

Tautness for riemannian foliations on non-compact manifolds

José Ignacio Royo Prieto*

Universidad del País Vasco.

Martintxo Saralegi-Aranguren†

Université d’Artois

Robert Wolak‡

Uniwersytet Jagiellonski

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Abstract

For a riemannian foliation \mathcal{F} on a closed manifold M , it is known that \mathcal{F} is taut (i.e. the leaves are minimal submanifolds) if and only if the (tautness) class defined by the mean curvature form κ_μ (relatively to a suitable riemannian metric μ) is zero (cf. [1]). In the transversally orientable case, tautness is equivalent to the non-vanishing of the top basic cohomology group $H^n(M/\mathcal{F})$, where $n = \text{codim } \mathcal{F}$ (cf. [10]). By the Poincaré Duality (cf. [9]) this last condition is equivalent to the non-vanishing of the basic twisted cohomology group $H_{\kappa_\mu}^0(M/\mathcal{F})$, when M is oriented.

When M is not compact, the tautness class is not even defined in general. In this work, we recover the previous study and results for a particular case of riemannian foliations on non compact manifolds: the regular part of a singular riemannian foliation on a compact manifold (CERF).

The study of taut foliations (foliations where all the leaves are minimal submanifolds for some riemannian metric) has been an important part of the research in regular foliations on riemannian manifolds: F. Kamber and Ph. Tondeur (cf. [8]), H. Rummeler (cf. [15]), as well as D. Sullivan (cf. [20]) were the first ones to present algebraical or variational characterizations of such foliations. A. Haefliger’s paper [7] proved to be an important step in the development of the theory. He showed that “being taut” is a transverse property, i.e. it depends only on the holonomy pseudogroup of the foliation \mathcal{F} . This led Y. Carrière (cf. [3]) to propose a characterization of taut riemannian foliations on a compact manifold M as those foliations for which the top dimensional basic cohomology group $H^n(M/\mathcal{F})$ is non-trivial, i.e. isomorphic to \mathbb{R} , being $n = \text{codim } \mathcal{F}$. For a concise presentation of the history of the basic cohomology and tautness we refer to V. Sergiescu’s appendix [18] in [14], which shows a close relation between the finiteness of basic cohomology, Poincaré Duality Property in basic cohomology and tautness.

*Departamento de Matemática Aplicada. Escuela Superior de Ingeniería. Universidad del País Vasco. Alameda de Urquijo s/n. 48013 Bilbao. España. joseignacio.royo@ehu.es. Partially supported by the UPV - EHU grant 127.310-E-14790/2002 by a PostGrant from the Gobierno Vasco - Eusko Jaurlaritza and by the MCyT of the Spanish Government.

†Fédération CNRS Nord-Pas-de-Calais FR 2956. UPRES-EA 2462 LML. Faculté Jean Perrin. Université d’Artois. Rue Jean Souvraz SP 18. 62 307 Lens Cedex - France. saralegi@euler.univ-artois.fr.

‡Instytut Matematyki. Uniwersytet Jagiellonski. Władysława Reymonta 4, 30 059 Krakow - Poland. robert.wolak@im.uj.edu.pl Partially supported by the KBN grant 2 PO3A 021 25.

The problem was positively solved by X. Masa in [10], in the following way: if M is compact and oriented and \mathcal{F} is riemannian and transversally oriented, then

$$(1) \quad \mathcal{F} \text{ is taut if and only if } H^n(M/\mathcal{F}) = \mathbb{R}.$$

The work of A. Álvarez (cf. [1]) removes the orientability condition on M and gives another characterization of taut riemannian foliations on a compact manifold. He constructs a cohomological class $\kappa \in H^1(M/\mathcal{F})$, the tautness class, whose vanishing is equivalent to the tautness of \mathcal{F} . This class is defined from the mean curvature form κ_μ of a bundle-like metric μ , which we can suppose to be basic due to D. Domínguez (cf. [5]).

We have a third cohomological characterization of the tautness of \mathcal{F} . The Poincaré Duality of [9] implies that this property is equivalent to the non-vanishing of the basic twisted cohomology group $H_\kappa^0(M/\mathcal{F})$ when M is orientable.

The situation is more complicated when the manifold M is not compact. For example, the mean curvature form κ_μ may be a basic form without being closed (cf. [4]).

We consider in this work a particular case of a riemannian foliation \mathcal{F} on a non-compact manifold M : the Compactly Embeddable Riemannian Foliations or CERFs. In this context, we have a compact manifold N endowed with a singular riemannian foliation \mathcal{H} (in the sense of [14]) in such a way that M is the regular stratum of N with $\mathcal{F} = \mathcal{H}|_M$. We consider a class of bundle-like metrics on M for which we construct a tautness class $\kappa = [\kappa_\mu] \in H^1(M/\mathcal{F})$ which is independent of the choice of μ . We prove that the tautness of \mathcal{F} is equivalent to any of the following three properties:

- $\kappa = 0$,
- $H_{\kappa_\mu}^0(M/\mathcal{F}) \neq 0$,
- $H_c^n(M/\mathcal{F}) = \mathbb{R}$, where $n = \text{codim } \mathcal{F}$, when \mathcal{F} is transversally oriented.

Notice that in the second characterization we have eliminated the orientation hypothesis. We also prove that the cohomology groups $H_\kappa^0(M/\mathcal{F})$ and $H_c^n(M/\mathcal{F})$ are 0 or \mathbb{R} .

The standard method to prove the equivalence of the second and third conditions is to use the Poincaré Duality Property (PDP in short). However, although reasonable, the PDP for the basic cohomology has been proved neither for (M, \mathcal{F}) nor for (N, \mathcal{H}) . So, we shall proceed by proving that the first condition is equivalent to either of the two remaining ones. The proof of the equivalence of the first and second condition is purely algebraic. To obtain the second equivalence we use Molino's desingularisation $(\tilde{N}, \tilde{\mathcal{H}})$ of (N, \mathcal{H}) . The key point is the following.

$$\mathcal{F} \text{ is taut} \iff \tilde{\mathcal{H}} \text{ is taut.}$$

Note that this equivalence cannot be extended to the singular riemannian foliation \mathcal{H} itself, since the existence of leaves with different dimensions implies the non-existence of "minimal metrics." Then the comparison of the corresponding basic cohomology groups completes the proof. In this way we have avoided in the proof any reference to the PDP of the basic cohomology of the foliated manifolds we are studying.

In the sequel M and N are connected, second countable, Hausdorff, without boundary and smooth (of class C^∞) manifolds of dimension m . All the maps considered are smooth unless something else is indicated.

1 Riemannian foliations¹

The framework category of this work is that of CERFs. They are riemannian foliations embedded in singular riemannian foliations on compact manifolds. Before introducing this notion, we need to recall some important facts about singular riemannian foliations.

1.1 The SRFs . A *Singular Riemannian Foliation*² (SRF for short) on a manifold N is a partition \mathcal{H} by connected immersed submanifolds, called *leaves*, verifying the following properties:

- I- The module of smooth vector fields tangent to the leaves is transitive on each leaf.
- II- There exists a riemannian metric μ on N , called *adapted metric*, such that each geodesic that is perpendicular at one point to a leaf remains perpendicular to every leaf it meets.

The first condition implies that (N, \mathcal{H}) is a singular foliation in the sense of [19] and [21]. Notice that the restriction of \mathcal{H} to a saturated open subset produces an SRF. Each (regular) Riemannian Foliation (RF for short) is an SRF, but the first interesting examples are the following:

- The orbits of the action by isometries of a Lie group.
- The closures of the leaves of a regular riemannian foliation.

1.2 Stratification. Classifying the points of N following the dimension of the leaves one gets a stratification $\mathbf{S}_{\mathcal{H}}$ of N whose elements are called *strata*. The restriction of \mathcal{H} to a stratum S is a RF \mathcal{H}_S . The strata are ordered by: $S_1 \preceq S_2 \Leftrightarrow S_1 \subset \overline{S_2}$. The minimal (resp. maximal) strata are the closed strata (resp. open strata). Since N is connected, there is just one open stratum, denoted $R_{\mathcal{H}}$. It is a dense subset. This is the *regular stratum*, the other strata are the *singular strata*. The family of singular strata is written $\mathbf{S}_{\mathcal{H}}^{sin}$. The *dimension of the foliation* \mathcal{H} is the dimension of the biggest leaves of \mathcal{H} .

The *depth* of $\mathbf{S}_{\mathcal{H}}$, written $\text{depth } \mathbf{S}_{\mathcal{H}}$, is defined to be the largest i for which there exists a chain of strata $S_0 \prec S_1 \prec \dots \prec S_i$. So, $\text{depth } \mathbf{S}_{\mathcal{H}} = 0$ if and only if the foliation \mathcal{H} is regular.

The *depth* of a stratum $S \in \mathbf{S}_{\mathcal{H}}$, written $\text{depth}_{\mathcal{H}} S$, is defined to be the largest i for which there exists a chain of strata $S_0 \prec S_1 \prec \dots \prec S_i = S$. So, $\text{depth}_{\mathcal{H}} S = 0$ (resp. $\text{depth}_{\mathcal{H}} S = \text{depth } \mathbf{S}_{\mathcal{H}}$) if and only if the stratum S is minimal (resp. regular). For each $i \in \mathbb{Z}$ we write

$$\Sigma_i = \Sigma_i(N) = \cup\{S \in \mathbf{S}_{\mathcal{H}} \mid \text{depth}_{\mathcal{H}} S \leq i\}.$$

We have $\Sigma_{<0} = \emptyset$, $\Sigma_{\text{depth } \mathbf{S}_{\mathcal{H}}-1} = N \setminus R_{\mathcal{H}}$ and $\Sigma_i = N$ if $i \geq \text{depth } \mathbf{S}_{\mathcal{H}}$. The union of closed (minimal) strata is Σ_0 .

1.3 The CERFs. Consider a riemannian foliation \mathcal{F} on a manifold M . We say that \mathcal{F} is a *Compactly Embeddable Riemannian Foliation* (or CERF) if there exists a connected compact manifold N , endowed with an SRF \mathcal{H} , and a foliated imbedding $(M, \mathcal{F}) \subset (N, \mathcal{H})$ such that M is the regular stratum of $\mathbf{S}_{\mathcal{H}}$, that is, $M = R_{\mathcal{H}}$. We shall say that (N, \mathcal{H}) is a *zipper* of (M, \mathcal{F}) . Both manifolds, M and N , are connected or not at the same time.

When M is compact and \mathcal{F} is regular then (M, \mathcal{F}) is a CERF and (M, \mathcal{F}) the zipper. But in the general case, the zipper may not be unique. The foliated manifold $(M, \mathcal{F}) = (\mathbb{S}^3 \times]0, 1[, \text{Hopf} \times \mathcal{I})$, with *Hopf* the Hopf foliation of \mathbb{S}^3 and \mathcal{I} the foliation of $]0, 1[$ by points, is a CERF possessing two natural zippers (N_i, \mathcal{H}_i) , $i = 1, 2$:

¹For the notions related to riemannian foliations we refer the reader to [14, 23].

²For the notions related to singular riemannian foliations we refer the reader to [2, 13, 14].

- $N_1 = \mathbb{S}^4$ and \mathcal{H}_1 is given by the orbits of the \mathbb{S}^1 -action: $z \cdot (z_1, z_2, t) = (z \cdot z_1, z \cdot z_2, t)$, where $\mathbb{S}^4 = \{(z_1, z_2, t) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R} / |z_1|^2 + |z_2|^2 + t^2 = 1\}$.
- $N_2 = \mathbb{C}\mathbb{P}^2$ and \mathcal{H}_2 is given by the orbits of the \mathbb{S}^1 -action: $z \cdot [z_1, z_2, z_3] = [z \cdot z_1, z \cdot z_2, z_3]$.

We consider in the sequel a manifold M endowed with a CERF \mathcal{F} and we fix (N, \mathcal{H}) a zipper. We present the Molino's desingularisation of (N, \mathcal{H}) in several steps.

1.4 Foliated tubular neighborhood. A singular stratum $S \in \mathbf{S}_{\mathcal{H}}^{\text{sin}}$ is a proper submanifold of the riemannian manifold (N, \mathcal{H}, μ) . So it possesses a tubular neighborhood (T_S, τ_S, S) . Recall that associated with this neighborhood we have the following smooth maps:

- + The *radius map* $\rho_S: T_S \rightarrow [0, 1[$, which is defined fiberwise by $z \mapsto |z|$. Each $t \neq 0$ is a regular value of the ρ_S . The pre-image $\rho_S^{-1}(0)$ is S .
- + The *contraction* $H_S: T_S \times [0, 1] \rightarrow T_S$, which is defined fiberwise by $(z, r) \mapsto r \cdot z$. The restriction $(H_S)_t: T_S \rightarrow T_S$ is an imbedding for each $t \neq 0$ and $(H_S)_0 \equiv \tau_S$.

These maps verify $\rho_S(r \cdot u) = r \cdot \rho_S(u)$ and $\tau_S(r \cdot u) = \tau_S(u)$. This tubular neighborhood can be chosen so that the two following important properties are verified (cf. [14]):

- (a) Each $(\rho_S^{-1}(t), \mathcal{H})$ is a SRF, and
- (b) Each $(H_S)_t: (T_S, \mathcal{F}) \rightarrow (T_S, \mathcal{F})$ is a foliated map.

When this happens, we shall say that (T_S, τ_S, S) is a *foliated tubular neighborhood* of S . The hypersurface $D_S = \rho_S^{-1}(1/2)$ is the *core* of the tubular neighborhood. We have $\text{depth } \mathbf{S}_{\mathcal{H}_D} < \text{depth } \mathbf{S}_{\mathcal{H}_{T_S}}$.

There is a particular type of singular stratum we shall use in this work. A stratum S is a *boundary stratum* if $\text{codim}_N \mathcal{H} = \text{codim}_S \mathcal{H}_S - 1$. The reason for this name is well illustrated by the following example. The usual \mathbb{S}^1 -action on \mathbb{S}^2 by rotations defines a singular riemannian foliation \mathcal{H} with two singular leaves, two fixed points of the action. These points are the boundary strata and we have $N/\mathcal{H} = [0, 1]$. The boundary $\partial(N/\mathcal{H})$ is given by the boundary strata. In fact, the link of a boundary stratum is a sphere with the one leaf foliation (see, for example, [16] for the notion of link).

In the sequel, we shall use the *partial blow up*

$$\mathfrak{L}_S: (D_S \times [0, 1[, \mathcal{H} \times \mathcal{I}) \rightarrow (T_S, \mathcal{H}),$$

which is the foliated smooth map defined by $\mathfrak{L}_S(z, t) = H_S(z, 2t)$. Here, \mathcal{I} denotes the pointwise foliation. The restriction

$$(2) \quad \mathfrak{L}_S: (D_S \times]0, 1[, \mathcal{H} \times \mathcal{I}) \rightarrow (T_S \setminus S, \mathcal{H})$$

is a foliated diffeomorphism.

1.5 Foliated Thom-Mather system. In the proof of Lemma 2.3.1 we find two strata $S' \preceq S$ endowed with two tubular neighborhoods T and $T_{S'}$. We shall need $T_{S'} \setminus T_S$ to be a tubular neighborhood of $S' \setminus T_S$, but this is not always achieved. To guarantee this property, we introduce the following notion, which is inspired in the abstract stratified objects of [11, 22].

A family of foliated tubular neighborhoods $\{T_S \mid S \in \mathbf{S}_{\mathcal{F}}^{\text{sin}}\}$ is a *foliated Thom-Mather system* of (N, \mathcal{H}) if the following conditions are verified.

(TM1) For each pair of singular strata S, S' we have

$$T_S \cap T_{S'} \neq \emptyset \iff S \preceq S' \text{ or } S' \preceq S.$$

Let us suppose that $S' \prec S$. The two other conditions are:

$$(TM2) \quad T_S \cap T_{S'} = \tau_{S'}^{-1}(T_S \cap S').$$

$$(TM3) \quad \rho_S = \rho_{S'} \circ \tau_{S'} \text{ on } T_S \cap T_{S'}.$$

1.6 Remarks.

(a) Given two singular strata S and S' with $S \not\prec S'$ and $S' \not\prec S$ then condition (TM1) gives $T_S \setminus T_{S'} \equiv T_S$.

(b) Given two singular strata S and S' with $S \prec S'$ we have $\rho_S(H_{S'}(z, r)) = \rho_S(z)$, for each $r \in [0, 1]$ and $z \in T_S \cap T_{S'}$. We conclude that the restriction

$$\tau_{S'}: \left(T_{S'} \setminus \rho_{S'}^{-1}(I) \equiv \tau_{S'}^{-1}(S' \setminus \rho_{S'}^{-1}(I)) \right) \longrightarrow (S' \setminus \rho_{S'}^{-1}(I))$$

is a foliated tubular neighborhood of $S' \setminus \rho_{S'}^{-1}(I)$ on $N \setminus \rho_{S'}^{-1}(I)$, where $I \subset [0, 1[$ is a closed subset.

Proposition 1.6.1 *Each compact manifold endowed with an SRF possesses a foliated Thom-Mather system.*

Proof. See Appendix. ♣

1.7 Blow up. The Molino's blow up of the foliation \mathcal{H} produces a new foliation $\widehat{\mathcal{H}}$ of the same kind but with smaller depth (see [13] and also [17]). The main idea is to replace each point of the closed strata by a sphere.

We suppose that depth $S_{\mathcal{H}} > 0$. The union of closed (minimal) strata we denote by Σ_0 . We choose T_0 a disjoint family of foliated tubular neighborhoods of the closed strata. The union of the associated cores is denoted by D_0 . Let $\mathfrak{L}_0: (D_0 \times [0, 1[, \mathcal{H} \times \mathcal{I}) \rightarrow (T_0, \mathcal{H})$ be the associated partial blow up. The *blow up* of (N, \mathcal{H}, μ) is

$$\mathfrak{L}: (\widehat{N}, \widehat{\mathcal{H}}, \widehat{\mu}) \longrightarrow (N, \mathcal{H}, \mu)$$

where

- The manifold \widehat{N} is

$$\widehat{N} = \left\{ (D_0 \times]-1, 1[) \amalg ((N \setminus \Sigma_0) \times \{-1, 1\}) \right\} / \sim,$$

where $(z, t) \sim (\mathfrak{L}_0(z, |t|), t/|t|)$. Notice that $D_0 \times]-1, 1[$ and $(N \setminus \Sigma_0) \times \{-1, 1\}$ are open subsets of \widehat{N} with

$$(D_0 \times]-1, 1[) \cap ((N \setminus \Sigma_0) \times \{-1, 1\}) = D_0 \times (]-1, 0[\cup]0, 1[).$$

- The foliation $\widehat{\mathcal{H}}$ is determined by

$$\widehat{\mathcal{H}}|_{D_0 \times]-1, 1[} = \mathcal{H}|_{D_0} \times \mathcal{I} \quad \text{and} \quad \widehat{\mathcal{H}}|_{(N \setminus \Sigma_0) \times \{-1, 1\}} = \mathcal{H}|_{N \setminus \Sigma_0} \times \mathcal{I}.$$

Here, \mathcal{I} denotes the 0-dimensional foliation of $]-1, 1[$.

- The riemannian metric $\widehat{\mu}$ is

$$f \cdot (\mu|_{D_0} + dt^2) + (1 - f) \cdot \mu|_{N \setminus \Sigma_0},$$

where $f: \widehat{N} \rightarrow [0, 1]$ is the smooth map defined by

$$f(v) = \begin{cases} \xi(|t|) & \text{if } v = (z, t) \in D_0 \times]-1, 1[\\ 0 & \text{if } v = (z, j) \in (N \setminus \rho_0^{-1}([0, 3/4])) \times \{-1, 1\}, \end{cases}$$

with $\xi: [0, 1] \rightarrow [0, 1]$ a smooth map verifying $\xi \equiv 1$ on $[0, 1/4]$ and $\xi \equiv 0$ on $[3/4, 1[$.

- The map \mathfrak{L} is defined by

$$\mathfrak{L}(v) = \begin{cases} \mathfrak{L}_0(z, |t|) & \text{if } v = (z, t) \in D_0 \times]-1, 1[\\ z & \text{if } v = (z, j) \in (N \setminus \Sigma_0) \times \{-1, 1\}, \end{cases}$$

Notice that the blow up of (N, \mathcal{H}, μ) depends just on the choice of ξ . So, we fix from now on such a ξ .

1.7.1 Remarks.

(a) The blow up of $(N \times \mathbb{R}, \mathcal{H} \times \mathcal{I}, \mu + dt^2)$ is just $\mathfrak{L} \times \text{Identity} : (\widehat{N} \times \mathbb{R}, \widehat{\mathcal{H}} \times \mathcal{I}, \widehat{\mu} + dt^2) \longrightarrow (N \times \mathbb{R}, \mathcal{H} \times \mathcal{I}, \mu + dt^2)$.

(b) The manifold \widehat{N} is connected and compact, the foliation $\widehat{\mathcal{H}}$ is an SRF and $\widehat{\mu}$ is an adapted metric.

(c) The map \mathfrak{L} is a foliated continuous map whose restriction to $(\widehat{N} \setminus \mathfrak{L}^{-1}(\Sigma_0), \widehat{\mathcal{H}}) \equiv ((N \setminus \Sigma_0) \times \{-1, 1\}, \mathcal{H} \times \mathcal{I})$ is the canonical projection on the first factor.

(d) We shall denote by M_1 the regular stratum $R_{\widehat{\mathcal{H}}}$ and \mathcal{F}_1 the restriction of $\widehat{\mathcal{H}}$ to M_1 . In fact, the foliation \mathcal{F}_1 is a CERF on M_1 and a zipper is given by $(\widehat{N}, \widehat{\mathcal{H}})$. Notice that we have the inclusion $\mathfrak{L}^{-1}(M) = M \times \{-1, 1\} \subset M_1$. We choose $\mathcal{S}_1: M \rightarrow M_1$ a smooth foliated imbedding with $\mathfrak{L} \circ \mathcal{S}_1 = \text{Identity}$. There are two of them.

(e) *The stratification $\mathcal{S}_{\widehat{\mathcal{H}}}$.* For each non-minimal stratum $S \in \mathcal{S}_{\mathcal{H}}$ there exists a unique stratum $S^s \in \mathcal{S}_{\widehat{\mathcal{H}}}$, with $\mathfrak{L}^{-1}(S) \subset S^s$, in fact, $S^s = \left\{ \left((D_0 \cap S) \times]-1, 1[\right) \amalg \left(S \times \{-1, 1\} \right) \right\} / \sim$. This gives $\mathcal{S}_{\widehat{\mathcal{H}}} = \{S^s \mid S \in \mathcal{S}_{\mathcal{H}} \text{ and non-minimal}\}$. We have the following important properties

- $\text{depth}_{\mathcal{H}} S - 1 = \text{depth}_{\widehat{\mathcal{H}}} S^s$, for each non-minimal stratum $S \in \mathcal{S}_{\mathcal{H}}$,
- $\mathfrak{L}^{-1}(\Sigma_i \setminus \Sigma_0) = \Sigma_{i-1}(\widehat{N}) \setminus \mathfrak{L}^{-1}(\Sigma_0)$ for each $i \in \mathbb{Z}$, and
- $\text{depth } \mathcal{S}_{\widehat{\mathcal{H}}} < \text{depth } \mathcal{S}_{\mathcal{H}}$.

(f) We shall use the diffeomorphism $\sigma: \widehat{N} \rightarrow \widehat{N}$ defined by

$$\sigma(v) = \begin{cases} (z, -t) & \text{if } v = (z, t) \in D_0 \times]-1, 1[\\ (z, -j) & \text{if } v = (z, j) \in (N \setminus \Sigma_0) \times \{-1, 1\}. \end{cases}$$

In fact, the diffeomorphism σ is a foliated isometry verifying $\mathfrak{L} \circ \sigma = \mathfrak{L}$. It induces the smooth foliated action $\Phi: \mathbb{Z}_2 \times M_1 \rightarrow M_1$ defined by $\zeta \cdot v = \sigma(v)$, where ζ is the generator of \mathbb{Z}_2 .

1.8 Molino's desingularisation. If the depth of $S_{\tilde{\mathcal{H}}}$ is not 0 then the blow up can be continued (cf. 1.6.1 (b)). In the end, we have a riemannian foliated manifold $(\tilde{N}, \tilde{\mathcal{H}}, \tilde{\mu})$ and a foliated continuous map $\mathfrak{N}: (\tilde{N}, \tilde{\mathcal{H}}) \rightarrow (N, \mathcal{H})$, whose restriction $\mathfrak{N}: \mathfrak{N}^{-1}(M) \rightarrow M$ is a smooth trivial bundle (cf. 1.6.1 (d)). Notice that \tilde{N} is connected and compact. This type of construction is a *Molino's desingularisation* of (N, \mathcal{H}, μ) (cf. [13]).

We choose $\mathcal{S}: M \rightarrow \tilde{N}$ a smooth foliated imbedding verifying $\mathfrak{N} \circ \mathcal{S} = \text{Identity}$. It always exists.

2 Tautness

The tautness of a RF on a compact manifold can be detected by using the basic cohomology. This is not the case when the manifold is not compact. In this section we recover this result for a CERF.

We consider in the sequel a manifold M endowed with a CERF \mathcal{F} and we fix a zipper (N, \mathcal{H}) . We also consider a Molino's desingularisation $\mathfrak{N}: (\tilde{N}, \tilde{\mathcal{H}}) \rightarrow (N, \mathcal{H})$. We shall write $n = \text{codim } \mathcal{F}$. We fix $\mathcal{S}: M \rightarrow \tilde{N}$ a smooth foliated imbedding verifying $\mathfrak{N} \circ \mathcal{S} = \text{Identity}$.

2.1 Basic cohomology. Recall that the *basic cohomology* $H^*(M/\mathcal{F})$ is the cohomology of the complex $\Omega^*(M/\mathcal{F})$ of *basic forms*. A differential form ω is basic when $i_X \omega = i_X d\omega = 0$ for every vector field X tangent to \mathcal{F} .

An open covering $\{U, V\}$ of M by saturated open subsets possesses a subordinated partition of the unity made up of basic functions (see Lemma below). For such a covering we have the Mayer-Vietoris short sequence

$$(3) \quad 0 \rightarrow \Omega^*(M/\mathcal{F}) \rightarrow \Omega^*(U/\mathcal{F}) \oplus \Omega^*(V/\mathcal{F}) \rightarrow \Omega^*((U \cap V)/\mathcal{F}) \rightarrow 0,$$

where the maps are defined by restriction. The third map is onto since the elements of the partition of the unity are basic functions. Thus, the sequence is exact.

The *compactly supported basic cohomology* $H_c^*(M/\mathcal{F})$ is the cohomology of the basic subcomplex $\Omega_c^*(M/\mathcal{F}) = \{\omega \in \Omega^*(M/\mathcal{F}) \mid \text{the support of } \omega \text{ is compact}\}$.

The *twisted basic cohomology* $H_\kappa^*(M/\mathcal{F})$, relatively to the cycle $\kappa \in \Omega^1(M/\mathcal{F})$, is the cohomology of the basic complex $\Omega^*(M/\mathcal{F})$ relatively to the differential $\omega \mapsto d\omega - \kappa \wedge \omega$. This cohomology does not depend on the choice of the cycle: we have $H_\kappa^*(M/\mathcal{F}) \cong H_{\kappa+df}^*(M/\mathcal{F})$ through the isomorphism: $[\omega] \mapsto [e^f \omega]$.

Given V , a \mathbb{Z}_2 -invariant saturated open subset of M , we shall write

$$\begin{aligned} (H^*(V/\mathcal{F}))^{\mathbb{Z}_2} &= \{\omega \in H^*(V/\mathcal{F}) \mid \sigma^* \omega = \omega\} \\ (H^*(V/\mathcal{F}))^{-\mathbb{Z}_2} &= \{\omega \in H^*(V/\mathcal{F}) \mid \sigma^* \omega = -\omega\}. \end{aligned}$$

For the existence of the Mayer-Vietoris sequence (3) we need the following folk result, well-known for compact Lie group actions and regular riemannian foliations.

Lemma 2.1.1 *Any covering of M by saturated open subsets possesses a subordinated partition of the unity made up of basic functions.*

Proof. The closure \bar{L} of a leaf $L \in \mathcal{F}$ is a saturated submanifold of M whose leaves are dense (cf. [14]). So, the open subsets U and V are in fact $\bar{\mathcal{F}}$ -saturated subsets.

The closure \bar{L} of a leaf $L \in \mathcal{F}$ possesses a tubular neighborhood as in 1.4 (cf. [14]). Since the family of these tubular neighborhoods is a basis for the family of saturated open subsets then it suffices to construct the partition of unity relatively to the tubular neighborhoods. This is done by using the radius maps. ♣

2.2 Tautness reminder (compact case). Given a bundle-like metric μ on (M, \mathcal{F}) , the mean curvature form $\kappa_\mu \in \Omega^1(M)$ is defined as follows (see for example [23]). Consider the second fundamental form of the leaves and W the corresponding Weingarten map. Then,

$$\kappa_\mu(X) = \begin{cases} \text{trace } W(X) & \text{if } X \text{ is orthogonal to the foliation } \mathcal{F} \\ 0 & \text{if } X \text{ is tangent to the foliation } \mathcal{F}. \end{cases}$$

When the manifold is compact (and then $(N, \mathcal{H}) = (M, \mathcal{F})$), the following properties of κ_μ are well-known:

(a) The form κ_μ can be supposed to be basic, i.e., there exists a bundle-like metric μ such that its mean curvature form is basic (see [5]).

(b) If κ_μ is basic, then κ_μ is a closed form (see [23]). The Example 2.4 of [4] shows that the compactness assumption cannot be removed: there, the mean curvature form is basic, but not closed.

(c) The class $\kappa = [\kappa_\mu] \in H^1(M/\mathcal{F})$ does not depend on the metric, but just on \mathcal{F} (see [1]). This is the *tautness class* of \mathcal{F} .

The mean curvature form contains some geometric information about \mathcal{F} . Recall that the foliation \mathcal{F} is *taut* if there exists a riemannian metric μ on M such that every leaf is a minimal submanifold of M . It is known (see [23]) that

$$\mathcal{F} \text{ is taut} \iff \text{the tautness class } \kappa \text{ vanishes.}$$

We also have the following cohomological characterizations for the tautness of \mathcal{F} :

$$\mathcal{F} \text{ is taut} \iff H^n(M/\mathcal{F}) \neq 0,$$

when \mathcal{F} is transversally oriented. We also have that

$$\mathcal{F} \text{ is taut} \iff H_{\kappa_\mu}^0(M/\mathcal{F}) \neq 0,$$

when M is oriented and \mathcal{F} is transversally oriented.

Immediate examples of taut foliations are isometric flows (i.e. 1-foliations induced by the orbits of a nonvanishing Killing vector field), isometric actions on compact manifolds and compact foliations with locally bounded volume of leaves (foliations where every leaf is compact, see [6],[15]).

In the example of [4] referred above, we find a non-compact manifold where the tautness class may not exist.

These results are not directly extendable to the framework of singular riemannian foliations, as we can see in the following example: the usual \mathbb{S}^1 -action on \mathbb{S}^2 by rotations defines a singular riemannian foliation \mathcal{H} with two singular leaves, two fixed points. Notice that $H^1(N/\mathcal{H}) = 0$. But there cannot exist a metric on \mathbb{S}^2 such that the one dimensional orbits are geodesics (in dimension

1, “minimal” implies “geodesic”). To see this, it suffices to consider a totally convex neighborhood U of one of the fixed points: it would contain as geodesics some orbits (full circles, indeed), apart from rays, which would contradict uniqueness of geodesics connecting any two points of U . In higher dimensions (i.e. the cone of a sphere), one may consider the volume of every leaf. It should be constant by minimality, which would give a positive volume to the vertex (for more details see [12]).

2.3 Construction of κ . We would like to define a cohomological class $\kappa \in H^1(M/\mathcal{F})$ which would play a rôle similar to that of the tautness class of a regular riemannian foliation on a regular manifold. First, we introduce the notion of D-metric. A bundle-like metric μ on (M, \mathcal{F}) is a *D-metric* if the mean curvature form κ_μ is a basic cycle. The *tautness class* of \mathcal{F} is the cohomological class $\kappa = [\kappa_\mu] \in H^1(M/\mathcal{F})$. The next Proposition proves that this class is well defined and independent of the D-metric.

Lemma 2.3.1 *There exists a saturated open subset $U \subset M$ such that*

- (a) *the inclusion $v: U \hookrightarrow M$ induces the isomorphism $v^*: H^*(M/\mathcal{F}) \rightarrow H^*(U/\mathcal{F})$, and*
- (b) *the closure \bar{U} (in N) is included in M .*

Proof. We consider $\{T_S \mid S \in \mathcal{S}_{\mathcal{F}}\}$ a foliated Thom-Mather system of (N, \mathcal{H}) (cf. Proposition 1.6.1). For each $i \in \mathbb{Z}$ we write:

- $\tau_i: (T_i = \cup\{S \in \mathcal{S}_{\mathcal{K}} \mid \text{depth}_{\mathcal{H}} S \leq i\}) \rightarrow \Sigma_i \setminus \Sigma_{i-1}$ the foliated tubular neighborhood,
- $\rho_i: T_i \rightarrow [0, 1[$ its radius function, and
- $D_i = \rho_i^{-1}(0)$ the core of T_i .

The family $\{M \cap T_0, M \setminus \rho_0^{-1}([0, 7/8])\}$ is a saturated open covering of M . Notice that the inclusion $((M \cap T_0) \setminus \rho_0^{-1}([0, 7/8]), \mathcal{F}) \hookrightarrow (M \cap T_0, \mathcal{F})$ induces an isomorphism for the basic cohomology since it is foliated diffeomorphic to the inclusion

$$((M \cap D_0) \times]7/8, 1[, \mathcal{F} \times \mathcal{I}) \hookrightarrow ((M \cap D_0) \times]0, 1[, \mathcal{F} \times \mathcal{I})$$

(cf. (2)). From the Mayer-Vietoris sequence we conclude that the inclusion $M \setminus \rho_0^{-1}([0, 7/8]) \hookrightarrow M$ induces the isomorphism

$$H^*(M/\mathcal{F}) = H^*(M \setminus \rho_0^{-1}([0, 7/8])/\mathcal{F})$$

(cf. (3)).

Notice now that $M \setminus \rho_0^{-1}([0, 7/8])$ is the regular part of $(N \setminus \rho_0^{-1}([0, 7/8]), \mathcal{H})$. Moreover, the family

$$\{T_S \setminus \rho_0^{-1}([0, 7/8]) \mid S \in \mathcal{S}_{\mathcal{F}}, \text{depth}_{\mathcal{H}} S > 0\}$$

is a foliated Thom-Mather system of $(N \setminus \rho_0^{-1}([0, 7/8]), \mathcal{H})$ (cf. (1.6)). The same previous argument gives

$$H^*(M \setminus \rho_0^{-1}([0, 7/8])/\mathcal{F}) = H^*(M \setminus (\rho_0^{-1}([0, 7/8]) \cup \rho_1^{-1}([0, 7/8]))) / \mathcal{F}.$$

So, one gets the isomorphisms

$$H^*(M/\mathcal{F}) = \dots = H^*(M \setminus (\rho_0^{-1}([0, 7/8]) \cup \dots \cup \rho_{p-1}^{-1}([0, 7/8]))) / \mathcal{F},$$

where $p = \text{depth } \mathcal{S}_{\mathcal{H}}$. Take $U = M \setminus (\rho_0^{-1}([0, 7/8]) \cup \dots \cup \rho_{p-1}^{-1}([0, 7/8]))$, which is an open saturated subset of \mathcal{F} included on M . This gives (a).

Consider $K = M \setminus (\rho_0^{-1}([0, 6/8]) \cup \dots \cup \rho_{p-1}^{-1}([0, 6/8]))$, which is a subset of M containing U . We compute its closure on N :

$$\begin{aligned} \overline{K} &= \overline{M \setminus (\rho_0^{-1}([0, 6/8]) \cup \dots \cup \rho_{p-1}^{-1}([0, 6/8]))} \subset \overline{M} \setminus \left((\rho_0^{-1}([0, 6/8]))^\circ \cup \dots \cup (\rho_{p-1}^{-1}([0, 6/8]))^\circ \right) \\ &= N \setminus (\rho_0^{-1}([0, 6/8]) \cup \dots \cup \rho_{p-1}^{-1}([0, 6/8])) = M \setminus (\rho_0^{-1}([0, 6/8]) \cup \dots \cup \rho_{p-1}^{-1}([0, 6/8])), \end{aligned}$$

since $N \setminus M = \Sigma_{p-1} = \rho_0^{-1}(\{0\}) \cup \dots \cup \rho_{p-1}^{-1}(\{0\})$. This implies that K is a closed subset of N and therefore compact. This gives (b). \clubsuit

Proposition 2.3.2 *The tautness class of a CERF exists and it does not depend on the choice of the D-metric.*

Proof. We proceed in two steps.

(i) - Existence of D-metrics. Since \tilde{N} is compact then there exists a D-metric ν on \tilde{N} . The metric $\mathcal{S}^*\nu$ is a bundle-like metric on M . Since $\kappa_{\mathcal{S}^*\nu} = \mathcal{S}^*\kappa_\nu$ then $\mathcal{S}^*\nu$ is a D-metric on M .

(ii) - Uniqueness of κ . Consider μ a D-metric on M and let $\tilde{\kappa}$ be the tautness class of $\tilde{\mathcal{H}}$. It suffices to prove $[\kappa_\mu] = \mathcal{S}^*\tilde{\kappa}$. Take U as in the previous Lemma and $\{f, g\}$ a subordinated partition of unity associated to the covering $\{M, N \setminus \overline{U}\}$ made up of basic functions (cf. Lemma 2.1.1). Notice that $f \equiv 1$ on U . So, it suffices to prove

$$(4) \quad v^*[\kappa_\mu] = v^*\mathcal{S}^*\tilde{\kappa}$$

(cf Lemma 2.3.1 (a)). Consider the following diagram:

$$\begin{array}{ccccc} \mathfrak{N}^{-1}(U) & \xrightarrow{v'} & \mathfrak{N}^{-1}(M) & \xrightarrow{\iota} & \tilde{N} \\ \mathfrak{N}'' \downarrow \uparrow \mathcal{S}'' & & \mathfrak{N}' \downarrow \uparrow \mathcal{S}' & \nearrow \mathcal{S} & \mathfrak{N} \downarrow \\ U & \xrightarrow{v} & M & \xrightarrow{\iota'} & N, \end{array}$$

where v', ι, ι' are the natural inclusions, $\mathfrak{N}', \mathfrak{N}''$ are the restrictions of \mathfrak{N} and $\mathcal{S}', \mathcal{S}''$ are the restrictions of \mathcal{S} . Notice the equalities $\mathcal{S} \circ v = \iota \circ v' \circ \mathcal{S}''$ and $v = \mathfrak{N}' \circ v' \circ \mathcal{S}''$.

The maps v', \mathfrak{N}' are foliated local diffeomorphism so $\kappa_{v'^*\mathfrak{N}'^*\mu} = v'^*\mathfrak{N}'^*\kappa_\mu$. This differential form is a cycle since μ is a D-metric. Take on \tilde{N} the riemannian metric

$$\lambda = \mathfrak{N}^*f \cdot \mathfrak{N}^*\mu + (1 - \mathfrak{N}^*f) \cdot \nu$$

It is a bundle-like metric since f is basic and the support of f is included on M . Notice the equality: $v'^*\iota^*\lambda = v'^*\mathfrak{N}'^*\mu$.

We can use this metric for the computation of $\tilde{\kappa}$ in the following way. Let $(\kappa_\lambda)_b$ be the basic part of the mean curvature form κ_λ , relatively to the λ -orthogonal decomposition $\Omega^*(\tilde{N}) = \Omega^*(\tilde{N}/\tilde{\mathcal{H}}) \oplus \Omega^*(\tilde{N}/\tilde{\mathcal{H}})^\perp$. It is a basic cycle and we have $\tilde{\kappa} = [(\kappa_\lambda)_b]$ (see [1]).

Since $\mathfrak{N}^{-1}(U)$ is $\tilde{\mathcal{H}}$ -saturated open subset of \tilde{N} then it is also a $\tilde{\mathcal{H}}$ -saturated subset of \tilde{N} (see the proof of Lemma 2.1.1). From the definition of the basic component $(\kappa_\lambda)_b$ we get that the restriction $v'^*\iota^*(\kappa_\lambda)_b$ is defined by using just $(\mathfrak{N}^{-1}(U), v'^*\iota^*\tilde{\mathcal{H}}, v'^*\iota^*\lambda)$. As a consequence, we have $v'^*\iota^*((\kappa_\lambda)_b) = \kappa_{v'^*\iota^*\lambda} = \kappa_{v'^*\mathfrak{N}'^*\mu} = v'^*\mathfrak{N}'^*\kappa_\mu$.

Finally, we get

$$v^*\mathcal{S}^*\tilde{\kappa} = \mathcal{S}''^*v'^*\iota^*\tilde{\kappa} = \mathcal{S}''^*v'^*\iota^*[(\kappa_\lambda)_b] = \mathcal{S}''^*[v'^*\mathfrak{N}'^*\kappa_\mu] = \mathcal{S}''^*v'^*\mathfrak{N}'^*[\kappa_\mu] = v^*[\kappa_\mu].$$

This gives (4). \clubsuit

2.3.3 Remarks.

(a) The tautness class κ of (M, \mathcal{F}) and the tautness class $\tilde{\kappa}$ of a Molino's desingularisation $(\tilde{N}, \tilde{\mathcal{H}})$ are related by the formula $\kappa = \mathcal{S}^* \tilde{\kappa}$, where $\mathcal{S}: M \rightarrow \tilde{N}$ is any smooth foliated embedding verifying $\mathfrak{N} \circ \mathcal{S} = \text{Identity}$.

(b) Let κ_1 be the tautness class of (M_1, \mathcal{F}_1) . This class is \mathbb{Z}_2 -invariant (cf. 1.6). This comes from the fact that the diffeomorphism σ preserves \mathcal{H}_1 and therefore $\kappa_{\sigma^* \mu} = \sigma^* \kappa_\mu$ for a D-metric μ on M_1 . Thus, the metric $\sigma^* \mu$ is also a D-metric and we obtain: $\sigma^* \kappa_1 = [\sigma^* \kappa_\mu] = [\kappa_{\sigma^* \mu}] = \kappa_1$.

2.4 First characterization of tautness: vanishing of κ .

We give the first characterization of the tautness of \mathcal{F} through the vanishing of κ . We lift the question to the Molino's desingularisation $(\tilde{N}, \tilde{\mathcal{H}})$.

Lemma 2.4.1 *The map $\mathcal{S}_1: M \rightarrow M_1$ (cf. 1.6) induces the isomorphism:*

$$(H^*(M_1/\mathcal{F}_1))^{\mathbb{Z}_2} \cong H^*(M/\mathcal{F}).$$

Proof. The open covering $\{(D_0 \cap M) \times]-1, 1[, M \times \{-1, 1\}\}$ of M_1 is a \mathbb{Z}_2 -equivariant one (cf. 1.6.1 (f)). So, from Mayer-Vietoris (cf. (3)) we get the long exact sequence

$$\begin{aligned} \dots &\rightarrow \left(H^j((D_0 \cap M) \times]-1, 1[/ \mathcal{F} \times \mathcal{I}) \right)^{\mathbb{Z}_2} \oplus \left(H^j(M \times \{-1, 1\} / \mathcal{F} \times \mathcal{I}) \right)^{\mathbb{Z}_2} \xrightarrow{I} \\ &\rightarrow \left(H^j((D_0 \cap M) \times (]-1, 0[\cup]0, 1[) / \mathcal{F} \times \mathcal{I}) \right)^{\mathbb{Z}_2} \rightarrow \left(H^{j+1}(M_1/\mathcal{F}_1) \right)^{\mathbb{Z}_2} \xrightarrow{I} \\ &\rightarrow \left(H^{j+1}((D_0 \cap M) \times]-1, 1[/ \mathcal{F} \times \mathcal{I}) \right)^{\mathbb{Z}_2} \oplus \left(H^{j+1}(M \times \{-1, 1\} / \mathcal{F} \times \mathcal{I}) \right)^{\mathbb{Z}_2} \rightarrow \dots, \end{aligned}$$

where I denotes the restriction map (i.e. induced by the inclusion). Since the natural projection $(D_0 \cap M) \times]-1, 1[\rightarrow (D_0 \cap M)$ is \mathbb{Z}_2 -invariant, then we get isomorphisms

$$\begin{aligned} H^*((D_0 \cap M)/\mathcal{F}) &\cong \left(H^*((D_0 \cap M) \times]-1, 1[/ \mathcal{F} \times \mathcal{I}) \right)^{\mathbb{Z}_2} \\ H^*((D_0 \cap M)/\mathcal{F}) &\cong \left(H^*((D_0 \cap M) \times (]-1, 0[\cup]0, 1[) / \mathcal{F} \times \mathcal{I}) \right)^{\mathbb{Z}_2}. \end{aligned}$$

We conclude that the inclusion $M \times \{-1, 1\} \hookrightarrow M_1$ induces the isomorphism

$$(H^*(M_1/\mathcal{F}_1))^{\mathbb{Z}_2} \cong (H^*(M \times \{-1, 1\} / \mathcal{F} \times \mathcal{I}))^{\mathbb{Z}_2}.$$

Since the natural projection $P: M \times \{-1, 1\} \rightarrow M$ is \mathbb{Z}_2 -invariant, then we get isomorphism

$$H^*(M/\mathcal{F}) \cong (H^*(M \times \{-1, 1\} / \mathcal{F} \times \mathcal{I}))^{\mathbb{Z}_2}.$$

Since $P \circ \mathcal{S}_1 = \text{Identity}$ then \mathcal{S}_1 induces the isomorphism $(H^*(M_1/\mathcal{F}_1))^{\mathbb{Z}_2} \cong H^*(M/\mathcal{F})$. ♣

The main result of this section is the following.

Theorem 2.4.2 *Let M be a manifold endowed with a CERF \mathcal{F} . Then, the following two statements are equivalent:*

- (a) *The foliation \mathcal{F} is taut.*
- (b) *The tautness class $\kappa \in H^1(M/\mathcal{F})$ vanishes.*

Proof. We prove the two implications.

(a) \Rightarrow (b). There exists a D-metric μ on M with $\kappa_\mu = 0$. Then $\kappa = [\kappa_\mu] = 0$.

(b) \Rightarrow (a). We proceed by induction on depth $S_{\mathcal{H}}^{sin}$. When this depth is 0 we have the regular case of [1]. When this depth is not 0 then we can consider the Molino's desingularisation $\mathfrak{N} = \mathfrak{L} \circ \mathfrak{N}_1: (\tilde{N}, \tilde{\mathcal{H}}) \rightarrow (N, \mathcal{H})$ of (N, \mathcal{H}) , where $\mathfrak{N}_1: (\tilde{N}, \tilde{\mathcal{H}}) \rightarrow (\hat{N}, \hat{\mathcal{H}})$ is a Molino's desingularisation of $(\hat{N}, \hat{\mathcal{H}})$ (cf. 1.6 and 1.7).

Write $\mathcal{S}^1: M_1 \rightarrow \tilde{N}$ a smooth foliated imbedding verifying $\mathfrak{N}_1 \circ \mathcal{S}^1 = \text{Identity}$. The composition $\mathcal{S} = \mathcal{S}^1 \circ \mathcal{S}_1: M \rightarrow \tilde{N}$ is a smooth foliated imbedding verifying $\mathfrak{N} \circ \mathcal{S} = \text{Identity}$ (cf. 1.6). From Remark 2.3.3 we get that $\mathcal{S}_1^* \kappa_1 = \kappa$ with $\kappa_1 \in \left(H^1(M_1/\mathcal{F}_1)\right)^{\mathbb{Z}_2}$. The above Lemma gives $\kappa_1 = 0$. By induction hypothesis (depth $S_{\tilde{\mathcal{H}}} < \text{depth } S_{\mathcal{H}}$) we get that \mathcal{H}_1 is taut. So, the restriction of \mathcal{H}_1 to $\mathcal{L}^{-1}(M) = M \times \{-1, 1\}$ is also taut. We conclude that \mathcal{F} is also taut (cf. 1.6). \clubsuit

2.4.3 Remark. The proof of the above Theorem shows that the tautness of \mathcal{F} and $\tilde{\mathcal{H}}$ are closely related. In fact,

$$\begin{array}{ccc} \text{The foliation } \mathcal{F} \text{ is taut} & \iff & \text{The foliation } \tilde{\mathcal{H}} \text{ is taut.} \\ \updownarrow & & \updownarrow \end{array}$$

$$\text{The tautness class } \kappa \in H^1(M/\mathcal{F}) \text{ vanishes} \iff \text{The tautness class } \tilde{\kappa} \in H^1(\tilde{N}/\tilde{\mathcal{H}}) \text{ vanishes.}$$

2.5 Second characterization of tautness: the bottom group $H_\kappa^0(M/\mathcal{F})$.

We give a characterization of the tautness of \mathcal{F} using $H_\kappa^0(M/\mathcal{F})$. Notice that, in the compact case, this result comes directly from (1) and the Poincaré Duality of [8, 9] when M is oriented and \mathcal{F} is transversally oriented. In fact, we shall not need these orientability conditions.

Theorem 2.5.1 *Let M be a manifold endowed with a CERF \mathcal{F} . Consider μ a D-metric on M . Then, the following two statements are equivalent:*

- (a) *The foliation \mathcal{F} is taut.*
- (b) *The cohomology group $H_{\kappa_\mu}^0(M/\mathcal{F})$ is \mathbb{R} (cf. 2.1).*

Otherwise, $H_{\kappa_\mu}^0(M/\mathcal{F}) = 0$.

Proof. We proceed in two steps.

(a) \Rightarrow (b). If \mathcal{F} is taut then $\kappa = [\kappa_\mu] = 0$. So, $H_{\kappa_\mu}^0(M/\mathcal{F}) \cong H^0(M/\mathcal{F}) = \mathbb{R}$.

(b) \Rightarrow (a). If $H_{\kappa_\mu}^0(M/\mathcal{F}) \neq 0$ then there exists a function $0 \neq f \in \Omega^0(M/\mathcal{F})$ with $df = f\kappa_\mu$. The set $Z(f) = f^{-1}(0)$ is clearly a closed subset of M . Let us see that it is also an open subset. Take $x \in Z(f)$ and consider a contractible open subset $U \subset M$ containing x . So, there exists a smooth map $g: U \rightarrow \mathbb{R}$ with $\kappa_\mu = dg$ on U . The calculation

$$d(fe^{-g}) = e^{-g}df - fe^{-g}dg = e^{-g}f\kappa_\mu - e^{-g}f\kappa_\mu = 0$$

shows that fe^{-g} is constant on U . Since $x \in Z(f)$ then $f \equiv 0$ on U and therefore $x \in U \subset Z(f)$. We get that $Z(f)$ is an open subset.

By connectedness we have that $Z(f) = \emptyset$ and $|f|$ is a smooth function. From the equality $d(\log |f|) = \frac{1}{f}df = \kappa_\mu$ we conclude that $\kappa = 0$ and then \mathcal{F} is taut.

Notice that we have also proved: $H_{\kappa_\mu}^0(M/\mathcal{F}) \neq 0 \Rightarrow H_{\kappa_\mu}^0(M/\mathcal{F}) = \mathbb{R}$. This ends the proof. ♣

2.6 Third characterization of tautness: the top group $H_c^n(M/\mathcal{F})$.

We give a characterization of the tautness of \mathcal{F} by using $H_c^n(M/\mathcal{F})$. We lift the question to a Molino's desingularisation of \mathcal{F} , where the result is known. But we need to formulate an orientability condition on \mathcal{F} .

Lemma 2.6.1

The foliation \mathcal{F} is transversally orientable \iff The foliation $\tilde{\mathcal{H}}$ is transversally orientable.

Proof. Since $\mathcal{S}: M \rightarrow M_1$ is a smooth foliated imbedding then we get " \Leftarrow ".

Consider \mathfrak{D} a transverse orientation on (M, \mathcal{F}) . The tubular neighborhood $(T_0 \cap M, \mathcal{F})$ inherits the transverse orientation \mathfrak{D} . Since $((D_0 \cap M) \times]0, 1[, \mathcal{F} \times \mathcal{I})$ is foliated diffeomorphic to $(T_0 \cap M, \mathcal{F})$ then it inherits a transverse orientation, written \mathfrak{D} . This transverse orientation induces on the product $((D_0 \cap M) \times]-1, 1[, \mathcal{F} \times \mathcal{I})$ a transverse orientation, written \mathfrak{D} . Notice that the involution $(x, t) \mapsto (x, -t)$ reverses the orientation \mathfrak{D} .

Since

$$M_1 = \left\{ \left((D_0 \cap M) \times]-1, 1[\right) \amalg \left(M \times \{-1, 1\} \right) \right\} / \sim,$$

where $(z, t) \sim (2|t| \cdot z, t/|t|)$, then it suffices to define on M_1 the transverse orientation \mathfrak{D}_1 by:

- * \mathfrak{D} on $(D_0 \cap M) \times]-1, 1[$,
- * \mathfrak{D} on $M \times \{1\}$, and
- * $-\mathfrak{D}$ on $M \times \{-1\}$.

This gives " \Rightarrow ".

♣

Before passing to the third characterization, we need two computational Lemmas.

Lemma 2.6.2 *The inclusion*

$$\Omega_c^*(((D_0 \cap M) \times]0, 1[)/\mathcal{F} \times \mathcal{I}) \hookrightarrow \Omega_c^*(((D_0 \cap M) \times]-1, 1[)/\mathcal{F} \times \mathcal{I})$$

induces an isomorphism in cohomology.

Proof. For the sake of simplicity, we write $E = (D_0 \cap M)$. Let $f \in \Omega_c^0(]-1, 1[)$ be a function with $f \equiv 0$ on $]-1, 1/3[$ and $f \equiv 1$ on $[2/3, 1[$. So, $df \in \Omega_c^1(]0, 1[) \subset \Omega_c^1(]-1, 1[)$. Lemma will be proved if we show that the assignment $[\gamma] \mapsto [df \wedge \gamma]$ establishes the isomorphisms of degree $+1$

$$H_c^*(E/\mathcal{F}) \cong H_c^*((E \times]0, 1[)/\mathcal{F} \times \mathcal{I}) \quad \text{and} \quad H_c^*(E/\mathcal{F}) \cong H_c^*((E \times]-1, 1[)/\mathcal{F} \times \mathcal{I}).$$

Let us prove the first one (the second one is proved in the same way). Consider the following differential complexes:

$$\begin{aligned}
- A^*(]0, 3/4[) &= \left\{ \omega \in \Omega^*(E \times]0, 3/4[/ \mathcal{F} \times \mathcal{I}) / \left[\begin{array}{l} \text{supp } \omega \subset K \times [c, 3/4[\\ \text{for a compact } K \subset E \text{ and } 0 < c < 3/4 \end{array} \right] \right\}. \\
- A^*(]1/4, 1[) &= \left\{ \omega \in \Omega^*(E \times]1/4, 1[/ \mathcal{F} \times \mathcal{I}) / \left[\begin{array}{l} \text{supp } \omega \subset K \times]1/4, c[\\ \text{for a compact } K \subset E \text{ and } 1/4 < c < 1 \end{array} \right] \right\}. \\
- A^*(]1/4, 3/4[) &= \left\{ \omega \in \Omega^*(E \times]1/4, 3/4[/ \mathcal{F} \times \mathcal{I}) / \left[\begin{array}{l} \text{supp } \omega \subset K \times]1/4, 3/4[\\ \text{for a compact } K \subset E \end{array} \right] \right\}.
\end{aligned}$$

Proceeding as in (3) we get the short exact sequence

$$0 \longrightarrow \Omega_c^*(E \times]0, 1[/ \mathcal{F} \times \mathcal{I}) \longrightarrow A^*(]0, 3/4[) \oplus A^*(]1/4, 1[) \longrightarrow A^*(]1/4, 3/4[) \longrightarrow 0.$$

The associated long exact sequence is

$$\begin{aligned}
\cdots \rightarrow H^{i-1}(A^*(]1/4, 3/4[)) \xrightarrow{\delta} H_c^i(E \times]0, 1[/ \mathcal{F} \times \mathcal{I}) \rightarrow H^i(A^*(]0, 3/4[)) \oplus \\
\oplus H^i(A^*(]1/4, 1[)) \rightarrow H^i(A^*(]1/4, 3/4[)) \rightarrow \cdots,
\end{aligned}$$

where the connecting morphism is $\delta([\omega]) = [df \wedge \omega]$.

Before executing the calculation let us introduce some notation. Let β be a differential form on $\Omega^i(D_S \times]a, b[)$ which does not include the dt factor. By $\int_-^c \beta(s) \wedge ds$ and $\int_c^- \beta(s) \wedge ds$ we denote the forms on $\Omega^i(D_S \times]a, b[)$ obtained from β by integration with respect to s , that is, $\left(\int_-^c \beta(s) \wedge ds \right) (x, t)(\vec{v}_1, \dots, \vec{v}_i) = \int_t^c (\beta(x, s)(\vec{v}_1, \dots, \vec{v}_i)) ds$ and on the other hand $\left(\int_c^- \beta(s) \wedge ds \right) (x, t)(\vec{v}_1, \dots, \vec{v}_i) = \int_c^t (\beta(x, s)(\vec{v}_1, \dots, \vec{v}_i)) ds$ where $c \in]a, b[$, $(x, t) \in D_S \times]a, b[$ and $(\vec{v}_1, \dots, \vec{v}_i) \in T_{(x,t)}(D_S \times]a, b[)$.

(i) Computing δ .

Each differential form $\omega \in A^*(Interval)$ can be written $\omega = \alpha + \beta \wedge dt$ where α and β do not contain dt .

Consider a cycle $\omega = \alpha + \beta \wedge dt \in A^i(]0, 3/4[)$ with $\text{supp } \omega \subset K \times [c, 3/4[$ for a compact $K \subset E$ and $0 < c < 3/4$. We have $\omega = \alpha(c/2) - d \left(\int_-^{c/2} \beta(s) \wedge ds \right) = -d \left(\int_-^{c/2} \beta(s) \wedge ds \right)$.

Since $\text{supp } \int_-^{c/2} \beta(s) \wedge ds \subset K \times [c, 3/4[$ then we get $H^*(A^i(]0, 3/4[)) = 0$. In the same way, we get $H^*(A^i(]1/4, 1[)) = 0$.

We conclude that δ is an isomorphism.

(i) Computing $H_c^*(E \times]0, 1[/ \mathcal{F} \times \mathcal{I})$.

Consider a cycle $\omega = \alpha + \beta \wedge dt \in A^*(]1/4, 3/4[)$. We have $\omega = \alpha(1/2) + d \left(\int_{1/2}^- \beta(s) \wedge ds \right)$.

Notice that $\text{supp } \int_{1/2}^- \beta(s) \wedge ds \subset K \times]1/4, 3/4[$ and $\text{supp } \alpha(1/2) \subset K$. A standard procedure shows that the operator $\Delta: H^*(A^i(]1/4, 3/4[)) \rightarrow H_c^{*-1}(E/\mathcal{F})$, defined by $\Delta([\omega]) = [\alpha(1/2)]$, is an isomorphism. The inverse is $\Delta^{-1}([\gamma]) = [\gamma]$.

So, the composition $\delta \circ \Delta^{-1}: H_c^{*-1}(E/\mathcal{F}) \rightarrow H_c^*((E \times]0, 1[)/\mathcal{F} \times \mathcal{I})$ is an isomorphism. It is exactly the operator: $[\gamma] \mapsto [df \wedge \gamma]$. \clubsuit

The reason why we use the $(-\mathbb{Z}_2)$ -invariant classes in the next Lemma instead of the more natural \mathbb{Z}_2 -invariant classes is the following: we have $(H_c^*(]-1, 0[\cup]0, 1[/\mathcal{I}))^{-\mathbb{Z}_2} \cong (H_c^*(]-1, 1[/\mathcal{I}))^{-\mathbb{Z}_2}$ but also $(H_c^*(]-1, 0[\cup]0, 1[/\mathcal{I}))^{\mathbb{Z}_2} \not\cong (H_c^*(]-1, 1[/\mathcal{I}))^{\mathbb{Z}_2}$.

Lemma 2.6.3 *The inclusion*

$$\Omega_c^*((D_0 \cap M) \times (]-1, 0[\cup]0, 1[)/\mathcal{F} \times \mathcal{I}) \hookrightarrow \Omega_c^*((D_0 \cap M) \times]-1, 1[/\mathcal{F} \times \mathcal{I})$$

induces the isomorphism

$$(H_c^*((D_0 \cap M) \times (]-1, 0[\cup]0, 1[)/\mathcal{F} \times \mathcal{I}))^{-\mathbb{Z}_2} \cong (H_c^*((D_0 \cap M) \times]-1, 1[/\mathcal{F} \times \mathcal{I}))^{-\mathbb{Z}_2}.$$

Proof. For the sake of simplicity, we write $E = (D_0 \cap M)$. Consider $f:]-1, 1[\rightarrow \mathbb{R}$ a smooth function with $f \equiv 0$ on $]-1, 1/3[$ and $f \equiv 1$ on $[2/3, 1[$. The function $f \circ \sigma + f - 1:]-1, 1[\rightarrow \mathbb{R}$ is a smooth function whose support is in $[-2/3, 2/3]$. So, $[df] = -[d(f \circ \sigma)] \in H_c^*(]-1, 1[)$. The above Lemma gives

- $H_c^*(E/\mathcal{F}) \cong H_c^*((E \times]-1, 1[)/\mathcal{F} \times \mathcal{I})$ by $[\gamma] \mapsto [df] \wedge [\gamma]$,
- $H_c^*(E/\mathcal{F}) \oplus H_c^*(E/\mathcal{F}) \cong H_c^*((E \times (]-1, 0[\cup]0, 1[)/\mathcal{F} \times \mathcal{I}))$ by $([\gamma_1], [\gamma_2]) \mapsto [d(f \circ \sigma)] \wedge [\gamma_1] + [df] \wedge \gamma_2$.

Notice that $\zeta \cdot df = -\sigma^* df = -d(f \circ \sigma)$ on $]-1, 1[$ and $]-1, 0[\cup]0, 1[$. This gives the isomorphisms of degree +1:

- $H_c^*(E/\mathcal{F}) \cong (H_c^*((E \times]-1, 1[)/\mathcal{F} \times \mathcal{I}))^{-\mathbb{Z}_2}$ by $[\gamma] \mapsto [df] \wedge [\gamma]$,
- $H_c^*(E/\mathcal{F}) \cong (H_c^*((E \times (]-1, 0[\cup]0, 1[)/\mathcal{F} \times \mathcal{I}))^{-\mathbb{Z}_2}$ by $[\gamma] \mapsto -\frac{1}{2}[d(f \circ \sigma)] \wedge [\gamma] + \frac{1}{2}[df] \wedge [\gamma]$.

We obtain the result, since on $]-1, 1[$ we have $-\frac{1}{2}[d(f \circ \sigma)] + \frac{1}{2}[df] = \frac{1}{2}[df] + \frac{1}{2}[df] = [df]$. \clubsuit

The way to lift the question to the Molino's desingularisation is the following.

Lemma 2.6.4

$$H_c^n(M_1/\mathcal{F}_1) \neq 0 \iff H_c^n(M/\mathcal{F}) \neq 0.$$

Proof. We prove the two implications.

\Leftarrow From the open covering $\{(D_0 \cap M) \times]-1, 1[, M \times \{-1, 1\}\}$ of M_1 we obtain the short exact sequence

$$\begin{aligned} 0 &\rightarrow \Omega_c^*((D_0 \cap M) \times (]-1, 0[\cup]0, 1[)/\mathcal{F} \times \mathcal{I}) \rightarrow \\ &\rightarrow \Omega_c^*((D_0 \cap M) \times]-1, 1[/\mathcal{F} \times \mathcal{I}) \oplus \Omega_c^*((M \times \{-1, 1\})/\mathcal{F} \times \mathcal{I}) \rightarrow \Omega_c^*(M_1/\mathcal{F}_1) \rightarrow 0 \end{aligned}$$

(same argument as in (3)). The associated long exact sequence is

$$\begin{aligned} \dots &\rightarrow H_c^j((D_0 \cap M) \times (]-1, 0[\cup]0, 1[)/\mathcal{F} \times \mathcal{I}) \xrightarrow{(I, J)} H_c^j((D_0 \cap M) \times]-1, 1[/\mathcal{F} \times \mathcal{I}) \oplus \\ &H_c^j((M \times \{-1, 1\})/\mathcal{F} \times \mathcal{I}) \rightarrow H_c^j(M_1/\mathcal{F}_1) \rightarrow H_c^{j+1}((D_0 \cap M) \times (]-1, 0[\cup]0, 1[)/\mathcal{F} \times \mathcal{I}) \rightarrow \dots \end{aligned}$$

Lemma 2.6.2 gives that I is an onto map. We also have

$$H_c^{n+1}((D_0 \cap M) \times (]-1, 0[\cup]0, 1[) / \mathcal{F} \times \mathcal{I}) = 0$$

for degree reasons. Therefore

$$H_c^n(M/\mathcal{F}) \oplus H_c^n(M/\mathcal{F}) = H_c^n((M \times \{-1, 1\})/\mathcal{F} \times \mathcal{I}) \rightarrow H_c^n(M_1/\mathcal{F}_1)$$

is a surjective map. This gives the result.

\Rightarrow The above short exact sequence is \mathbb{Z}_2 -equivariant. So, we get the long exact sequence

$$\begin{aligned} \cdots &\rightarrow \left(H_c^j((D_0 \cap M) \times (]-1, 0[\cup]0, 1[) / \mathcal{F} \times \mathcal{I}) \right)^{-\mathbb{Z}_2} \xrightarrow{(I, J)} \\ &\rightarrow \left(H_c^j((D_0 \cap M) \times]-1, 1[/ \mathcal{F} \times \mathcal{I}) \right)^{-\mathbb{Z}_2} \oplus \left(H_c^j((M \times \{-1, 1\}) / \mathcal{F} \times \mathcal{I}) \right)^{-\mathbb{Z}_2} \rightarrow \\ &\rightarrow \left(H_c^j(M_1/\mathcal{F}_1) \right)^{-\mathbb{Z}_2} \rightarrow \left(H_c^{j+1}((D_0 \cap M) \times (]-1, 0[\cup]0, 1[) / \mathcal{F} \times \mathcal{I}) \right)^{-\mathbb{Z}_2} \rightarrow \cdots \end{aligned}$$

Lemma 2.6.3 ensures that I is an isomorphism. We conclude that

$$\left(H_c^*(M_1/\mathcal{F}_1) \right)^{-\mathbb{Z}_2} \cong \left(H_c^*((M \times \{-1, 1\}) / \mathcal{F} \times \mathcal{I}) \right)^{-\mathbb{Z}_2} \cong H_c^*(M/\mathcal{F}) \otimes \left(H^0(-1, 1) \right)^{-\mathbb{Z}_2} \cong H_c^*(M/\mathcal{F}),$$

which ends the proof. ♣

The main result of this section is the following.

Theorem 2.6.5 *Let M be a manifold endowed with a CERF \mathcal{F} . Let us suppose that \mathcal{F} is transversally orientable. Put $n = \text{codim } \mathcal{F}$. Then, the two following statements are equivalent:*

- (a) *The foliation \mathcal{F} is taut.*
- (b) *The cohomology group $H_c^n(M/\mathcal{F})$ is \mathbb{R} .*

Otherwise, $H_c^n(M/\mathcal{F}) = 0$.

Proof. We proceed by induction on depth $S_{\mathcal{H}}$. When depth $S_{\mathcal{H}} = 0$ then we have a regular foliation and the result comes from [1, Theorem 6.4]. For the induction step, we can suppose that the result is true for $S_{\tilde{\mathcal{H}}}$, since depth $S_{\tilde{\mathcal{H}}} < \text{depth } S_{\mathcal{H}}$. We have in particular $H_c^n(M_1/\mathcal{F}_1) = \mathbb{R}$ or 0. So, Lemma 2.6.4 ensures that

$$H_c^n(M_1/\mathcal{F}_1) = \mathbb{R} \text{ (resp. } 0) \iff H_c^n(M/\mathcal{F}) = \mathbb{R} \text{ (resp. } 0).$$

By Lemma 2.6.1 the foliation $\tilde{\mathcal{H}}$ is transversally orientable. Since $(\tilde{H}, \tilde{\mathcal{H}})$ is a Molino's desingularisation of (N_1, \mathcal{H}_1) then the foliation \mathcal{H}_1 is also transversally orientable. So, we have

$$\mathcal{F} \text{ is taut} \xleftrightarrow{2.4.3} \tilde{\mathcal{H}} \text{ is taut} \xleftrightarrow{2.4.3} \mathcal{F}_1 \text{ is taut} \xleftrightarrow{\text{Induc}} H_c^n(M_1/\mathcal{F}_1) = \mathbb{R} \iff H_c^n(M/\mathcal{F}) = \mathbb{R},$$

which ends the proof. ♣

2.7 Reading the tautness of \mathcal{F} on N and \tilde{N} . The tautness of \mathcal{F} and $\tilde{\mathcal{H}}$ are closely related. We have already seen

$$H_c^n(M/\mathcal{F}) = \mathbb{R} \iff H^n(\tilde{N}/\tilde{\mathcal{H}}) = \mathbb{R}.$$

The situation at the zipper level is more complicated. In fact, we don't have the expected equivalence³ $H_c^n(M/\mathcal{F}) = \mathbb{R} \iff H^n(N/\mathcal{H}) = \mathbb{R}$ (for instance, in Example 1.4 we have $H_c^n(M/\mathcal{F}) = \mathbb{R}$ and $H^n(N/\mathcal{H}) = 0$). This situation comes from the existence of boundary strata but we are going to prove that it is the only obstruction.

Lemma 2.7.1 *Consider (T_S, τ, S) a foliated tubular neighborhood of a singular stratum $S \in \mathcal{S}_{\mathcal{H}}^{sin}$. If S is not a boundary stratum then the inclusion $(T_S \setminus S) \hookrightarrow T_S$ induces the isomorphism $\iota: H_c^n((T_S \setminus S)/\mathcal{H}) \rightarrow H_c^n(T_S/\mathcal{H})$.*

Proof. We proceed in two steps.

(a) Approaching $H_c^n(T_S/\mathcal{F})$. Consider the complexes

$$\begin{aligned} - A^*([0, 3/4]) &= \left\{ \omega \in \Omega^*(\rho_S^{-1}([0, 3/4])/\mathcal{H}) \mid \left[\begin{array}{l} \text{supp } \omega \subset \tau^{-1}(K) \\ \text{for a compact } K \subset S. \end{array} \right] \right\}. \\ - A^*([1/4, 1]) &= \left\{ \omega \in \Omega^*(\rho_S^{-1}([1/4, 1])/\mathcal{H}) \mid \left[\begin{array}{l} \text{supp } \omega \subset \tau^{-1}(K) \cap \rho_S^{-1}([1/4, \epsilon]) \\ \text{for a compact } K \subset S \text{ and } 1/4 < \epsilon < 1. \end{array} \right] \right\}. \\ - A^*([1/4, 3/4]) &= \left\{ \omega \in \Omega^*(\rho_S^{-1}([1/4, 3/4])/\mathcal{H}) \mid \left[\begin{array}{l} \text{supp } \omega \subset \tau^{-1}(K) \\ \text{for a compact } K \subset S. \end{array} \right] \right\}. \end{aligned}$$

The short exact sequence

$$0 \longrightarrow \Omega_c^*(T_S/\mathcal{H}) \longrightarrow A^*([0, 3/4]) \oplus A^*([1/4, 1]) \longrightarrow A^*([1/4, 3/4]) \longrightarrow 0$$

produces the long exact sequence

$$\begin{aligned} H^{n-1}(A^*([0, 3/4])) \oplus H^{n-1}(A^*([1/4, 1])) &\longrightarrow H^{n-1}(A^*([1/4, 3/4])) \xrightarrow{\delta} H_c^n(T_S/\mathcal{H}) \longrightarrow \\ &\longrightarrow H^n(A^*([0, 3/4])) \oplus H^n(A^*([1/4, 1])) \end{aligned}$$

where the connecting morphism is $\delta([\omega]) = [df \wedge \omega]$ for a smooth map $f: [0, 1] \rightarrow [0, 1]$ with $f \equiv 0$ on $[0, 3/8]$ and $f \equiv 1$ on $[5/8, 1]$.

(b) Computing $H^*(A(\text{Interval}))$. Consider a cycle $\omega = \alpha + \beta \wedge dt \in A^*([1/4, 1])$, where α and β do not contain dt . We have $\omega = -d \left(\int_{-}^{1} \beta(s) \wedge ds \right)$. Since $\text{supp } \int_{-}^{1} \beta(s) \wedge ds \subset \tau^{-1}(K) \times]1/4, \epsilon]$ we get $H^*(A([1/4, 1])) = 0$.

The contraction $H_S: \rho_S^{-1}([0, 3/4]) \times [0, 1] \rightarrow \rho_S^{-1}([0, 3/4])$ (cf. 1.4) is a foliated proper homotopy between $\rho_S^{-1}([0, 3/4])$ and S . This homotopy preserves the fibers of τ . Then $H^*(A([0, 3/4])) \cong H_c^*(S/\mathcal{H}_S)$. Since the stratum S is a not boundary stratum, then $\text{codim}_S \mathcal{H}_S < n - 1$ (cf. 1.4). Then, by degree reasons, we get $H^{n-1}(A^*([0, 3/4])) = H^n(A^*([0, 3/4])) = 0$.

Since $\rho_S^{-1}([1/4, 3/4]) = D_S \times]1/4, 3/4[$, then we have that the assignment $[\gamma] \mapsto [\gamma \wedge df]$ induces the isomorphism $H_c^*(D_S/\mathcal{H}) \cong H^*(A([1/4, 3/4]))$ (cf. proof of the Lemma 2.6.2).

³The basic cohomology of an SRF is defined as in 2.1.

(c) *Last step.* The above exact sequence gives that $\delta: H_c^{n-1}(D_S/\mathcal{H}) \rightarrow H_c^n(T_S/\mathcal{H})$, defined by $\delta([\omega]) = [df \wedge \omega]$, is an isomorphism. We have completed the proof as $\delta: H_c^{n-1}(D_S/\mathcal{H}) \rightarrow H_c^n((T_S \setminus S)/\mathcal{H}) \equiv (D_S \times]0, 1])/\mathcal{H} \times \mathcal{I}$ is an isomorphism (cf. proof of the Lemma 2.6.2). ♣

The tautness of \mathcal{F} can be read on a zipper as follows.

Theorem 2.7.2 *Let M be a manifold endowed with a CERF \mathcal{F} , which is transversally oriented. Consider (N, \mathcal{H}) a zipper of \mathcal{F} . Let us suppose that $S_{\mathcal{H}}$ does not possess any boundary stratum. Put $n = \text{codim } \mathcal{F}$. Then, the following two statements are equivalent:*

- (a) *The foliation \mathcal{F} is taut.*
- (b) *The cohomology group $H^n(N/\mathcal{H})$ is \mathbb{R} .*

Otherwise, $H^n(N/\mathcal{H}) = 0$.

Proof. It suffices to prove that $H^n(N/\mathcal{H}) = H_c^n(M/\mathcal{F})$ (cf. Theorem 2.6.5). Consider Σ_i as in 2.3.1. We have $N \setminus \Sigma_{-1} = N$ and $N \setminus \Sigma_{m-1} = M$, where $m = \dim M$. We will get the result if we prove that $H_c^n((N \setminus \Sigma_i)/\mathcal{H}) \cong H_c^n((N \setminus \Sigma_{i-1})/\mathcal{H})$, for $i \in \{0, \dots, m-1\}$.

From the open covering $\left\{ N \setminus \Sigma_i, \bigcup_{\dim S=i} T_S \right\}$ of $N \setminus \Sigma_{i-1}$, we obtain the Mayer-Vietoris sequence

$$\bigoplus_{\dim S=i} H_c^n((T_S \setminus S)/\mathcal{H}) \longrightarrow H_c^n((N \setminus \Sigma_i)/\mathcal{H}) \oplus \bigoplus_{\dim S=i} H_c^n(T_S/\mathcal{H}) \longrightarrow H_c^n((N \setminus \Sigma_{i-1})/\mathcal{H}) \rightarrow 0.$$

Now, the Lemma 2.7.1 gives the result. ♣

When boundary strata appear, it can be shown that this Theorem still holds considering $H^n(N/\mathcal{H}, \partial(N/\mathcal{H}))$ in (b). Here, $\partial(N/\mathcal{H}) = \cup\{S \mid S \preceq S' \text{ for a boundary stratum } S'\}$ and the relative basic cohomology must be understood in a suitable way.

3 Appendix

We prove the existence of foliated Thom-Mather systems for an SRF \mathcal{H} defined on a compact manifold N announced in Proposition 1.6.1. First of all, we need a more accurate presentation of the Molino's desingularisation of 1.7. We fix an adapted metric μ on N .

3.1 Molino's desingularisation. A *Molino's desingularisation* of (N, \mathcal{H}, μ) (see [13]) is a sequence \mathfrak{N} of blow ups (cf. 1.6):

$$(N_p, \mathcal{H}_p, \mu_p) \xrightarrow{\mathcal{L}_p} (N_{p-1}, \mathcal{H}_{p-1}, \mu_{p-1}) \longrightarrow \dots \longrightarrow (N_1, \mathcal{H}_1, \mu_1) \xrightarrow{\mathcal{L}_1} (N_0, \mathcal{H}_0, \mu_0)$$

with $(N_0, \mathcal{H}_0, \mu_0) = (N, \mathcal{F}, \mu)$ and $(N_i, \mathcal{H}_i, \mu_i) \equiv (\widehat{N_{i-1}}, \widehat{\mathcal{H}_{i-1}}, \widehat{\mu_{i-1}})$, for $i \in \{1, \dots, p\}$ ⁴. Here, $p = \text{depth } S_{\mathcal{F}}$. The triple $(\widetilde{N}, \widetilde{\mathcal{H}}, \widetilde{\mu}) \equiv (N_p, \mathcal{H}_p, \mu_p)$ is a regular riemannian foliated manifold.

Notice that the blow up of $(N \times \mathbb{R}, \mathcal{F} \times \mathcal{I}, \mu + dt^2)$ is just

$$(N_p \times \mathbb{R}, \mathcal{H}_p \times \mathcal{I}, \mu_p + dt^2) \xrightarrow{\mathcal{L}_p} (N_{p-1} \times \mathbb{R}, \mathcal{H}_{p-1} \times \mathcal{I}, \mu_{p-1} + dt^2) \longrightarrow \dots \xrightarrow{\mathcal{L}_1} (N_0 \times \mathbb{R}, \mathcal{H}_0 \times \mathcal{I}, \mu_0 + dt^2)$$

(cf. 1.6.1 (a)).

⁴The map ξ necessary for the construction of the blow up (cf. 1.6) is supposed to be the same for each i .

3.2 Construction of the foliated Thom-Mather system. We first construct a tubular neighborhood of each singular stratum and then, we prove that these neighborhoods satisfy the compatibility conditions (TM1), (TM2) and (TM3).

3.2.1 Take $S' \in \mathcal{S}_{\mathcal{H}}$ a singular stratum and let $i \in \mathbb{N}$ be its depth. Take the partial desingularisation $\mathfrak{N}_i = \mathfrak{L}_1 \circ \mathfrak{L}_2 \circ \dots \circ \mathfrak{L}_i: (N_i, \mathcal{H}_i, \mu_i) \rightarrow (N, \mathcal{F}, \mu)$. Notice that each restriction

$$(5) \quad \mathfrak{N}_i: \mathfrak{N}_i^{-1}(N \setminus \Sigma_{i-1}) \rightarrow N \setminus \Sigma_{i-1}$$

is a trivial foliated smooth bundle (cf. 1.6.1 (c),(e)). We fix $\mathfrak{s}_i: N \setminus \Sigma_{i-1} \rightarrow \mathfrak{N}_i^{-1}(N \setminus \Sigma_{i-1})$ a smooth foliated section of (5). For $i = 0$ we put $\mathfrak{N}_i = \mathfrak{s}_i = \text{Identity}: N \rightarrow N$.

The stratum S' is a proper submanifold of the foliated riemannian manifold $(N \setminus \Sigma_{i-1}, \mathcal{H}_i, \mathfrak{s}_i^* \mu_i)$. Let $(T_{S'}, \tau_{S'}, S')$ be a foliated tubular neighborhood of S' constructed as in 1.4. It is also a foliated tubular neighborhood of S' in (N, \mathcal{H}) . We need to shrink it in order to assure the Thom-Mather compatibility of these neighborhoods.

Following 1.6.1 (e), there exists a family of strata $\{S^j \in \mathcal{S}_{\mathcal{H}_j} \mid j \in \{0, \dots, i\}\}$ such that:

- + $S^0 = S'$,
- + $(S^j)^{\mathfrak{L}} = S^{j+1}$ for $j \in \{0, \dots, i-1\}$ and
- + S^i is a minimal stratum of $\mathcal{S}_{\mathcal{H}_i}$.

Notice that $\mathfrak{s}_i(S) \subset S^i$. By construction, there exists a foliated tubular neighborhood $(U_{S^i}, \nu_{S^i}, S^i)$ of S^i on (N_i, \mathcal{H}_i) such that its restriction to S^i is isomorphic to $(T_{S'}, \tau_{S'}, S')$ through \mathfrak{s}_i . We can take these neighborhoods small enough such that:

$$(6) \quad (\mathfrak{L}_{i+1}^{-1}(U_{S^i}), \mathcal{H}_{i+1}, \mu_{i+1}) = (D_{S^i} \times]-1, 1[, \mathcal{H}_{i+1} \times \mathcal{I}, \mu|_{D_{S^i}} + dt^2),$$

where D_{S^i} is the core of $(U_{S^i}, \nu_{S^i}, S^i)$ (cf. 1.6).

We prove now that the family $\{T_S \mid S \in \mathcal{S}_{\mathcal{F}}^{\text{sin}}\}$ which we have constructed is a foliated Thom-Mather system of (N, \mathcal{H}) .

3.2.2 Properties TM2 and TM3. We consider two singular strata $S', S \in \mathcal{S}_{\mathcal{F}}^{\text{sin}}$ with $S' \prec S$. Without loss of generality we can suppose that S' is a minimal stratum. The Thom-Mather conditions involve just $T_{S'}$ and not the whole manifold N . So, we can suppose $N = T_{S'}$. The blow up $\mathfrak{L}: (\widehat{N}, \widehat{\mathcal{H}}, \widehat{\mu}) \rightarrow (N, \mathcal{H}, \mu)$ becomes the map $\mathfrak{L}: (D_{S'} \times]-1, 1[, \mathcal{H} \times \mathcal{I}, \mu|_{D_{S'}} + dt^2) \rightarrow (T_{S'}, \mathcal{H}, \mu)$, defined by $\mathfrak{L}(u, t) = H_{S'}(u, 2|t|)$ (cf. (6)). Recall that the restriction

$$\mathfrak{L}: (D_{S'} \times]0, 1[, \mathcal{H} \times \mathcal{I}) \rightarrow ((T_{S'} \setminus S'), \mathcal{H})$$

is a foliated diffeomorphism. We check properties (TM2) and (TM3) at the $(D_{S'} \times]0, 1[)$ -level. This makes sense since $S' \cap (T_{S'} \cap T_S) = \emptyset$.

The foliated tubular neighborhood (T_S, τ_S, S) can be described as follows. There exists a foliated tubular neighborhood $(T_{S''}, \tau_{S''}, S'')$ of the stratum $S'' = D_{S'} \cap S \in \mathcal{S}_{\mathcal{H}_{D_{S'}}}$ such that (T_S, τ_S, S) can be identified with $(T_{S''} \times]0, 1[, \tau_{S''} \times \text{Identity}, S'' \times]0, 1[)$ through \mathfrak{L} (cf. 3.1). By using the foliated diffeomorphism \mathfrak{L} , the conditions (TM2) and (TM3) are clear since:

- + $T_{S'} \cap S$ becomes $S'' \times]0, 1[$,
- + $T_{S'} \cap T_S$ becomes $T_{S''} \times]0, 1[$,
- + τ_S becomes the map $(y, t) \mapsto (\tau_{S''}(y), t)$, and
- + $\rho_{S'}$ becomes the map $(y, t) \mapsto |t|$.

3.2.3 Property TM1.

For each $\epsilon \in]0, 1[$, the family $\mathcal{T}_\epsilon = \{\rho_S^{-1}([0, \epsilon]) / S \in \mathcal{S}_{\mathcal{F}}^{sin}\}$ is, after reparametrization, a foliated Thom-Mather system. We prove that there exists an ϵ such that \mathcal{T}_ϵ verifies (TM1).

Consider two singular strata $S, S' \in \mathcal{S}_{\mathcal{F}}^{sin}$ with $S \not\leq S'$ and $S' \not\leq S$. We prove that $\rho_{S'}^{-1}([0, \epsilon]) \cap \rho_S^{-1}([0, \epsilon]) = \emptyset$ for an $\epsilon \in]0, 1[$. Since the number of singular strata is finite, then we shall end the proof.

By using the blow up and an inductive argument on the depth, we can suppose that S' is a minimal (compact) stratum. Since $\overline{S} = \cup\{S'' / S'' \preceq S\}$ we have two disjoint compacts $S' \cap \overline{S} = \emptyset$ and then we can find an $\epsilon \in]0, 1[$ with $\rho_{S'}^{-1}([0, \epsilon]) \cap \rho_S^{-1}([0, \epsilon]) = \emptyset$. ♣

References

- [1] J.A. Álvarez: *The basic component of the mean curvature form for a riemannian foliation.* - Ann. Global Anal. and Geom. **10**(1992), 179–194.
- [2] H. Boualem and P. Molino: *Modèles locaux saturés de feuilletages riemanniens singuliers.* - C. R. Acad. Sci. Paris **316**(1993), 913-916.
- [3] Y. Carrière: *Flots riemanniens.* - Astérisque **116**(1984), 31-52.
- [4] G. Cairns and R. Escobales: *Further geometry of the mean curvature one-form and the normal plane field one-form on a foliated riemannian manifold* - J. Austral. Math. Soc. **62**(1997), 46–63.
- [5] D. Domínguez: *Finiteness and tensesness theorems for riemannian foliations* - Amer.J. Math. **120**(1998), 1237–1276.
- [6] R. Edwards, K. Millett and D. Sullivan: *Foliations with all leaves compact* - Topology **16**(1977), 13-32.
- [7] A. Haefliger: *Some remarks on foliations with minimal leaves* - J. Diff. Geom. **15**(1980), 269-284.
- [8] F.W. Kamber et Ph. Tondeur: *Harmonic foliations.* - Springer LN in Math. **949**(1982), 87-121.
- [9] F.W. Kamber et Ph. Tondeur: *Duality theorems for foliations.* - Astérisque **18**(1984), 458-471.
- [10] X. Masa: *Duality and minimality in Riemannian Foliations.* - Comment. Math. Helv. **67**(1992), 17-27.
- [11] J. Mather: *Notes on topological stability* - Harvard 1970.
- [12] V. Miguel and R. Wolak: *Minimal singular riemannian foliations.* - C.R.A.S. **342**(2006), 33-36.
- [13] P. Molino: *Feuilletages riemanniens réguliers et singuliers* - Colloque Géométrie et Physique, Paris 1986. Travaux en cours **33**(1988), 173-201.
- [14] P. Molino: *Riemannian Foliations* - Progress in Math. **73**. Birkhäuser, 1988.

- [15] H. Rummier: *Quelques notions simples en géométrie riemannienne et leurs applications aux feuilletages compacts.* - Comment. Math. Helv. **54**(1979), 224-239.
- [16] M. Saralegi-Aranguren and R. Wolak: *The BIC of a conical fibration.* - Math. Notes **77**(2005), 213-231.
- [17] M. Saralegi-Aranguren and R. Wolak: *The BIC of a defined by an abelian group of isometries.* - Ann. Polon. Math. **89**(2006), 203-246.
- [18] V. Sergiescu: *Basic cohomology and tautness of riemannian foliations. Appendix B* in P. Molino, *Riemannian Foliations.* - Progress in Math. **73**, Birkhäuser, 1988.
- [19] P. Stefan: *Accesible sets, orbits, and foliations with singularities* - Proc. London Math. Soc. **29**(1974), 699-713.
- [20] D. Sullivan: *A homological characterization of foliations consisting of minimal surfaces.* - Comment. Math. Helv. **54**(1979), 218-223.
- [21] H.J. Sussmann: *Orbit of families of vector fields and integrability of distributions* - Trans. Amer. Math. J. **180**(1973), 171-188.
- [22] R. Thom: *Ensembles et morphismes stratifiés* - Bull. Amer. Math. Soc., **75**(1969), 240-284.
- [23] Ph. Tondeur: *Geometry of Foliations* - Mon. in Math., **90**, Birkhäuser 1997.