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Cohomology of riemannian flows

[1]José Ignacio Royo Prieto and [2]Martintxo Saralegi Aranguren

[1]*Departamento de Matemáticas, Universidad del País Vasco–Euskal Herriko Unibertsitatea*
[2]*LaboGA, Université d'Artois*
emails: mtbroprj@lg.ehu.es, saralegi@euler.univ-artois.fr

Abstract

Let $\mathcal{F}$ be a riemannian flow on a closed manifold M. We stablish a Gysin sequence relating the de Rham cohomology of M and the basic cohomology of $\mathcal{F}$. We also give a geometric characterization of the vanishing of the Euler class. These results generalize the isometric case.

Key words: riemannian foliations, de Rham Cohomology
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1 Introduction

Given a smooth free action of the circle $\mathbb{S}^1$ on a manifold M the de Rham cohomologies of M and that of the orbit space B are related by a long exact sequence

$$\cdots \to H^i(B) \stackrel{e}{\to} H^{i+2}(B) \to H^{i+2}(M) \to H^{i+1}(B) \to \cdots,$$

called the *Gysin sequence.*

A more general Gysin sequence is obtained by considering a smooth action $\Phi\colon \mathbb{R} \times M \to M$ preserving a riemannian metric μ on M, that is, an *isometric action.* Since the orbit space can be very wild (even totally disconnected), the right cohomology to study the transverse structure is the *basic cohomology* $H^*(M/\mathcal{F})$ of the flow determined by the action. Of course, when the action is periodic we are in the previous case and moreover $H^*(M/\mathcal{F}) \equiv H^*(B)$. In this context there exists the Gysin sequence [7]

$$\cdots \to H^i(M/\mathcal{F}) \stackrel{e}{\to} H^{i+2}(M/\mathcal{F}) \to H^{i+2}(M) \to H^{i+1}(M/\mathcal{F}) \to \cdots,$$

which we construct in section 3.

In section 4, we describe a third Gysin sequence, which is obtained in the case of a smooth action $\Phi\colon \mathbb{R} \times M \to M$ preserving not a riemannian metric μ on M, but just the restriction of μ to the normal bundle of $\mathcal{F}$, that is, a *riemannian action.* In this context we have constructed the following Gysin sequence

$$\cdots \to H^i_\kappa(M/\mathcal{F}) \stackrel{e}{\to} H^{i+2}(M/\mathcal{F}) \to H^{i+2}(M) \to H^{i+1}_\kappa(M/\mathcal{F}) \to \cdots,$$

where κ is the mean curvature of the flow and $H^*_\kappa(M/\mathcal{F})$ is the twisted basic cohomology (this result has been developed in [13]).

The vanishing of the *Euler class* e of the foliation in the isometric case has a geometrical interpretation: the flow is orthogonal to a fibration (see [16]). We show that in the riemannian case the vanishing of the corresponding Euler class also has a geometrical interpretation.

2 Gysin problem

In this paper, M denotes a smooth compact connected manifold without boundary and $\mathcal{F}$ a flow, that is:

- a smooth vector field without zeroes, or

- a smooth action $\varphi\colon \mathbb{R} \times M \to M$ without fixed points.

The right cohomology to study the quotient space $M/\mathcal{F}$ (or transverse structure of $\mathcal{F}$) is the *basic cohomology* $H^*(M/\mathcal{F})$. It is defined from the complex of *basic differential forms*

$$\Omega^*(M/\mathcal{F}) = \{\omega \in \Omega^*(M)\ i_X\omega = i_X d\omega = 0\}.$$

Notice that when the flow is periodic, $M/\mathcal{F}$ is an orbifold B (or a manifold if all the isotropy groups are trivial) and $H^*(M/\mathcal{F}) \equiv H^*(B)$.

We shall study two particular cases of flows: the isometric flows and more generally the riemannian flows. For these kind of flows, the cohomological behavior of the basic cohomology is similar to the cohomological behavior of a riemannian manifold. The basic cohomology is finite dimensional (cf. [4]) and it verifies both the Poincaré duality (cf. [7]) and the Hodge theory (cf. [5]).

Since any basic form is a differential form, we have the short exact sequence

$$0 \longrightarrow \Omega^*(M/\mathcal{F}) \longrightarrow \Omega^*(M) \longrightarrow \frac{\Omega^*(M)}{\Omega^*(M/\mathcal{F})} \longrightarrow 0,$$

and therefore the long exact sequence, *Gysin sequence*

$$\cdots \to H^{i}(M/\mathcal{F}) \to H^{i}(M) \to \mathfrak{G}^{i}(M,\mathcal{F}) \xrightarrow{\delta} H^{i+1}(M/\mathcal{F}) \to \cdots.$$

We want to compute the third term of this sequence, the *Gysin term*, and the connecting morphism δ (cf. [15]).

The periodic case has been treated in [11]. There,

- $\mathfrak{G}^{*}(M,\mathcal{F}) = H^{*-1}(M/\mathcal{F})$,
- the connecting map is the product by the Euler class $e \in H^{2}(M/\mathcal{F})$, and
- the vanishing of the Euler class is equivalent to the fact that, up to a finite covering, $M = B \times \mathbb{S}^{1}$ endowed with the action $\varphi(t,b,z) = (b, e^{2\pi i t}z)$.

3 Isometric flows

The flow $\mathcal{F}$ is *isometric* when it preserves a riemannian metric μ of M, that is,

$$L_X\mu = 0 \quad \text{or } \varphi_t^*\mu = \mu \quad \text{for each } t.$$

Example 1 *The first example is the* linear flow *of the torus $M = \mathbb{T}^n$, which is defined by*

$$\varphi_t(z_1,\ldots,z_n) = (z_1 \cdot e^{2\pi a_1 t}, \ldots, z_n \cdot e^{2\pi a_n t}),$$

where $a_1,\ldots,a_n \in \mathbb{R}$.

The closure of an orbit on a isometric flow is always a torus and the induced flow is the linear flow (this a direct consequence of the below (i)). These tori may have different dimensions. Take for example $M = \mathbb{S}^3$ and $\varphi(t,(z_1,z_2)) = (z_1 \cdot e^{2\pi t}, z_2 \cdot e^{2\pi\sqrt{2}t})$. Here, the closures of the orbits are 2-tori ($|z_1|^2 + |z_2|^2 = r$, $0 < r < 1$) and two circles ($\{z_1 = 0\}, \{z_2 = 0\}$).

The cohomological study of isometric flows is based in the two following relevant properties:

(i) The flow of X lives in $\mathrm{Iso}\,(M,\mu)$, which is a compact Lie group.

(ii) The differential of the *characteristic form* $\chi = i_X\mu \in \Omega^{1}(M)$ is a basic form, that is,

$$d\chi \in \Omega^{2}(M/\mathcal{F}).$$

The cohomological class $e = [d\chi] \in H^{2}(M/\mathcal{F})$ is the *Euler class*. It is important to notice that this class does not depend on the choice of μ, up to the normalisation $\mu(X,X) = 1$.

The property (i) implies that the subcomplex of differential forms of M invariant by the flow computes the cohomology of M. The property (ii) gives the isomorphism

$$(\Omega^*(M/\mathcal{F}) \oplus \Omega^{*-1}(M/\mathcal{F}), D) \overset{\Phi}{\cong} (\Omega^*(M), d),$$

defined by $\Phi(\alpha, \beta) = \alpha + \chi \wedge \beta$, where $D(\alpha, \beta) = (d\alpha + d\chi \wedge \beta, -d\beta)$. From the short exact sequence

$$0 \longrightarrow \Omega^*(M/\mathcal{F}) \longrightarrow \Omega^*(M/\mathcal{F}) \oplus \Omega^{*-1}(M/\mathcal{F}) \longrightarrow \Omega^{*-1}(M/\mathcal{F}) \longrightarrow 0,$$

we get the Gysin sequence (cf. [8])

$$\cdots \to H^i(M/\mathcal{F}) \to H^i(M) \to H^{i-1}(M/\mathcal{F}) \xrightarrow{e} H^{i+1}(M/\mathcal{F}) \to \cdots.$$

In other words, we get the following solution for the Gysin problem

- $\mathfrak{G}^*(M, \mathcal{F}) = H^{*-1}(M/\mathcal{F})$,
- the connecting map is the product by the Euler class $e \in H^2(M/\mathcal{F})$.

Relatively to the vanishing of the Euler class we have (cf. [16])

Proposition 3.1 *The Euler class $e \in H^2(M/\mathcal{F})$ vanishes if and only if there exists a foliation $\mathcal{G}$ transverse to $\mathcal{F}$ which is defined by a cycle.*

Proof. If $\gamma \in \Omega^1(M/\mathcal{F})$ with $d\chi = d\gamma$ we take $\omega = \chi - \gamma$ and we consider the foliation $\mathcal{G}$ defined by ω.

On the other hand, let $\omega \in \Omega^*(M)$ be the cycle defining $\mathcal{G}$. From (i) we can consider that ω is an invariant cycle. The flow X is isometric with respect to the metric

$$\nu = \omega \otimes \omega + \mu_{\mathcal{G}}.$$

Since the new characteristic form is $\chi_\nu = \omega$ then we have $d\chi_\nu = d\omega = 0$. ♣

Remark 1 *The foliation $\mathcal{G}$ can be choosen a fibration. It may exist $\mathcal{G}$ transverse to $\mathcal{F}$ although e is not 0.*

4 Riemannian flows

4.1 Definitions

A riemannian metric μ on M is said to be *bundle-like* when a geodesic perpendicular at one point to a leaf of $\mathcal{F}$ remains perpendicular to the leaves at all of its points (cf. [12]).

Let Q be the normal bundle $TM/T\mathcal{F}$ and let μ_Q be the induced metric. Then, μ is bundle-like if the metric μ_Q is invariant by the flow, that is,

$$L_X \mu_Q = 0.$$

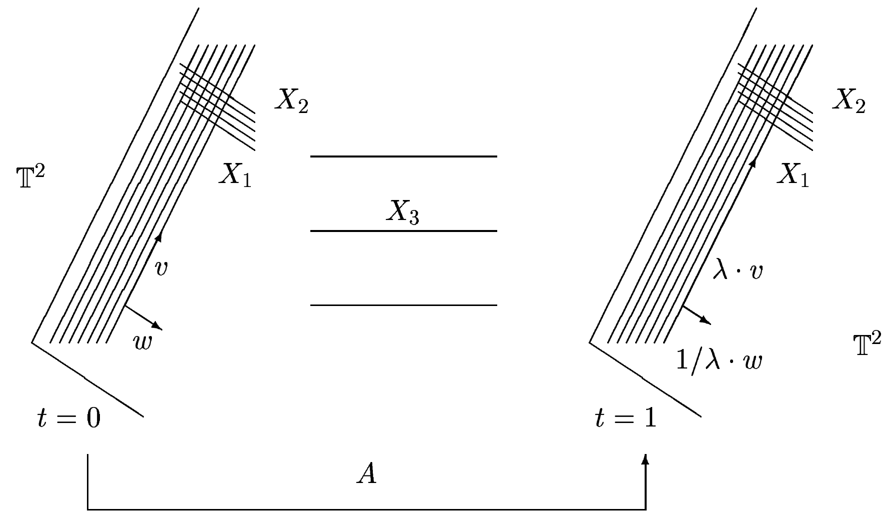
It is clear that any isometric flow is a riemannian flow. When a such metric exists, we shall say that the flow $\mathcal{F}$ is *riemannian*. We can choose a nonsingular vectorfield X defining $\mathcal{F}$ with $\mu(X, X) = 1$. The *characteristic form* is the one-form $\chi = i_X \mu$.

The geometry of a riemannian flow is similar to the geometry of an isometric flow. For example, the closure $\overline{L}$ of an orbit is a torus and the induced flow is linear as in the riemannian case (cf. [2]). Moreover, in both cases, the closure $\overline{L}$ possesses an isometric neighborhood. The difference between riemannian and isometric flows is a global matter. Let us see the classical example of a riemannian flow which is not an isometric flow.

4.2 An example (cf. [2])

The manifold $M = \mathbb{T}^3_A$ is obtained by suspending the diffeomorphism $A\colon \mathbb{T}^2 \to \mathbb{T}^2$ with $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$:

$$\mathbb{T}^3_A = \mathbb{T}^2 \times [0,1]/_{(u,0)\sim(Au,1)}.$$



Choose the metric μ on M by saying that $\{X_1, X_2, X_3\}$ is an orthonormal parallelism. Put $\{\chi_1, \chi_2, \chi_3,\} \subset \Omega^1(M)$ the dual forms. We have

$$L_{X_1}\chi_1 = \chi_3 \text{ and } L_{X_1}\chi_2 = L_{X_1}\chi_3 = 0.$$

Relatively to μ we have:

- Since $L_{X_1}\mu_Q = L_{X_1}(\chi_2 \otimes \chi_2 + \chi_3 \otimes \chi_3) = 0$ then X_1 is riemannian.

- Since $L_{X_1}\mu = \chi_3 \otimes \chi_1 + \chi_1 \otimes \chi_3 \neq 0$ then X_1 is not isometric.

So, there appears the following natural question: Is there another metric for which the flow $\mathcal{F}$ would be isometric? In other words, is the flow $\mathcal{F}$ *geodesible*?

This is not the case. In fact, since $A_* X_2 = (1/\lambda) \cdot X_2$ then there is not a two basic cycle not zero and therefore $H^2\left(\mathbb{T}^3_A/\mathcal{F}\right) = 0$, but on the other hand we know that

Proposition 4.1 *(cf. [10]) Let M be a closed manifold of dimension m endowed with a riemannian flow $\mathcal{F}$. Then $\mathcal{F}$ is isometric iff $H^{m-1}(M/\mathcal{F}) \neq 0$.*

Taking $A =$ Identity we get an isometric flow on $\mathbb{T}^3_A = \mathbb{T}^3$, but both foliations have the same local (saturated) geometrical structure.

4.3 Two main actors

The Gysin sequence that we construct uses the differential forms κ and e that we introduce now. Both of them are *semi-basic* ($i_X\kappa = i_X e = 0$):

- the *mean curvature one-form* $\kappa = L_X \chi \in \Omega^1(M)$ (L_X stands for the Lie derivative) and

- the *Euler form* $e \in \Omega^2(M)$ which is determined by the condition

$$e = d\chi + \kappa \wedge \chi.$$

Notice that the flow is isometric (relatively to μ) if and only if $\kappa = 0$. We shall weaken this condition in the next paragraph.

4.4 Two key points

A riemannian foliation does not necessarily verify the conditions (i) and (ii) of isometric flows. In this context, the conditions we are going to use are the following.

(iii) For each leaf $L \in \mathcal{F}$ there exists a saturated neighborhood U of the closure $\overline{L}$, called *Carrière's neighborhood*, such that

  - there is a diffeomorphism $U \to \mathbb{S}^1 \times \mathbb{T}^k \times \mathbb{D}^{n-k}$ mapping $\overline{L}$ onto $\mathbb{S}^1 \times \mathbb{T}^k \times \{0\}$,
  - the flow restricted to U is conjugated to the flow obtained by the suspension of a diffeomorphism $T \times R$ of $\mathbb{T}^k \times \mathbb{D}^{n-k}$ where T is an irrational translation and R is a rotation of $\mathbb{R}^{n-k}$ (cf. [2]).

Note that the flow is isometric with the canonical metric. In the case of the Example 4.2 we have that T is an irrational rotation on $\mathbb{T}^k = \mathbb{S}^1$, R is the identity on $\mathbb{D}^{n-k} =]-1, 1[$ and therefore U is the product $]-1, 1[\times \mathbb{T}^2$.

iv) There exists a bundle-like metric μ such that κ is a basic cycle and e is a basic form (cf. [3]).

Consequently, without loss of generality, we shall work with such a bundle-like metric. The cohomological class $[\kappa] \in H^*(M/\mathcal{F})$ is an invariant of the flow (cf. [1]). We call it the *Álvarez class*. This class vanishes if and only if the flow is geodesible. Since the natural inclusion $H^*(M/\mathcal{F}) \to H^*(M)$ is a monomorphism then, any riemannian flow on a simply connected manifold is an isometric flow.

4.5 Twisted cohomology

To solve the Gysin problem we use the twisted cohomology $H^*_\kappa(M/\mathcal{F})$. This cohomology is defined from the complex of basic forms $\Omega^*(M/\mathcal{F})$ using the twisted derivative

$$d_\kappa \omega = d\omega - \kappa \wedge \omega.$$

This cohomology depends on κ but only through its class:

$$H^*_{\kappa+df}(M/\mathcal{F}) \cong H^*_\kappa(M/\mathcal{F}),$$

the isomorphism is just given by $\omega \mapsto e^f\omega$. Similar definitions apply to $-\kappa$. In particular, $H^*_\kappa(M/\mathcal{F}) = H^*(M/\mathcal{F})$ when the flow is geodesible. The Euler form e is in fact a $(-\kappa)$-twisted cycle. We define the *Euler class* as $e = [e] \in H^2_{-\kappa}(M/\mathcal{F})$. Notice that $H^*_\kappa(M/\mathcal{F})$ is not an algebra but we have the wedge product

$$\wedge \colon H^*_\kappa(M/\mathcal{F}) \times H^*_{-\kappa}(M/\mathcal{F}) \longrightarrow H^*(M/\mathcal{F}).$$

The twisted cohomology is finite dimensional for a compact manifold and it appears naturally when one stablishes a Poincar Duality Theorem (cf. [7]).

Proposition 4.2 *Let M be an oriented closed manifold of dimension m endowed with a riemannian flow $\mathcal{F}$. The pairing*

$$\Pi \colon H^i(M/\mathcal{F}) \times H^{m-i-1}_\kappa(M/\mathcal{F}) \to \mathbb{R},$$

defined by $\Pi(\alpha,\beta) = \int_M \alpha \wedge \beta \wedge \chi$, is perfect.

5 The Gysin sequence

The first step to construct the Gysin sequence in the isometric case was the computation of the cohomology of M by using just the basic data. This was possible because the flow of X lives on a compact Lie group. In the riemannian case this is not longer true, so a more sophisticated tool is needed. We use the local description of the riemannian flow given by Carrière [2] and we get the following result:

Proposition 5.1 *We have the isomorphism:*

$$H^*\left(\Omega^*(M/\mathcal{F}) \oplus \Omega^{*-1}(M/\mathcal{F}), D\right) \cong H^*(M),$$

where $D(\alpha,\beta) = (d\alpha + \mathfrak{e} \wedge \beta, -d\beta + \kappa \wedge \beta)$.

Proof. Consider the differential operator

$$F_M\colon (\Omega^*(M/\mathcal{F}) \oplus \Omega^{*-1}(M/\mathcal{F}), D) \longrightarrow (\Omega^*(M), d)$$

defined by $F_M(\alpha, \beta) = \alpha + \chi \wedge \beta$. A direct calculation using (iv) proves that F_M is a differential operator. We prove that F_M induces an isomorphism in cohomology, which concludes the proof.

Using the usual Mayer-Vietoris techniques, one reduces the problem to $M = U$, a Carrière's neighborhood. Here the flow is geodesible and we can find a basic function with $\kappa = df$ on U. Consider the new riemannian metric $\mu' = e^f \mu$. Then $X' = e^{-f} X$, $\chi' = e^f \chi$, $\kappa' = 0$ and $e' = e^f e$. Since the flow $\mathcal{F}$ is isometric relatively to the metric μ', then we have already seen that $F'_U = \Phi$ induces an isomorphism in cohomology. Consider now the differential operator

$$\varepsilon\colon (\Omega^*(M/\mathcal{F}) \oplus \Omega^{*-1}(M/\mathcal{F}), D) \longrightarrow (\Omega^*(M/\mathcal{F}) \oplus \Omega^{*-1}(M/\mathcal{F}), D'),$$

defined by

$$\varepsilon(\alpha, \beta) = (\alpha, e^{-f}\beta).$$

One checks directly that this operators is an isomorphism verifying $F_U = F'_U \circ \varepsilon$. This completes the proof. ♣

Now, we arrive at the following:

Theorem 5.2 *Given a riemannian flow $\mathcal{F}$ on a compact manifold M we have the long exact sequence*

$$\cdots \to H^i(M/\mathcal{F}) \to H^i(M) \to H^{i-1}_\kappa(M/\mathcal{F}) \stackrel{\delta}{\to} H^{i+1}(M/\mathcal{F}) \to \cdots,$$

where the connecting map δ is the product by the Euler class $e \in H^2_{-\kappa}(M/\mathcal{F})$ up to sign.

Proof. The long exact sequence comes from the result from the above Proposition and from the short exact sequence

$$0 \to (\Omega^*(M/\mathcal{F}), d) \to (\Omega^*(M/\mathcal{F}) \oplus \Omega^{*-1}(M/\mathcal{F}), D) \to (\Omega^{*-1}(M/\mathcal{F}), d_\kappa) \to 0.$$

For the connecting map we consider a basic differential p-form β with $d_\kappa \beta = 0$. Since $D(0, \beta) = (e \wedge \beta, 0)$ then $\delta[\beta] = [(-1)^p e \wedge \beta]$. ♣

In other words, we get the following solution for the Gysin problem:

- $\mathfrak{G}^*(M, \mathcal{F}) = H^{*-1}_\kappa(M/\mathcal{F})$,
- the connecting map is the product by the Euler class $e \in H^2_{-\kappa}(M/\mathcal{F})$.

5.1 Vanishing of the Euler class

The riemannian flow determines the Euler class $e \in H^2_{-\kappa}(M/\mathcal{F})$. This class depends *a priori* on the choice of the metric μ. However, we obtain the following:

Proposition 5.3 *Let μ_1 and μ_2 be two bundle-metrics with basic mean curvature forms κ_1 and κ_2. Consider the canonical (up to a multiplicative positive constant) isomorphism*

$$T^*: H^2_{-\kappa_2}(M/\mathcal{F}) \longrightarrow H^2_{-\kappa_1}(M/\mathcal{F}),$$

defined from the differential operator $T(\omega) = e^f \omega$, where $df = \kappa_2 - \kappa_1$. Then, $T^*[e_2]$ and $[e_1]$ are proportional. In particular, the vanishing of the Euler class does not depend on the choice of the bundle-like metric, but just on $\mathcal{F}$.

Proof. Using the same techniques employed in the proof of the above Theorem we can find an isomorphism

$$H^*\left(\Omega^*_{-\kappa_j}(M/\mathcal{F}) \oplus \Omega^{*-1}(M/\mathcal{F}), D_j\right) \cong H^*_{-\kappa_j}(M),$$

with differential $D_j(\alpha, \beta) = (d_{-\kappa_j}\alpha + e_j \wedge \beta, -d\beta)$, for $j = 1, 2$. This leads us to the *twisted Gysin sequence*

$$\cdots \to H^i_{-\kappa_j}(M/\mathcal{F}) \to H^i_{-\kappa_j}(M) \to H^{i-1}(M/\mathcal{F}) \xrightarrow{\delta_j} H^{i+1}_{-\kappa_j}(M/\mathcal{F}) \to \cdots.$$

The connecting morphism is the multiplication by the Euler class e_j and then $e_j = \delta_j(1)$. The differential isomorphism

$$(T^*, \mathrm{Id.}): (\Omega^*_{-\kappa_2}(M/\mathcal{F}) \oplus \Omega^{*-1}(M/\mathcal{F}), D_2) \to (\Omega^*_{-\kappa_1}(M/\mathcal{F}) \oplus \Omega^{*-1}(M/\mathcal{F}), D_1)$$

induces a chain isomorphism between both exact sequences and therefore $T(e_2) = T\delta_2(1) = \delta_1(1) = e_1$. ♣

In both the periodic and the isometric cases the vanishing of the Euler class indicates the existence of a particular foliation transverse to the flow. This is also the case for a riemannian foliation. Recall that a foliation $\mathcal{G}$ transverse to X is defined by a *connection form* ω satisfying:

$$\omega(X) = 1 \quad \text{and} \quad d\omega = \tau \wedge \omega.$$

The form τ is said to be the *torsion* of $\mathcal{G}$.

Proposition 5.4 *An Euler class $e \in H^2_{-\kappa}(M/\mathcal{F})$ vanishes if and only if there exists a foliation $\mathcal{G}$ transverse to $\mathcal{F}$ whose torsion τ is basic.*

Proof. If $e = 0$ then there exists $\gamma \in \Omega^1(M/\mathcal{F})$ with $\mathfrak{e} = d\gamma + \kappa \wedge \gamma$. Take $\mathcal{G}$ defined by $\omega = \chi - \gamma$. It verifies $\omega(X) = 1$ and $d\omega = d\chi - d\gamma = \kappa \wedge \omega$ which gives the basic torsion $\tau = \kappa$.

Consider on the other hand the metric $\nu = \omega \otimes \omega + \mu_{\mathcal{G}}$. It is a bundle-like metric since $\mu_Q = \nu_Q$. The characteristic form is $\chi_\nu = \omega$ and therefore $d\chi_\nu = \tau \wedge \chi_\nu$. So $\kappa_\nu = \tau$ is a basic cycle and $e_\nu = 0$. The metric ν verifies (iv) and therefore $e_\nu = 0$. ♣

5.2 An example (cf. [6], [14])

The vanishing of the Euler class and the Álvarez class are independent. A trivial bundle and the Hopf fibration are immediate examples of isometric flows with zero and nonzero Euler class, respectively. Example 4.1 shows a non-isometric riemannian flow with zero Euler class. Now, we describe a riemannian flow which is not isometric and has nonzero Euler class.

Consider the matrix A of example 4.1 and the matrixes $B, I \in SL(4,\mathbb{Z})$ given by:

$$B = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} Id_2 & 0 \\ 0 & Id_2 \end{pmatrix},$$

which determine automorphisms of the torus $\mathbb{T}^4$. We define M^6 as the orbit space of the action $\Psi : \mathbb{Z}^2 \times (\mathbb{T}^4 \times \mathbb{R}^2) \longrightarrow (\mathbb{T}^4 \times \mathbb{R}^2)$, given by:

$$\Psi((k,l),[y_1,y_2,z_1,z_2],(t,x)) = (B^k \circ I^l([y_1,y_2,z_1,z_2]),(x+k,t+l)).$$

We can think of it as a fibration $\pi : M^6 \longrightarrow \mathbb{T}^2$ with fiber $\mathbb{T}^4$. A parallelization of M^6 is given by: aligned=¿array

$$\begin{aligned} X &= \frac{\partial}{\partial x}, \qquad T = \frac{\partial}{\partial t}, \qquad Y_i = \lambda_i^t (a_i \frac{\partial}{\partial y_1} + b_i \frac{\partial}{\partial y_2}), \\ Z_i &= x\lambda_i^t (a_i \frac{\partial}{\partial y_1} + b_i \frac{\partial}{\partial y_2}) + \lambda_i^t (a_i \frac{\partial}{\partial z_1} + b_i \frac{\partial}{\partial z_2}), \end{aligned} \qquad (i = 1,2)$$

where λ_1 and λ_2 are the eigenvalues of A, and $\{(a_1,b_1),(a_2,b_2)\}$ an orthonormal basis of eigenvectors. Denote by $\{\alpha,\beta,\gamma_1,\gamma_2,\delta_1,\delta_2\}$ the dual basis of 1-forms. Choose a metric μ on M^6 by saying that $\{X,T,Y_1,Y_2,Z_1,Z_2\}$ is an orthonormal parallelism. The flow $\mathcal{F}$ defined by Y_1 is riemannian respect to μ, for $L_{Y_1}\mu_Q = 0$. We also obtain from the formulae above:

$$\chi = \gamma_1, \quad \kappa = (\log \lambda_1)\alpha, \quad e = -\beta \wedge \delta_1.$$

Notice that both κ and e are basic. As $[\alpha] = \pi^*([\eta])$, where $[\eta]$ is one of the generators of $H^1(M^6)$, we have that the Álvarez class of $\mathcal{F}$ is not zero. To see that the Euler class of $\mathcal{F}$ does not vanish, we first notice that the $-\kappa$–twisted cohomology of $\mathcal{F}$ can be computed using π-basic functions as coefficients. A direct computation shows that no basic 1-form ω can satisfy $d_{-\kappa}\omega = e$.

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