed as the direct sum of $\Omega(M/\mathcal{F})$ (of basic forms) and its orthogonal complement $\Omega(M/\mathcal{F})^{\perp}$. Therefore, the mean curvature form κ_μ of $(M, \mu, \mathcal{F})$ can be decomposed into the basic component $\kappa_{\mu,b}$ and the orthogonal one. The 1-form $\kappa_{\mu,b}$ is closed. It defines the 1-basic cohomology class $\boldsymbol{\kappa} = [\kappa_{\mu,b}]$ which does not depend on μ. Moreover, as was proved in [1], any form cohomology equivalent to $\kappa_{\mu,b}$ (in the complex of basic forms) can be realized as the basic component of the mean curvature form for some bundle-like metric of $\mathcal{F}$ with the same transverse Riemannian metric. Additionally, one can prove that changing the orthogonal complement of $\mathcal{F}$ does not change the form $\kappa_{\mu,b}$.

As an application, the orientability assumption for M was removed from the original formulation of Theorem 1.3.4.

For some time, the condition that the mean curvature form is basic seemed to be a major obstacle to the existence of such a Riemannian metric. However, in 1995, Domínguez published his theorem [8, 9] with the following statement.

Theorem 1.4.5. *Let $\mathcal{F}$ be a Riemannian foliation on a closed manifold M. Then there is a bundle-like metric for $\mathcal{F}$ for which the mean curvature form is basic.*

These Riemannian metrics are very important in the remaining part of the paper. Therefore, a bundle-like metric for which the mean curvature form is basic is referred to as a *D-metric*.

This theorem, together with Proposition 1.4.1, ensures that a "taut" Riemannian metric can be chosen to be a D-metric. Below we shall use the following fact:

$$\text{if } U \text{ is a saturated open subset of } M \text{ such that } \mu|_U \text{ is a D-metric, then } \kappa_{\mu,b}|_U = \kappa_{\mu|_U}. \tag{3}$$

The final characterization of taut Riemannian foliations of closed manifolds can be summarized in the following theorem [43, 7.56].

Theorem 1.4.6. *Let $\mathcal{F}$ be a transversally oriented Riemannian foliation on a closed oriented Riemannian manifold (M, μ). Then $H^n_{\kappa_\mu}(M/\mathcal{F}) \cong \mathbb{R}$, and the following conditions are equivalent:*

(i) $H^n(M/\mathcal{F}) \cong \mathbb{R}$;
(ii) $\mathcal{F}$ *is taut*;
(iii) $\boldsymbol{\kappa} = 0$;
(iv) $H^0_{\kappa_\mu}(M/\mathcal{F}) = \mathbb{R}$.

Moreover, in this case, the basic cohomology of the foliated manifold $(M/\mathcal{F})$ has the Poincaré duality property.

1.5. Open manifolds. The theory has not been well developed for open manifolds. There is a fine and very general version of Poincaré duality theorem published in 1985 by Sergiescu [38]. Cairns and Escobales (1997) presented a very interesting example [7] of a Riemannian foliation on an open manifold for which the mean curvature form is basic but not closed.

1.5.1. SRFs. A singular Riemannian foliation[2] (SRF for short) on a connected manifold X is a partition $\mathcal{K}$ by connected immersed submanifolds, the so-called *leaves*, verifying the following properties.

I- The module of smooth vector fields tangent to the leaves is transitive on each leaf.
II- There is a Riemannian metric ν on N, the so-called *adapted metric*, such that each geodesic perpendicular at a point to a leaf remains perpendicular to every leaf it meets.

The first condition implies that $(X, \mathcal{K})$ is a singular foliation in the sense of [40] and [42]. Note that the restriction of $\mathcal{K}$ to a saturated open subset produces an SRF. Each (regular) Riemannian foliation (RF in short) is an SRF; however, the first interesting examples are as follows.

- The orbits of an action by isometries of a Lie group.
- The closures of the leaves of a regular Riemannian foliation.

1.5.2. Stratification. Classifying the points of X by the dimension of the leaves, one obtains a stratification $\mathsf{S}_{\mathcal{K}}$ of X whose elements are called *strata*. The restriction of $\mathcal{K}$ to a stratum S is the RF $\mathcal{K}_S$. The strata are ordered by $S_1 \preceq S_2 \Leftrightarrow S_1 \subset \overline{S_2}$. The minimal (maximal) strata are the closed strata (open strata). Denote by $S_{\min}$ the union of the closed strata. Since X is connected,

[2] For the notions related to singular Riemannian foliations, we refer the reader to [2, 26–28].

there is just one open stratum, which is denoted by $R_{\mathcal{K}}$. It is a dense subset. This is the *regular stratum*, and the other strata are said to be *singular*.

The depth of $\mathsf{S}_{\mathcal{K}}$, denoted by $\operatorname{depth} \mathsf{S}_{\mathcal{K}}$, is defined to be the largest i for which there is a chain of strata, $S_0 \prec S_1 \prec \cdots \prec S_i$. Thus, $\operatorname{depth} \mathsf{S}_{\mathcal{K}} = 0$ if and only if the foliation $\mathcal{K}$ is regular. The *depth* of a stratum $S \in \mathsf{S}_{\mathcal{H}}$, denoted by $\operatorname{depth}_{\mathcal{H}} S$, is defined to be the largest i for which there is a chain of strata of the form $S_0 \prec S_1 \prec \cdots \prec S_i = S$.

The basic cohomology of such foliations on closed manifolds is finite-dimensional, and it is a topological invariant [44]. However, as far as the tautness property is concerned, the situation is totally different.

1.5.3. Example. Consider the isometric action $\Phi\colon \mathbb{R} \times \mathbb{S}^{2d+2} \to \mathbb{S}^{2d+2}$ given by the formula $\Phi(t, (z_0, \dots, z_d, x)) = (e^{a_0 \pi i t} \cdot z_0, \dots, e^{a_d \pi i t} \cdot z_d, x)$, where $(a_0, \dots, a_d) \neq (0, \dots, 0)$. Here $\mathbb{S}^{2d+2} = \{(z_0, \dots, z_d, x) \in \mathbb{C}^{d+1} \times \mathbb{R} \mid |z_0|^2 + \cdots + |z_d|^2 + x^2 = 1\}$. There are two singular strata, namely, the north pole $S_1 = (0, \dots, 0, 1)$ and the south pole $S_2 = (0, \dots, -1)$. The regular stratum is $\mathbb{S}^{2d+1} \times]-1, 1[$. Let r be the variable of $]-1, 1[$. The basic cohomology $H^*(\mathbb{S}^{2d+2}/\mathcal{F})$ of the foliation is

$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=\cdots$	$i=2d$	$i=2d+1$
1	0	0	$[dr \wedge e]$	0	$[dr \wedge e^2]$	$\cdots$	0	$[dr \wedge e^d]$

where $e \in \Omega^2_{\overline{2}}\big(\mathbb{S}^{2d+2}/\mathcal{F}\big)$ is an Euler form.

The top-dimensional basic cohomology group is isomorphic to $\mathbb{R}$; however, this cohomology fails to have the Poincaré duality property, despite the flow being isometric. Certainly, the foliation is not minimal for any adapted (bundle-like) Riemannian metric.

Moreover, in [24], the authors proved that a singular foliation on a closed manifold admitting an adapted Riemannian metric for which all leaves are minimal must be regular. This fact led us to study singular Riemannian foliations closer. We have introduced basic intersection cohomology to recover some kind of Poincaré duality [36, 37, 33]. We hope to complete our task soon and prove the perverse version of the Poincaré duality property for basic intersection cohomology for singular Riemannian foliations of closed manifolds. In his thesis, [32], written under the supervision of Saralegi and Macho, Royo Prieto established (among other results on singular Riemannian flows) the Poincaré duality for basic intersection cohomology and the singular version of the Molino–Sergiescu theorem. Inspired by these results, we have started to investigate possible generalizations to the SRF case and, at the same time, we have found that our research gives some interesting insights into the problem on noncompact manifolds [33, 34]. The second part of this work is concerned with this problem.

We complete the section with the presentation of the BIC for the above example, in which the PD property can easily be seen.

When considering the BIC of our example, the picture changes. The following table presents the BIC $IH^*_{\overline{p}}\big(\mathbb{S}^{k=2d+2}/\mathcal{F}\big)$ for the constant perversities.

$i=$	0	1	2	3	4	5	6	7	$\cdots$	$k-2$	$k-1$
$\overline{p} < \overline{0}$	0	$[dr]$	0	$[e \wedge dr]$	0	$[e^2 \wedge dr]$	0	$[e^3 \wedge dr]$	$\cdots$	0	$[e^d \wedge dr]$
$\overline{p}=\overline{0}, \overline{1}$	1	0	0	$[e \wedge dr]$	0	$[e^2 \wedge dr]$	0	$[e^3 \wedge dr]$	$\cdots$	0	$[e^d \wedge dr]$
$\overline{p}=\overline{2}, \overline{3}$	0	0	$[e]$	0	0	$[e^2 \wedge dr]$	0	$[e^3 \wedge dr]$	$\cdots$	0	$[e^d \wedge dr]$
$\overline{p}=\overline{4}, \overline{5}$	1	0	$[e]$	0	$[e^2]$	0	0	$[e^3 \wedge dr]$	$\cdots$	0	$[e^d \wedge dr]$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\overline{p}=\overline{k-4}, \overline{k-3}$	1	0	$[e]$	0	$[e^2]$	0	$[e^3]$	0	$\cdots$	0	$[e^d \wedge dr]$
$\overline{p} \geqslant \overline{k-2}$	1	0	$[e]$	0	$[e^2]$	0	$[e^3]$	0	$\cdots$	$[e^d]$	0

Note that the top-dimensional basic cohomology group is isomorphic either to 0 or $\mathbb{R}$. These cohomology groups are finite-dimensional. We recover the Poincaré duality in the perverse sense, $IH^*_{\overline{p}}\big(\mathbb{S}^k/\mathcal{F}\big) \cong IH^{k-1-*}_{\overline{q}}\big(\mathbb{S}^k/\mathcal{F}\big)$ for two complementary perversities, $\overline{p} + \overline{q} = \overline{t} = \overline{k-3}$.

Theorem 1.5.4. *Let M be a connected closed manifold endowed with an SRF $\mathcal{F}$. If $\ell = \operatorname{codim}_M \mathcal{F}$ and $\overline{p}$ is a perversity on $(M/\mathcal{F})$, then $IH^{\ell}_{\overline{p}}(M/\mathcal{F}) = 0$ or $IH^{\ell}_{\overline{p}}(M/\mathcal{F}) = \mathbb{R}$.*

Corollary 1.5.5. *Let M be a connected compact manifold endowed with an SRF $\mathcal{F}$. Suppose that $\mathcal{F}$ is transversally orientable. Consider a perversity $\overline{p}$ on $(M,\mathcal{F})$ with $\overline{p} \leqslant \overline{t}$. If $\ell = \operatorname{codim}_M \mathcal{F}$, then the two following statements are equivalent.*

(1) *The foliation $\mathcal{F}_R$ is taut, where R is the regular stratum of $(M,\mathcal{F})$.*
(2) *The cohomology group $IH^{\ell}_{\overline{p}}(M/\mathcal{F})$ is $\mathbb{R}$.*

Otherwise, $IH^{\ell}_{\overline{p}}(M/\mathcal{F}) = 0$.

The BIC of a conical foliation $\mathcal{F}$ defined on M by an isometric action of an Abelian Lie group on an oriented manifold M satisfies the Poincaré duality,

$$IH^{\ell}_{\overline{p}}(M/\mathcal{F}) \cong IH^{\ell-*}_{\overline{q},c}(M/\mathcal{F})\,. \tag{4}$$

Here $\ell = \operatorname{codim}_M \mathcal{F}$, and the two perversities $\overline{p}$ and $\overline{q}$ are complementary.

Remark. Due to limited room devoted to this overview of the problem, we have not mentioned many partial results (e.g., [1, 19, 18]) and some survey papers (e.g., [6, 39]).

2. GEOMETRICAL PRELIMINARIES

In this section, we present foliations we are going to use in this work, namely, the CERFs. A CERF is essentially a Riemannian foliation defined on a noncompact manifold which is embeddable in a closed manifold in a nice way.

2.1. CERFs. We consider here a special case of Riemannian foliations defined on noncompact manifolds. They have an outside compact manifold (zipper) and an inside compact submanifold (reppiz). Consider a manifold M endowed with a Riemannian foliation $\mathcal{F}$.

A *zipper* of $\mathcal{F}$ is a closed manifold N endowed with a (regular) Riemannian foliation $\mathcal{H}$ verifying the following properties:

(a) The manifold M is a saturated open subset of N and $\mathcal{F}$ is the restriction of $\mathcal{H}$ to M.

The open subset M is also $\overline{\mathcal{F}}$-saturated. Thus, the closure $\overline{L}$ of a leaf $L \in \mathcal{F}$ is compact.

A *reppiz* of $\mathcal{F}$ is a saturated open subset U of M verifying the following properties:

(b) the closure $\overline{U}$ (in M) is compact,
(c) the inclusion mapping $U \hookrightarrow M$ induces an isomorphism $H^*(U/\mathcal{F}) \cong H^*(M/\mathcal{F})$.

A saturated open subset of M need not be a reppiz. Just consider $M = \mathbb{S}^1$ endowed with the pointwise foliation and take $U = \mathbb{S}^1 \backslash \{(\cos(2\pi/n), \sin(2\pi/n)) \ / \ n \in \mathbb{N}\backslash\{0\}\}$.

We say that $\mathcal{F}$ is a *Compactly Embeddable Riemannian Foliation* (or CERF)[3] if $(M,\mathcal{F})$ possesses a zipper and a reppiz. If M is closed, then $(M,\mathcal{F})$ is clearly a CERF if M by itself is a zipper and a reppiz. Neither the zipper nor the reppiz are unique.

The main example of a CERF is given by the strata of a singular Riemannian foliation defined on a closed manifold. This family is treated in the next section. The interior of a Riemannian foliation defined on a manifold with boundary is a CERF if the foliation is tangent to the boundary; we can consider the double of the manifold as a zipper. When the foliation is transverse to the boundary, the foliation is not a CERF.

Now let us present some geometrical tools needed to study an SRF $(X,\mathcal{K})$.

2.2. Tubular neighborhood. A singular stratum $S \in \mathsf{S}_{\mathcal{K}}$ is a proper submanifold of the Riemannian manifold (X,ν). Thus, it possesses a tubular neighborhood (T_S, τ_S, S). Recall that the following smooth mappings are assigned to this neighborhood.

+ The *radius mapping* $\rho_S \colon T_S \to [0,1[$ defined fiberwise by $z \mapsto |z|$. Each $t \neq 0$ is a regular value of the ρ_S. The pre-image $\rho_S^{-1}(0)$ is S.
+ The *contraction* $H_S \colon T_S \times [0,1] \to T_S$ defined fiberwise by $(z,r) \mapsto r\cdot z$. The restriction $(H_S)_t \colon T_S \to T_S$ is an embedding for each $t \neq 0$, and $(H_S)_0 \equiv \tau_S$.

These mappings verify $\rho_S(r\cdot z) = r\rho_S(z)$. The above tubular neighborhood can be chosen verifying the two following important properties [27].

(a) Each $(\rho_S^{-1}(t), \mathcal{K})$ is an SRF.

[3]The definition of CERF given in [34] is more restrictive; see Proposition 2.4.

(b) Each $(H_S)_t\colon (T_S,\mathcal{F}) \to (T_S,\mathcal{F})$ is a foliated mapping.

We then say that (T_S, τ_S, S) is a *foliated tubular neighborhood* of S.

The hypersurface $D_S = \rho_S^{-1}(1/2)$ is the *core* of the tubular neighborhood. Moreover, $\operatorname{depth} \mathsf{S}_{\mathcal{K}_{D_S}} = \operatorname{depth} \mathsf{S}_{\mathcal{K}_{T_S}} - 1$. Note that the mapping

$$\mathfrak{L}_S\colon (D_S\times]0,1[, \mathcal{K}\times\mathcal{I}) \to ((T_S\backslash S),\mathcal{K}), \tag{5}$$

defined by $\mathfrak{L}_S(z,t) = H_S(z,2t)$, is a foliated diffeomorphism. Here, and from now on, $\mathcal{I}$ stands for the foliation by points of $\mathbb{R}$, and the leaves of the product foliation $\mathcal{K}\times\mathcal{I}$ are of the form $L\times\{t\}$ with $L\in\mathcal{K}$.

A family of foliated tubular neighborhoods $\{T_S \mid S\in \mathsf{S}_{\mathcal{F}}^{\mathrm{sin}}\}$ is a *foliated Thom–Mather system* of $(N,\mathcal{H})$ if the following conditions hold.

(TM1) For each pair of singular strata S, S', we have $T_S\cap T_{S'} \neq \varnothing \Longleftrightarrow S \preceq S'$ or $S'\preceq S$.

Suppose that $S' \prec S$. The other two conditions are:

(TM2) $T_S \cap T_{S'} = \tau_S^{-1}(T_{S'}\cap S)$.
(TM3) $\rho_{S'} = \rho_{S'}\circ\tau_S$ on $T_S\cap T_{S'}$.

As was shown in [34], any closed manifold endowed with an SRF possesses a foliated Thom–Mather system. For the rest of the paper, we choose some foliated Thom–Mather system.

2.3. Blow up. Molino's blow up of an SRF produces a new SRF of the same generic dimension and of smaller depth (see [27] and also [37, 34]). The main idea is to replace every point of the closed strata by its link (a sphere).

In fact, for a given SRF $(X,\mathcal{K})$ with $\operatorname{depth}\mathsf{S}_{\mathcal{K}} > 0$, there is another SRF $(\widehat{X},\widehat{\mathcal{K}})$ and a continuous mapping $\mathfrak{L}\colon \widehat{X}\to X$, the so-called *blow up* of $(X,\mathcal{K})$, such that

- $\operatorname{depth}\mathsf{S}_{\widehat{\mathcal{K}}} = \operatorname{depth}\mathsf{S}_{\mathcal{K}} - 1$,
- there is a commutative diagram

$$\begin{tikzcd}
\mathfrak{L}^{-1}(X\backslash S_{\min}) \arrow[rr, "f_0"] \arrow[dr, "\mathfrak{L}"'] & & (X\backslash S_{\min})\times\{-1,1\} \arrow[dl, "\text{projection}"] \\
& X\backslash S_{\min} &
\end{tikzcd}$$

where $f_0\colon (\mathfrak{L}^{-1}(X\backslash S_{\min}),\widehat{\mathcal{K}}) \to (X\backslash S_{\min}\times\{-1,1\},\mathcal{K}\times\mathcal{I})$ is a foliated diffeomorphism,
- for any minimal (closed) stratum S_c, there is a commutative diagram

$$\begin{tikzcd}
\mathfrak{L}^{-1}(T_{S_c}) \arrow[rr, "f_{S_c}"] \arrow[dr, "\mathfrak{L}"'] & & D_{S_c}\times]-1,1[\arrow[dl, "\mathfrak{L}_{S_c}"] \\
& T_{S_c} &
\end{tikzcd}$$

where $f_{S_c}\colon (\mathfrak{L}^{-1}(T_{S_c}),\widehat{\mathcal{K}}) \to (D_{S_c}\times]-1,1[,\mathcal{K}\times\mathcal{I})$ is a foliated diffeomorphism and the mapping $\mathfrak{L}_{S_c}$ is defined by $\mathfrak{L}_{S_c}(z,t) = H_{S_c}(z,2|t|)$. Note that $f_{S_c}\colon(\mathfrak{L}^{-1}(S_c),\widehat{\mathcal{K}}) \to (D_{S_c}\times\{0\},\mathcal{K}\times\mathcal{I})$ is also a foliated diffeomorphism.

The stratification induced by $\widehat{\mathcal{K}}$ can be described as follows. For each nonminimal stratum $S\in\mathsf{S}_{\mathcal{K}}$, there is a unique stratum $S^{\mathfrak{L}}\in\mathsf{S}_{\widehat{\mathcal{K}}}$ with $\mathfrak{L}^{-1}(S)\subset S^{\mathfrak{L}}$ such that $\mathsf{S}_{\widehat{\mathcal{K}}} = \{S^{\mathfrak{L}} \;/\; S\in\mathsf{S}_{\mathcal{K}}$ and $S\cap S_{\min} = \varnothing\}$. In fact, $f_0\left(S^{\mathfrak{L}}\cap\mathfrak{L}^{-1}(X\backslash S_{\min})\right) = S\times\{-1,1\}$ and $f_{S_c}\left(S^{\mathfrak{L}}\cap\mathfrak{L}^{-1}(T_{S_c})\right) = (S\cap D_{S_c})\times]-1,1[$ if S_c is a closed stratum with $S_c\preceq S$.

The CERFs and the SRFs are related as follows.

Proposition 2.4. *Let X be a closed manifold endowed with an SRF $\mathcal{K}$. For any stratum S of $\mathsf{S}_{\mathcal{K}}$, the foliation $\mathcal{K}_S$ is a CERF.*

Proof. If S is a closed stratum, it suffices to take the zipper $(S, \mathcal{K})$ and the reppiz S. Consider now the case in which S is not closed (minimal). We proceed in two steps.

A zipper for $(S, \mathcal{K})$. Proceeding by induction on depth $\mathsf{S}_{\mathcal{K}}$, we know that there is a zipper $(N, \mathcal{H})$ of $(S^{\mathfrak{L}}, \widehat{\mathcal{K}})$. Since the mapping $\xi\colon (S, \mathcal{K}) \to (S^{\mathfrak{L}}, \widehat{\mathcal{K}})$ defined by $x \mapsto f_0^{-1}(x, 1)$ is an open foliated embedding, we can identify $(S, \mathcal{K})$ with its (open) image $(\xi(S), \widehat{\mathcal{K}})$. Thus, the foliated manifold $(N, \mathcal{H})$ is a zipper of $(S, \mathcal{K})$.

A reppiz for $(S, \mathcal{K})$. For each $i \in \{0, \dots, s-1\}$, where $s = \operatorname{depth}_{\mathcal{H}} S$, write

- Σ_i for $\cup\{S' \in \mathsf{S}_{\mathcal{H}} \mid \operatorname{depth}_{\mathcal{H}} S' \leqslant i\}$,
- T_i for the union of the disjoint tubular neighborhoods $\{T_{S'} \ / \ T_{S'} \subset \Sigma_i \backslash \Sigma_{i-1}\}$,
- $\rho_i \colon T_i \to [0, 1[$ for its radius function, and
- $D_i = \rho_i^{-1}(0)$ for the core of T_i.

The family $\{S \cap T_0, S \backslash \rho_0^{-1}([0, 7/8])\}$ is a saturated open covering of S. The inclusion mapping $I\colon ((S \cap T_0) \backslash \rho_0^{-1}([0, 7/8]), \mathcal{K}) \hookrightarrow (S \cap T_0, \mathcal{K})$ induces an isomorphism for the basic cohomology. This comes from the fact that the inclusion I is foliated diffeomorphic to the inclusion $J\colon ((S \cap D_0) \times]7/8, 1[, \mathcal{K} \times \mathcal{I}) \hookrightarrow ((S \cap D_0) \times]0, 1[, \mathcal{K} \times \mathcal{I})$ (by (5) and because $S \cap \Sigma_0 = \varnothing$). It follows from the Mayer–Vietoris sequence (see, e.g., [34]) that the inclusion $S \backslash \rho_0^{-1}([0, 7/8]) \hookrightarrow S$ induces the isomorphism $H^*(S/\mathcal{K}) \cong H^*\big(\big(S \backslash \rho_0^{-1}([0, 7/8])\big) /\mathcal{K}\big)$.

The family $\{T_{S'} \backslash \rho_0^{-1}([0, 7/8]) \mid S' \in \mathsf{S}_{\mathcal{F}},\ \operatorname{depth}_{\mathcal{H}} S' > 0\}$ is a foliated Thom–Mather system of $(S \backslash \rho_0^{-1}([0, 7/8]), \mathcal{H})$ [34, (1.6)]. The above argument applied to the stratum $S \backslash \rho_0^{-1}([0, 7/8])$ gives $H^*\big(\big(S \backslash \rho_0^{-1}([0, 7/8])\big) /\mathcal{K}\big) \cong H^*\big(\big(\big(S \backslash \rho_0^{-1}([0, 7/8])\big) \backslash \rho_1^{-1}([0, 7/8])\big) /\mathcal{K}\big)$. This procedure leads us to $H^*(S/\mathcal{K}) \cong H^*((S \backslash \rho_0^{-1}([0, 7/8]))/\mathcal{K}) \cong H^*((S \backslash (\rho_0^{-1}([0, 7/8]) \cup \rho_1^{-1}([0, 7/8])))/\mathcal{K}) \cong \cdots \cong H^*((S \backslash (\rho_0^{-1}([0, 7/8]) \cup \cdots \cup \rho_{s-1}^{-1}([0, 7/8])))/\mathcal{K})$. Take $U = S \backslash (\rho_0^{-1}([0, 7/8]) \cup \cdots \cup \rho_{s-1}^{-1}([0, 7/8]))$, which is an open saturated subset included into S. By construction, the inclusion $U \hookrightarrow S$ induces the isomorphism $H^*(S/\mathcal{K}) \cong H^*(U/\mathcal{K})$. This gives (a).

Let $K = S \backslash (\rho_0^{-1}([0, 1/8[) \cup \cdots \cup \rho_{s-1}^{-1}([0, 1/8[))$. This is a subset of S containing U. Its closure in S is $\overline{K} = \overline{S \backslash (\rho_0^{-1}([0, 1/8]) \cup \cdots \cup \rho_{s-1}^{-1}([0, 1/8]))} \subset \overline{S} \backslash \big((\rho_0^{-1}([0, 1/8[))^{\mathrm{i}} \cup \cdots \cup (\rho_{s-1}^{-1}([0, 1/8[))^{\mathrm{i}}\big) = \overline{S} \backslash \big(\rho_0^{-1}([0, 1/8[) \cup \cdots \cup \rho_{s-1}^{-1}([0, 1/8[)\big) = S \backslash (\rho_0^{-1}([0, 1/8[) \cup \cdots \cup \rho_{s-1}^{-1}([0, 1/8[))$, since $\overline{S} \backslash S \subset \Sigma_{s-1} = \rho_0^{-1}(\{0\}) \cup \cdots \cup \rho_{s-1}^{-1}(\{0\})$. Thus, K is closed in S, and hence a compact set. This gives (b).

2.5. Basic cohomology. As in the regular case, the *basic cohomology* $H^*(X/\mathcal{K})$ is the cohomology of the complex $\Omega^*(X/\mathcal{K})$ of *basic forms* (cf. [44]). A differential form ω is basic if $i_X \omega = i_X d\omega = 0$ for every vector field X tangent to $\mathcal{F}$.

Associated with a covering $\{U, V\}$ of X by saturated open subsets, we have the Mayer–Vietoris short exact sequence $0 \to (\Omega^*(X/\mathcal{K}), d) \to (\Omega^*(U/\mathcal{K}), d) \oplus (\Omega^*(V/\mathcal{K}), d) \to (\Omega^*((U \cap V)/\mathcal{K}), d) \to 0$, where the mappings are defined by restriction (the proof is the same as that of Lemma 2.1.1 in [34] for the regular case).

3. TAUTNESS IN THE NONCOMPACT CASE

We prove in this section that the above cohomological characterizations of the tautness of an RF $\mathcal{F}$ are still valid if the manifold is noncompact, whereas the foliation $\mathcal{F}$ is a CERF.

For the rest of this section, we fix a CERF $\mathcal{F}$ defined on a manifold M. We also fix a zipper $(N, \mathcal{H})$ and a reppiz U.

3.1. Tautness class of $\mathcal{F}$. Since N is compact, we see from Theorem 1.4.5 that M possesses a D-metric μ. The *tautness class* of $(M, \mathcal{F})$ is the class $\boldsymbol{\kappa} = [\kappa_\mu] \in H^1(M/\mathcal{F})$ (cf. the text below Theorem 1.4.4). This class is well defined by the following proposition.

Proposition 3.1.1. *Any two D-metrics on $(M, \mathcal{F})$ define the same tautness class.*

Proof. Fix a zipper $(N, \mathcal{H})$ and a reppiz U. Since N is compact, the tautness class $\boldsymbol{\kappa}_N$ is well defined. Let μ be a D-metric on M. The key point of the proof is to relate the class $[\kappa_\mu]$ to κ_N.

It follows from 2.1 (a) and (b) that $\{M, N \backslash \overline{U}\}$ is a saturated open covering of M. It possesses a subordinated partition of the unity $\{f, g\}$ made up by basic functions [34]. Consider a D-metric ν

on N, which always exists since N is compact. Thus, the metric $\lambda = f\mu + (1-f)\nu$ is a bundle-like metric on N with $\lambda|_U = \mu|_U$, which is a D-metric. This gives $\kappa_{\lambda,b}|_U = \kappa_{\mu|_U} = \kappa_\mu|_U$ ((3) and (2)). Denote by $I\colon U \to M$ and $J\colon U \to N$ the natural inclusions. We have

$$I^*[\kappa_\mu] = [\kappa_\mu|_U] = [\kappa_{\lambda,b}|_U] = J^*[\kappa_{\lambda,b}] = J^*\boldsymbol{\kappa}_N.$$

Consider another D-metric μ' on M. The above relation gives $I^*[\kappa_\mu] = I^*[\kappa_{\mu'}]$. It follows from 2.1 (c) that $[\kappa_\mu] = [\kappa_{\mu'}]$.

The first characterization of the tautness is the following assertion.

Theorem 3.2. *Let M be a manifold endowed with a CERF $\mathcal{F}$. Then the following two statements are equivalent.*

(a) *The foliation $\mathcal{F}$ is taut.*

(b) *The tautness class $\boldsymbol{\kappa} \in H^1(M/\mathcal{F})$ vanishes.*

Proof. Let us prove the two implications.

$(a) \Rightarrow (b)$. There is a D-metric μ on M with $\kappa_\mu = 0$. Then $\boldsymbol{\kappa} = [\kappa_\mu] = 0$.

$(b) \Rightarrow (a)$. See [43, Prop. 7.6].[4]

For the second characterization of the tautness, we use the twisted basic cohomology $H^*_{\kappa_\mu}(M/\mathcal{F})$, where μ is a D-metric. Note that this cohomology does not depend on the choice of the D-metric (cf. Proposition 3.1.1 and assertion (c) above Example 1.1).

Theorem 3.3. *Let M be a manifold endowed with a CERF $\mathcal{F}$. Consider a D-metric μ on M. Then the following two assertions are equivalent.*

(a) *The foliation $\mathcal{F}$ is taut.*

(b) *The cohomology group $H^0_{\kappa_\mu}(M/\mathcal{F})$ is $\mathbb{R}$.*

Otherwise, $H^0_{\kappa_\mu}(M/\mathcal{F}) = 0$.

Proof. We proceed in two steps.

$(a) \Rightarrow (b)$. If $\mathcal{F}$ is taut, then $\boldsymbol{\kappa} = [\kappa_\mu] = 0$. Thus, $H^0_{\kappa_\mu}(M/\mathcal{F}) \cong H^0(M/\mathcal{F}) = \mathbb{R}$.

$(b) \Rightarrow (a)$. If $H^0_{\kappa_\mu}(M/\mathcal{F}) \neq 0$, then there is an f, $0 \neq f \in \Omega^0(M/\mathcal{F})$, with $df = f\kappa_\mu$. The set $Z(f) = f^{-1}(0)$ is clearly a closed subset of M. Let us show that it is also open. Take $x \in Z(f)$ and consider a contractible open subset $V \subset M$ containing x. Thus, there is a smooth mapping $g\colon V \to \mathbb{R}$ with $\kappa_\mu = dg$ on V. The calculation

$$d(fe^{-g}) = e^{-g}df - fe^{-g}dg = e^{-g}f\kappa_\mu - e^{-g}f\kappa_\mu = 0$$

shows that fe^{-g} is constant on V. Since $x \in Z(f)$, we have $f \equiv 0$ on V, and therefore $x \in V \subset Z(f)$. This shows that $Z(f)$ is open.

Since M is connected, we have $Z(f) = \varnothing$. It follows from $d(\log|f|) = (1/f)df = \kappa_\mu$ that $\boldsymbol{\kappa} = 0$. The foliation $\mathcal{F}$ is taut.

We have also proved that $H^0_{\kappa_\mu}(M/\mathcal{F}) \neq 0 \Rightarrow H^0_{\kappa_\mu}(M/\mathcal{F}) = \mathbb{R}$.

3.3.1. Remark. To prove that $(b) \Rightarrow (a)$, we do not need $\mathcal{F}$ to be a CERF, whereas the existence of a D-metric μ is in use (see [43, Prop. 7.6]).

For the third characterization, we must extend the basic Poincaré duality to the noncompact case. We find in [38] another version of this Poincaré duality using the cohomological orientation sheaf instead of the twisted basic cohomology used above (compare also with [43, Prop. 7.54]).

Theorem 3.4. *Let M be a manifold endowed with a transversally oriented RF $\mathcal{F}$ possessing a zipper. Consider a D-metric μ on M. If $n = \operatorname{codim}\mathcal{F}$, then $H^*_c(M/\mathcal{F}) \cong H^{n-*}_{\kappa_\mu}(M/\mathcal{F})$.*

Proof. See Appendix.

The third characterization of tautness is as follows. Compare with [43, Prop. 7.56].

[4] At the beginning of Chapter 7 of [43], it is said that the manifold M must be compact. In fact, this condition is not necessary for the proof of Proposition 7.6.

Theorem 3.5. *Let M be a manifold endowed with a CERF $\mathcal{F}$. Suppose that $\mathcal{F}$ is transversally oriented. If $n = \operatorname{codim} \mathcal{F}$, then the following two assertions are equivalent.*

(a) *The foliation $\mathcal{F}$ is taut.*

(b) *The cohomology group $H^n_c(M/\mathcal{F})$ is $\mathbb{R}$.*

Otherwise, $H^n_c(M/\mathcal{F}) = 0$.

Proof. It suffices to apply Theorem 3.4 and Theorem 3.3.

As a direct application, we extend the scope of a well-known result for closed manifolds (see [1, Cor. 6.6]) to arbitrary CERFs.

Corollary 3.6. *Any codimension-one CERF is taut.*

Proof. Let $\mathcal{F}$ be a codimension-one CERF defined on a manifold M. Without loss of generality, we can suppose that $\mathcal{F}$ is transversally oriented [1, Lemma 6.3]. Proceeding by contradiction, suppose that $\mathcal{F}$ is not taut. Then $H^0_{\kappa_\mu}(M/\mathcal{F}) = 0$ by Theorem 3.3, and $\boldsymbol{\kappa} \neq 0$ by Theorem 3.2. Thus, $H^1(M/\mathcal{F}) \neq 0$. Now $H^0_{\kappa_\mu, c}(M/\mathcal{F}) \neq 0$ by Remark 4.11 (b), and hence $H^0_{\kappa_\mu}(M/\mathcal{F}) \neq 0$. Theorem 3.4 yields $H^1_c(M/\mathcal{F}) \neq 0$, a contradiction.

4. APPENDIX

This appendix is devoted to the proof of Theorem 3.4. We distinguish two cases following the orientability of M. Let us first introduce two technical tools.

4.1. Bredon's Trick. The Mayer–Vietoris sequence allows us to make computations if the manifold is equipped by a suitable finite covering. The passage from the finite case to the general one can be carried out by using an adapted version of Bredon's trick, [3, p. 289], which we present now.

Let X be a paracompact topological space and let $\{U_\alpha\}$ be an open covering, closed with respect to finite intersections. Suppose that $Q(U)$ is a statement about open subsets of X, satisfying the following three properties:

(BT1) $Q(U_\alpha)$ is true for each α;

(BT2) $Q(U)$, $Q(V)$, and $Q(U \cap V) \Longrightarrow Q(U \cup V)$, where U and V are open subsets of X;

(BT3) $Q(U_i) \Longrightarrow Q(\cup_i U_i)$, where $\{U_i\}$ is an arbitrary disjoint family of open subsets of X.

Then $Q(X)$ is true.

4.2. Mayer–Vietoris. Associated with a covering $\{U, V\}$ of M by saturated open subsets is the Mayer–Vietoris short exact sequence

$$0 \to (\Omega^*(M/\mathcal{F}), d) \to (\Omega^*(U/\mathcal{F}), d) \oplus (\Omega^*(V/\mathcal{F}), d) \to (\Omega^*((U \cap V)/\mathcal{F}), d) \to 0,$$

where the mappings are defined by restriction (see, e.g., [34]). In the context of compact support, we have the Mayer–Vietoris sequence

$$0 \to (\Omega^*_c((U \cap V)/\mathcal{F}), d) \to (\Omega^*_c(U/\mathcal{F}), d) \oplus (\Omega^*_c(V/\mathcal{F}), d) \to (\Omega^*_c(M/\mathcal{F}), d) \to 0,$$

where the mappings are defined by extension (see, e.g., [34]). Finally, for the twisted basic cohomology, we have the Mayer–Vietoris sequence

$$0 \to (\Omega^*(M/\mathcal{F}), d_{\kappa_\mu}) \to (\Omega^*(U/\mathcal{F}), d_{\kappa_\mu}) \oplus (\Omega^*(V/\mathcal{F}), d_{\kappa_\mu}) \to (\Omega^*((U \cap V)/\mathcal{F}), d_{\kappa_\mu}) \to 0,$$

where the mappings are defined by restriction.

Orientable case

4.3. Integration. To define the duality operator, we choose (a) an oriented manifold M, (b) a transversally oriented RF $\mathcal{F}$ (TORF for short) on M, and (c) a D-metric μ on $(M, \mathcal{F})$. We say that $(M, \mathcal{F}, \mu)$ is a *D-triple*. The associated tangent volume form is χ_μ (it exists since $\mathcal{F}$ is also oriented). With all these ingredients, we define a morphism $\int_M \colon H^*_c(M/\mathcal{F}) \longrightarrow \operatorname{Hom}\left(H^{n-*}_{\kappa_\mu}(M/\mathcal{F}); \mathbb{R}\right) = \left(H^{n-*}_{\kappa_\mu}(M/\mathcal{F})\right)^\star$ by $\int_M([\alpha])([\beta]) = \int_M \alpha \wedge \beta \wedge \chi_\mu$. Here $n = \operatorname{codim}_M \mathcal{F}$. This operator is well defined since M is oriented and we have the Rummler formula

$$i_{Y_1} \cdots i_{Y_r} d\chi_\mu + \chi_\mu(Y_1, \ldots, Y_r) \cdot \kappa_\mu = 0, \tag{6}$$

where $\{Y_1, \ldots, Y_r\}$ are vector fields tangent to $T\mathcal{F}$ and $r = \dim \mathcal{F}$ [43]. We prove in this section that the operator $\int_M$ is an isomorphism.

Before studying the general case, we consider first some special cases.

Lemma 4.4. *Suppose that the D-triple $(M, \mathcal{F}, \mu)$ is $(E \times \mathbb{R}, \mathcal{E} \times \mathcal{I}, \mu)$, where $(E, \mathcal{E})$ is a closed foliated manifold and the leaves of $\mathcal{E}$ are dense. Then the operator $\int_M$ is an isomorphism.*

Proof. For the proof of the lemma, we proceed in several steps. Introduce the following notation. For a differential form (or Riemannian metric) ω on $E \times \mathbb{R}$, denote by $\omega(t)$ the restriction $I_t^* \omega$, where $I_t \colon E \to E \times \mathbb{R}$ is defined by $I_t(x) = (x, t)$ for any $t \in \mathbb{R}$.

First Step. *The cohomology $H_c^*\left((E \times \mathbb{R})/\mathcal{E} \times \mathcal{I}\right)$.* Consider a smooth compactly supported function $f \colon \mathbb{R} \to [0,1]$ such that $\int_{\mathbb{R}} f dt = 1$. As is known, the correspondence $[\gamma] \mapsto [f\gamma \wedge dt]$ establishes an isomorphism between $H^*(E/\mathcal{E})$ and $H_c^{*+1}\left((E \times \mathbb{R})/\mathcal{E} \times \mathcal{I}\right)$ [34]. In fact, this isomorphism does not depend on choice of f.

Second Step. *The cohomology $H_{\kappa_\mu}^*\left((E \times \mathbb{R})/\mathcal{E} \times \mathcal{I}\right)$.* Note that the metric $\mu(0)$ is a D-metric (1). Thus, we have two D-metrics on $E \times \mathbb{R}$, μ and $\mu(0) + dt^2$, with $\kappa_{\mu(0)+dt^2} = \kappa_{\mu(0)}$. Since $\mathcal{E} \times \mathcal{I}$ is a CERF (it suffices to take $E \times]-1,1[$ as a reppiz and $(E \times \mathbb{S}^1, \mathcal{E} \times \mathcal{I})$ as a zipper), there is a function $g \in \Omega^0((E \times \mathbb{R})/\mathcal{E} \times \mathcal{I})$ with $\kappa_\mu = \kappa_{\mu(0)} + dg$ (cf. Proposition 3.1.1). Since the leaves of $\mathcal{E}$ are dense in N, the (basic) function g is smooth on $\mathbb{R}$.

We know (see (c) above Example 1.1) that the assignment $[\omega] \mapsto [e^g \omega]$ establishes an isomorphism between $H_{\kappa_{\mu(0)}}^*\left((E \times \mathbb{R})/\mathcal{E} \times \mathcal{I}\right)$ and $H_{\kappa_\mu}^*\left((E \times \mathbb{R})/\mathcal{E} \times \mathcal{I}\right)$. The usual technique shows that the assignment $[\omega] \mapsto [\omega]$ establishes an isomorphism between $H_{\kappa_{\mu(0)}}^*(E/\mathcal{E})$ and $H_{\kappa_{\mu(0)}}^*\left((E \times \mathbb{R})/\mathcal{E} \times \mathcal{I}\right)$.

Last Step. Note that $(E, \mathcal{E}, \mu(0))$ is a D-triple. Since E is compact, the morphism

$$\int_E \colon H^*(E/\mathcal{E}) \longrightarrow \left(H_{\kappa_{\mu(0)}}^{n-1-*}(E/\mathcal{E})\right)^\star, \tag{7}$$

defined by $\int_E([\gamma])([\zeta]) = \int_E \gamma \wedge \zeta \wedge \chi_\mu(0)$, is an isomorphism [21]. Following the previous steps, it suffices to show that the morphism $\int_E^\# \colon H^*(E/\mathcal{E}) \longrightarrow \left(H_{\kappa_{\mu(0)}}^{n-1-*}(E/\mathcal{E})\right)^\star$ defined by $\int_E^\#([\gamma])([\zeta]) = \int_{E\times\mathbb{R}} f e^g \gamma \wedge dt \wedge \zeta \wedge \chi_\mu$ is an isomorphism. Let us prove that *$\int_E^\#$ is a monomorphism.* Consider $[\gamma] \in H^*(E/\mathcal{E})$ with $\int_E^\#([\gamma]) \equiv 0$. We have $\int_{\mathbb{R}} f(t) e^{g(t)} \left(\int_E \gamma \wedge \zeta \wedge \chi_\mu(t)\right) dt = 0$ for every $[\zeta] \in H_{\kappa_{\mu(0)}}^{n-1-*}(E/\mathcal{E})$ and any smooth compactly supported function $f \colon \mathbb{R} \to [0,1]$ with $\int_{\mathbb{R}} f = 1$. Thus, $\int_E \gamma \wedge \zeta \wedge \chi_{\mu(t)} dt = 0$ for any $[\zeta] \in H_{\kappa_{\mu(0)}}^{n-1-*}(E/\mathcal{E})$ and any $t \in \mathbb{R}$. Hence $\int_E([\gamma])([\zeta]) = 0$ for every $[\zeta] \in H_{\kappa_{\mu(0)}}^{n-1-*}(E/\mathcal{E})$. Next, *$\int_E^\#$ is an epimorphism.* It follows from (7) that $\dim H^*(E/\mathcal{E}) = \dim \left(H_{\kappa_{\mu(0)}}^{n-1-*}(E/\mathcal{E})\right)^\star$, which is finite since E is compact [11]. Thus, $\int_E^\#$ is also an epimorphism.

Lemma 4.5. *Let $(M, \mu, \mathcal{F})$ be a D-triple. Suppose that $(M, \mathcal{F})$ possesses a transversally parallelizable zipper. Then $\int_M$ is an isomorphism.*

Proof. Denote by $(N, \mathcal{H})$ the zipper in question. Since $\mathcal{H}$ is transversally parallelizable, there is a fiber bundle $\pi \colon N \to B$ whose fibers are the closures of the leaves of $\mathcal{H}$. Since M is saturated for the leaves of $\mathcal{H}_M$, it is also saturated for the closures of these leaves. We obtain an open subset $V_M \subset B$ with $M = \pi^{-1}(V_M)$.

Choose an open subset V of V_M. The foliation $\mathcal{H}_{\pi^{-1}(V)}$ admits the zipper $(N, \mathcal{H})$. Note that $\pi^{-1}(V)$ is oriented and $\mathcal{H}_{\pi^{-1}(V)}$ is a TORF. Then $(\pi^{-1}(V), \mathcal{H}_{\pi^{-1}(V)}, \mu_{\pi^{-1}(V)})$ is a D-triple. Thus, the operator $\int_{\pi^{-1}(V)}$ is well defined. We claim that this operator is nondegenerate. Then the proof can be completed by taking $V = V_M$.

Let $(E, \mathcal{E})$ be a generic fiber of π. The manifold E is closed, and the leaves of $\mathcal{E}$ are dense in E. We know that the fibration $\pi \colon \pi^{-1}(V) \to V$ has a foliated atlas $\mathcal{A} = \{\varphi \colon (\pi^{-1}(U), \mathcal{H}) \longrightarrow (U \times E, \mathcal{I} \times \mathcal{E})\}$. Suppose that the covering $\mathcal{U} = \{U \mid \exists (U, \varphi) \in \mathcal{A}\}$ is a *good* covering of V (i.e., if $U_1, \dots, U_k \in \mathcal{U}$, then the intersection $V = U_1 \cap \dots \cap U_k$ is diffeomorphic to $\mathbb{R}^{\dim B}$ [4]) and is closed with respect to finite intersections. Consider the statement $Q(U)$ claiming that *the integration operator $\int_{\pi^{-1}(U)}$ is an isomorphism*, where $U \subset V$ is an open subset. Following Bredon's trick, it suffices to prove (BT1), (BT2), and (BT3) with respect to the covering $\mathcal{U}$.

+ (BT1). It follows directly from the Lemma 4.4.

+ (BT2). The integration operator $\int$ commutes with the restriction and inclusion operators. It suffices to apply the five–lemma to the Mayer–Vietoris sequences in 4.2.

+ (BT3). Straightforward.

4.6. Frame bundle. Let $(M, \mu, \mathcal{F})$ be a D-triple possessing a zipper $(N, \mathcal{H})$. Consider the bundle $p\colon \widetilde{N} \to N$ of transverse oriented orthonormal frames of N [25]. It is an $SO(n)$-principal bundle. The canonical lift $\widetilde{\mathcal{H}}$ of $\mathcal{H}$ is a transversally parallelizable foliation on the closed manifold $\widetilde{N}$ with $\operatorname{codim}_{\widetilde{N}} \mathcal{H} = n + \dim SO(n)$. The restriction bundle morphism $p_*\colon T\widetilde{\mathcal{H}} \to T\mathcal{H}$ is an isomorphism. We can lift $(M, \mathcal{F}, \mu)$ as follows.

- *Lifting* $\mathcal{F}$. Since M is a saturated open subset of $(N, \mathcal{H})$, $\widetilde{M} = p^{-1}(M)$ is a saturated open subset of $(\widetilde{N}, \widetilde{\mathcal{H}})$. Denote by $\widetilde{\mathcal{F}}$ the restriction of $\widetilde{\mathcal{H}}$ to $\widetilde{M}$. Note that $\widetilde{\mathcal{F}}$ is transversally parallelizable (and thus a TORF). The manifold $\widetilde{M}$ is oriented since $p\colon \widetilde{M} \to M$ is a $SO(n)$-bundle and M is oriented. The foliation $\widetilde{\mathcal{F}}$ possesses $(\widetilde{N}, \widetilde{\mathcal{H}})$ as a zipper.

- *Lifting* μ. Consider the decomposition $\mu = \mu_1 + \mu_2$ with respect to the orthogonal decomposition $TM = T\mathcal{F} \oplus (T\mathcal{F})^{\perp_\mu}$. Since the restriction bundle morphism $p_*\colon T\widetilde{\mathcal{F}} \to T\mathcal{F}$ is an isomorphism, we have the decomposition $T\widetilde{M} = T\widetilde{\mathcal{F}} \oplus p_*^{-1}(T\mathcal{F})^{\perp_\mu}$. Moreover, since $(\widetilde{M}, \widetilde{\mathcal{F}})$ is a Riemannian foliated manifold (TP in fact), there is a Riemannian metric ν_2 on $p_*^{-1}(T\mathcal{F})^{\perp_\mu}$ such that the Riemannian metric $\nu = p^*\mu_1 + \nu_2$ is bundle-like on $(\widetilde{M}, \widetilde{\mathcal{F}})$. Then the associated volume forms satisfy $\chi_\nu = p^*\chi_\mu$. Rummler's formula (6) gives

$$\kappa_\nu = p^*\kappa_\mu. \tag{8}$$

We conclude that $(\widetilde{M}, \widetilde{\mathcal{F}}, \nu)$ is a D-triple possessing a transversally parallelizable zipper. By Lemma 4.5, the integration operator $\int_{\widetilde{M}}\colon H_c^*(\widetilde{M}/\widetilde{\mathcal{F}}) \longrightarrow \left(H_{\kappa_\nu}^{n+\ell-*}(\widetilde{M}/\widetilde{\mathcal{F}})\right)^\star$ is an isomorphism. Here $\ell = \dim SO(n)$. We shall prove Theorem 3.4 by relating $(M, \mathcal{F}, \mu)$ with $(\widetilde{M}, \widetilde{\mathcal{F}}, \nu)$ with the help of two spectral sequences.

4.7. Spectral sequence.[5] Let $F^p\Omega_c^{p+q}(\widetilde{M}/\widetilde{\mathcal{F}}) = \{\omega \in \Omega_c^{p+q}(\widetilde{M}/\widetilde{\mathcal{F}}) \mid i_{X_{u_0}} \cdots i_{X_{u_q}}\omega = 0$ for each $\{u_0, \ldots, u_q\} \subset \mathfrak{g}\}$ be the usual filtration, where $X_u \in \mathfrak{X}(\widetilde{M})$ is determined by an element $u \in \mathfrak{g}$ (the Lie algebra of $SO(n)$). It induces a filtration in the differential complex ${}_IK^* = \left(\Omega_c^*(\widetilde{M}/\widetilde{\mathcal{F}})\right)^{SO(n)}$ by $F^p\,{}_IK^{p+q} = {}_IK^{p+q} \cap F^p\Omega_c^*(\widetilde{M}/\widetilde{\mathcal{F}})$, leading us to a first-quadrant spectral sequence ${}_IE_r^{p,q}$ such that

(a) ${}_IE_r^{p,q} \Rightarrow H_c^{p+q}\left(\widetilde{M}/\widetilde{\mathcal{F}}\right)$, and

(b) ${}_IE_2^{p,q} \cong H_c^p(M/\mathcal{F}) \otimes H^q(SO(n))$.

Let us prove this. The inclusion ${}_IK^* \hookrightarrow \Omega_c^*(\widetilde{M}/\widetilde{\mathcal{F}})$ induces an isomorphism in cohomology. This is a standard argument based on the fact that $SO(n)$ is a connected compact Lie group [15, Th. I, Chap. IV, Vol. II]. This gives (a).

Denote by $\gamma_u = i_{X_u}\nu$ the associated fundamental differential form. Note that the assignment $\alpha \otimes u \mapsto \alpha \wedge \gamma_u$ induces the identification

$$\oplus_{p+q=*}\left(F^p\Omega_c^p(\widetilde{M}/\widetilde{\mathcal{F}}) \otimes \wedge^q\mathfrak{g}\right) = \Omega_c^*(\widetilde{M}/\widetilde{\mathcal{F}}). \tag{9}$$

Thus, ${}_IE_0^{p,q} \cong \left(F^p\Omega_c^p(\widetilde{M}/\widetilde{\mathcal{F}}) \otimes \bigwedge^q\mathfrak{g}\right)^{SO(n)}$. A straightforward calculation gives (see also [15, (9.2), Vol. III]) $d_0 = -\,\mathrm{Identity} \otimes \delta$, where d_0 is the 0-differential of the spectral sequence and δ is the differential of $\wedge^*\mathfrak{g}$. This gives ${}_IE_1^{p,q} \cong \left(F^p\Omega_c^p(\widetilde{M}/\widetilde{\mathcal{F}})\right)^{SO(n)} \otimes H^q(SO(n))$ [15, 5.28 and 5.12, vol. III]. On the other hand, $\left(F^p\Omega_c^p(\widetilde{M}/\widetilde{\mathcal{F}})\right)^{SO(n)} = \{\omega \in \Omega_c^p(\widetilde{M}/\widetilde{\mathcal{F}}) \,/\, i_{X_u}\omega = L_{X_u}\omega = 0 \text{ for each } u \in \mathfrak{g}\} = p^*\Omega_c^p(M/\mathcal{F})$, and hence ${}_IE_1^{p,q} \cong \Omega_c^p(M/\mathcal{F}) \otimes H^q(SO(n))$. Since d_1, the 1-differential of the spectral sequence, becomes $d \otimes \mathrm{Identity}$, we conclude that ${}_IE_2^{p,q} \cong H_c^p(M/\mathcal{F}) \otimes H^q(SO(n))$. This gives (b).

4.8. Another spectral sequence.[6] Let $F^p\Omega^{p+q}(\widetilde{M}/\widetilde{\mathcal{F}}) = \{\omega \in \Omega^{p+q}(\widetilde{M}/\widetilde{\mathcal{F}}) \mid i_{X_{u_0}} \cdots i_{X_{u_q}}\omega = 0$ for each $\{u_0, \ldots, u_q\} \subset \mathfrak{g}\}$ be the usual filtration. The complex ${}_{II}K^* = \left(\left(\Omega^{n+\ell-*}(\widetilde{M}/\widetilde{\mathcal{F}})\right)^{SO(n)}\right)^\star$

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[5] This is the spectral sequence of [15, 9.1, Chap. IX, Vol. III].

[6] This is the spectral sequence of [15, 9.1, Chap. IX, Vol. III] associated with $\left(\Omega_{\kappa_\nu}^*\left(\widetilde{M}/\widetilde{\mathcal{F}}\right)\right)^\star$, which is *not* a differential graded algebra.

admits ∇_{κ_ν}, the dual of d_{κ_ν}, as a differential since (8). Consider the filtration $F^p{}_{II}K^{p+q} = \{L \in {}_{II}K^{p+q} \ / \ L \equiv 0$ on $\big(F^{n-p+1}\Omega^{n+\ell-(p+q)}\big(\widetilde{M}/\widetilde{\mathcal{F}}\big)\big)^{SO(n)}\}$. A straightforward calculation gives that $F^{p+1}{}_{II}K^* \subset F^p{}_{II}K^*$ and $\nabla_{\kappa_\nu}(F^p{}_{II}K^{p+q}) \subset F^p{}_{II}K^{p+q+1}$. Thus, it induces a first-quadrant spectral sequence ${}_{II}E_r^{p,q}$ verifying

(a) ${}_{II}E_r^{p,q} \Rightarrow \big(H_{\kappa_\nu}^{n+\ell-(p+q)}\big(\widetilde{M}/\widetilde{\mathcal{F}}\big)\big)^\star$,
(b) ${}_{II}E_2^{p,q} \cong \big(H_{\kappa_\mu}^{n-p}\big(M/\mathcal{F}\big)\big)^\star \otimes \big(H^{\ell-q}(SO(n))\big)^\star$.

Let us prove this. As in 4.7 (a), the inclusion $\big(\Omega^*\big(\widetilde{M}/\widetilde{\mathcal{F}}\big)\big)^{SO(n)} \hookrightarrow \Omega^*\big(\widetilde{M}/\widetilde{\mathcal{F}}\big)$ induces an isomorphism in the corresponding twisted basic cohomology. This yields (a).

The identification similar to (9) here is $\oplus_{p+q=*}\big(F^p\Omega^p\big(\widetilde{M}/\widetilde{\mathcal{F}}\big) \otimes \wedge^q\mathfrak{g}\big) = \Omega^*\big(\widetilde{M}/\widetilde{\mathcal{F}}\big)$. Thus, we obtain ${}_{II}E_0^{p,q} \cong \big(\big(F^{n-p}\Omega^{n-p}\big(\widetilde{M}/\widetilde{\mathcal{F}}\big) \otimes \wedge^{\ell-q}\mathfrak{g}\big)^{SO(n)}\big)^\star$. A straightforward calculation shows that the 0-differential of the spectral sequence is the dual of $-\,$Identity$\,\otimes\,\delta$. This gives ${}_{I}E_1^{p,q} \cong \big(\big(F^{n-p}\Omega^{n-p}\big(\widetilde{M}/\widetilde{\mathcal{F}}\big)\big)^{SO(n)}\big)^\star \otimes \big(H^{\ell-q}(SO(n))\big)^\star$. On the other hand, $\big(F^{n-p}\Omega^{n-p}\big(\widetilde{M}/\widetilde{\mathcal{F}}\big)\big)^{SO(n)} = \big\{\omega \in \Omega^{n-p}\big(\widetilde{M}/\widetilde{\mathcal{F}}\big) \,/\, i_{X_u}\omega = L_{X_u}\omega = 0 \text{ for each } u \in \mathfrak{g}\big\} = p^*\Omega^{n-p}(M/\mathcal{F})$, and hence ${}_{I}E_1^{p,q} \cong \big(\Omega^{n-p}\big(M/\mathcal{F}\big)\big)^\star \otimes \big(H^{\ell-q}\big(SO(n)\big)\big)^\star$. Since the 1-differential of the spectral sequence becomes the dual of $d_{\kappa_\mu} \otimes$ Identity (cf. (8)), we conclude that ${}_{I}E_2^{p,q} \cong \big(H_{\kappa_\mu}^{n-p}(M/\mathcal{F})\big)^\star \otimes \big(H^{\ell-q}(SO(n))\big)^\star$.

Proposition 4.9. *Let $(M,\mu,\mathcal{F})$ be a D-triple. Suppose that $\mathcal{F}$ possesses a zipper, then the operator $\int_M$ is an isomorphism.*

Proof. Let $\Delta\colon \big(\big(\Omega_c^*\big(\widetilde{M}/\widetilde{\mathcal{F}}\big)\big)^{SO(n)}; d\big) \longrightarrow \big(\big(\big(\Omega^{n+\ell-*}\big(\widetilde{M}/\widetilde{\mathcal{F}}\big)\big)^{SO(n)}\big)^\star; \nabla_{\kappa_\nu}\big)$ be the differential operator defined by $\Delta(\omega)(\eta) = \int_{\widetilde{M}} \omega \wedge \eta \wedge \chi_{\kappa_\nu}$ (cf. 4.3). By the degree reasons, it preserves the involved filtrations, i.e., $\Delta\,(F^p{}_{I}K^{p+q}) \subset F^p{}_{II}K^{p+q}$. It induces the morphisms

+ [at ∞-level] $\displaystyle\int_{\widetilde{M}}\colon H_c^{p+q}\big(\widetilde{M}/\widetilde{\mathcal{F}}\big) \longrightarrow \big(H_{\kappa_\nu}^{n+\ell-(p+q)}\big(\widetilde{M}/\widetilde{\mathcal{F}}\big)\big)^\star$,
+ [at 2-level] $\displaystyle\int_M \otimes \int_{SO(n)}\colon H_c^p(M/\mathcal{F}) \otimes H^q(SO(n)) \longrightarrow \big(H_{\kappa_\mu}^{n-p}(M/\mathcal{F})\big)^\star \otimes \big(H^{\ell-q}(SO(n))\big)^\star$.

Since the operators $\int_{\widetilde{M}}$ and $\int_{SO(n)}$ are isomorphisms (cf. Lemma 4.5), Zeeman's comparison theorem proves that $\int_M$ is an isomorphism (see, e.g., [23]).

Nonorientable case

For the nonorientable case, it suffices to consider an orientable covering in order to apply the previous results.

Proposition 4.10. *Let $\mathcal{F}$ be a TORF defined on a manifold M. Suppose that $\mathcal{F}$ possesses a zipper. Consider μ a D-metric on M. If $n = \operatorname{codim}\mathcal{F}$, then $H_c^*(M/\mathcal{F}) \cong H_{\kappa_\mu}^{n-*}(M/\mathcal{F})$.*

Proof. Suppose that M is not orientable. Choose a zipper $(N,\mathcal{H})$ of $\mathcal{F}$. Regard μ as a D-metric on $(M,\mathcal{F})$.

Consider the two-fold orientable covering $\oslash\colon \check{N} \to N$ of N. It gives an oriented closed manifold. Denote by $\check{\mathcal{H}}$ the lifted foliation, which is Riemannian. In fact, there is a smooth foliated action $\Phi\colon \mathbb{Z}_2 \times (\check{N},\check{\mathcal{H}}) \to (\check{N},\check{\mathcal{H}})$ such that $\oslash$ is $\mathbb{Z}_2$-invariant and $\check{N}/\mathbb{Z}_2 = N$. Write $\flat\colon (\check{N},\check{\mathcal{H}}) \to (\check{N},\check{\mathcal{H}})$ for the foliated diffeomorphism generating this action.

The restriction $\oslash\colon \oslash^{-1}(M) \to M$ is a two-fold orientable covering of M. The manifold $\check{M} = \oslash^{-1}(M)$ is oriented and $\check{\mathcal{H}}$-saturated. The diffeomorphism $\flat\colon \check{M} \to \check{M}$ preserves the foliation $\check{\mathcal{F}}$ induced by $\check{\mathcal{H}}$ by restriction. It preserves the D-metric $\check{\mu} = \oslash^*\mu$ as well.

Since the foliation $\mathcal{F}$ is transversally oriented, the foliation $\check{\mathcal{F}}$ is also transversally oriented. Moreover, the diffeomorphism $\flat\colon \check{M} \to \check{M}$ preserves the transversal orientation of $\check{\mathcal{F}}$. It does not preserve the orientation of $\check{M}$, because M is not an orientable manifold. We see that $\flat$ does not preserve the tangential orientation of $\check{\mathcal{F}}$, and this gives

$$\flat^* \chi_{\check{\mu}} = -\chi_{\check{\mu}}. \tag{10}$$

The foliated manifold $\check{\mathcal{F}}$ is a TORF on an oriented manifold $\check{M}$ with $(\check{N}, \check{\mathcal{H}})$ as a zipper. Thus, $(\check{M}, \check{\mathcal{F}}, \check{\mu})$ is a D-triple. It follows from Theorem 3.4 that $\int_{\check{M}}$ induces an isomorphism

$$H_c^*\left(\check{M}/\check{\mathcal{F}}\right) \cong H_{\kappa_{\check{\mu}}}^{n-*}\left(\check{M}/\check{\mathcal{F}}\right), \tag{11}$$

where $n = \operatorname{codim}_{\check{M}} \check{\mathcal{F}} = \operatorname{codim}_M \mathcal{F}$.

On the other hand, the mapping $\oslash$ induces the isomorphisms $H_c^*(M/\mathcal{F}) \cong \left(H_c^*\left(\check{M}/\check{\mathcal{F}}\right)\right)^{\mathbb{Z}_2}$ and $H_{\kappa_\mu}^*(M/\mathcal{F}) \cong \left(H_{\kappa_{\check{\mu}}}^*\left(\check{M}/\check{\mathcal{F}}\right)\right)^{\mathbb{Z}_2}$ since $\flat^*\kappa_{\check{\mu}} = \kappa_{\check{\mu}}$. By (11), it suffices to prove that $\int_{\check{M}}$ is $\mathbb{Z}_2$-invariant. This comes from the relation

$$\int_{\check{M}} \flat^*\alpha \wedge \beta \wedge \chi_{\check{\mu}} \overset{(10)}{=} -\int_{\check{M}} \flat^*\alpha \wedge \beta \wedge \flat^*\chi_{\check{\mu}} \xlongequal{\flat^{-1}=\flat} -\int_{\check{M}} \flat^*\alpha \wedge \flat^*\flat^*\beta \wedge \flat^*\chi_{\check{\mu}} \xlongequal{\flat \text{ not orient.}} \int_{\check{M}} \alpha \wedge \flat^*\beta \wedge \chi_{\check{\mu}},$$

where $\alpha \in \Omega_c^*\left(\check{M}/\check{\mathcal{F}}\right)$ and $\beta \in \Omega^{n-*}\left(\check{M}/\check{\mathcal{F}}\right)$.

4.11. Remarks.

(a) The above proof shows as a by-product that the pairing $I\colon H_c^*(M/\mathcal{F}) \oplus H_{\kappa_\mu}^{n-*}(M/\mathcal{F}) \longrightarrow \mathbb{R}$ defined by $I([\alpha], [\beta]) = \int_{\check{M}} \oslash^*\alpha \wedge \oslash^*\beta \wedge \chi_{\check{\mu}}$ is nondegenerate.

(b) Under the assumptions of Theorem 3.4, we also have $H^*(M/\mathcal{F}) \cong H_{\kappa_\mu, c}^{n-*}(M/\mathcal{F})$, where the twisted cohomology is with compact supports.

REFERENCES

1. J. A. Álvarez López, "The Basic Component of the Mean Curvature of Riemannian Foliations," Ann. Global Anal. Geom. **10** (2), 179–194 (1992).
2. H. Boualem and P. Molino, "Modèles locaux saturés de feuilletages riemanniens singuliers," C. R. Acad. Sci. Paris Ser. I Math. **316** (9), 913–916 (1993).
3. G. E. Bredon, *Topology and Geometry*, Grad. Texts in Math. **139** (Springer, New York–Berlin, 1993).
4. R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, Grad. Texts in Math. **82** (Springer, 1982).
5. Y. Carrière, "Flots riemanniens," in: *Transversal Structure of Foliations (Toulouse, 1982)*, Astérisque **116** (Société Mathèmatique de France, Paris, 1984), pp. 31–52.
6. Y. Carrière, "Variations on Riemannian Flows," Appendix A in [27].
7. G. Cairns and R. Escobales, "Further Geometry of the Mean Curvature One-Form and the Normal Plane Field One-Form on a Foliated Riemannian Manifold," J. Aust. Math. Soc. Ser. A **62** (1), 46–63 (1997).
8. D. Domínguez, "A Tenseness Theorem for Riemannian Foliations," C. R. Acad. Sci. Paris Ser. I Math. **320** (11), 1331–1335 (1995).
9. D. Domínguez, "Finiteness and Tenseness Theorems for Riemannian Foliations," Amer. J. Math. **120** (6), 1237–1276 (1998).
10. A. El Kacimi et G. Hector, "Decomposition de Hodge basique pour un feuilletage riemannien," Ann. Inst. Fourier **36** (3), 207–227 (1986).
11. A. El Kacimi, V. Sergiescu, and G. Hector, "La cohomologie basique d'un feuilletage riemannien est de dimension finie," Math. Z. **188** (4), 593–599 (1985).
12. A. El Kacimi and M. Nicolau, "On the Topological Invariance of the Basic Cohomology," Math. Ann. **295** (4), 627–634 (1993).
13. E. Ghys, "Un feuilletage analytique dont la cohomologie basique est de dimension infinie," Publ. de l'IRMA, Lille **7** (1985).
14. H. Gluck, "Dynamical Behavior of Geodesic Fields," in *Global Theory of Dynamical Systems*, Lecture Notes in Math. **819** (Springer, Berlin, 1980), pp. 190–215.
15. W. Greub, S. Halperin and R. Vanstone, *Connections, Curvature and Cohomology*, Vols. II, III (Academic Press, New York–London, 1973–1976).
16. A. Haefliger, "Some Remarks on Foliations with Minimal Leaves," J. Differential Geom. **15** (2), 269–284 (1980).
17. A. Haefliger, "Pseudogroups of Local Isometries," in *Differential Geometry (Santiago de Compostela, 1984)* (Pitman, Boston, Mass.-London, 1985), pp. 174–197.

18. A. Haefliger, "Leaf Closures in Riemannian Foliations," in *A fète of Topology* (Academic Press, Boston, 1988), pp. 3–32.
19. F. W. Kamber and Ph. Tondeur, "Harmonic Foliations," in: *Harmonic Maps (New Orleans, La., 1980)* Lecture Notes in Math. **949** (Springer, Berlin-New York, 1982), pp. 87–121.
20. F. Kamber and Ph. Tondeur, "Foliations and Metrics," in *Differential Geometry (College Park, Md., 1981/1982)*, Progr. Math. **32** (Birkhäuser, 1983), pp. 103–152.
21. F. W. Kamber and Ph. Tondeur, "Duality Theorems for Foliations," in *Transversal Structure of Foliations*, Astérisque **116** (Société Mathèmatique de France, Paris, 1984), pp. 108–116.
22. X. Masa, "Duality and Minimality in Riemannian Foliations," Comment. Math. Helv. **67** (1), 17–27 (1992).
23. J. McCleary, *User's Guide to Spectral Sequences* (Publish or Perish, Inc., Wilmington, Del., 1985).
24. V. Miquel and R. Wolak, "Minimal Singular Riemannian Foliations," C. R. Math. Acad. Sci. Paris **342** (1), 33–36 (2006).
25. P. Molino, "Feuilletages transversalement complets et applications," Ann. Sci. École Norm. Sup. **10**, 289–307 (1977).
26. P. Molino, "Feuilletages riemanniens ré guliers et singuliers," in *Colloque Gé omé trie et Physique, Paris 1986*, Travaux en cours **33**, 173–201 (1988).
27. P. Molino, *Riemannian Foliations*, Progr. Math. **73** (Birkhäuser Boston, Inc., Boston, MA, 1988).
28. P. Molino, "Orbit-Like Foliations," in *Geometric Study of Foliations* (World Sci. Publ., River Edge, NJ, 1994), pp. 97–119.
29. P. Molino and V. Sergiescu, "Deux remarques sur les flots riemanniens," Manuscripta Math. **51**, 145–161 (1985).
30. B. Reinhart, "Foliated Manifolds with Bundle-Like Metrics," Ann. of Math. **69**, 119–132 (1959).
31. B. Reinhart, "Harmonic Integrals on Foliated Manifolds," Amer. J. Math. **81**, 529–536 (1959).
32. J. I. Royo Prieto, "Estudio cohomológico de Flujos Riemannianos," Ph. D. Thesis. Universidad del País Vasco, Euskal Herriko Unibertsitatea, `http://www.ehu.es/joseroyo/pdf/tesis.pdf`.
33. J. I. Royo Prieto, M. Saralegi-Aranguren, and R. Wolak, "Top-Dimensional Group of the Basic Intersection Cohomology for Singular Riemannian Foliations," Bull. Pol. Acad. Sci. Math. **53** (4), 429–440 (2005).
34. J. I. Royo Prieto, M. Saralegi-Aranguren, and R. Wolak, "Tautness for Riemannian Foliations on Non-Compact Manifolds," Manuscripta Math. **126** (2), 177–200 (2008).
35. H. Rummler, "Quelques notions simples en gé omé trie riemannienne et leurs applications aux feuilletages compacts," Comment. Math. Helv. **54** (2), 224–239 (1979).
36. M. Saralegi-Aranguren and R. Wolak, "Basic Intersection Cohomology of Conical Fibrations," Mat. Zametki **77** (2), 235–257 (2005) [Math. Notes **77** (2), 213–231 (2005)].
37. M. Saralegi-Aranguren and R. Wolak, "The BIC of a Singular Foliation Defined by an Abelian Group of Isometries," Ann. Polon. Math. **89** (3), 203–246 (2006).
38. V. Sergiescu, "Cohomologie basique et dualité des feuilletages riemanniens," Ann. Inst. Fourier **3**, 137–158 (1985).
39. V. Sergiescu, "Basic Cohomology and Tautness of Riemannian Foliations," Appendix B in: P. Molino, *Riemannian Foliations*, Progr. Math. **73** (Birkhäuser Boston, Inc., Boston, MA, 1988).
40. P. Stefan, "Accessible Sets, Orbits, and Foliations with Singularities," Proc. London Math. Soc. **29**, 699–713 (1974).
41. D. Sullivan, "A Homological Characterization of Foliations Consisting of Minimal Surfaces," Comment. Math. Helv. **54**, 218–223 (1979).
42. H. J. Sussmann, "Orbit of Families of Vector Fields and Integrability of Distributions," Trans. Amer. Math. Soc. **180**, 171–188 (1973).
43. Ph. Tondeur, *Geometry of Foliations* (Birkhäuser Verlag, Basel, 1997).
44. R. Wolak, "Basic Cohomology for Singular Riemannian Foliations," Monatsh. Math. **128** (2), 159–163 (1999).