Contemporary Mathematics

The Gysin Sequence for Riemannian Flows

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Abstract. Let $\mathcal{F}$ be a riemannian flow on a closed manifold M. The Gysin sequence relating the de Rham cohomology of M and the basic cohomology of $\mathcal{F}$ has been constructed for isometric flows in [**7**]. In this paper, we extend this result to riemannian flows. We also give a geometric characterization of the vanishing of the Euler class, similar to the one given in [**11**], and describe an example of a non-isometric riemannian flow with nonzero Euler class.

1. Introduction

Let M be a $(n+1)$-dimensional closed smooth manifold. A smooth action $\Psi : \mathbb{R} \times M \longrightarrow M$ without fixed points defines a 1-dimensional foliation $\mathcal{F}$. We are interested in the relation between the de Rham cohomology of M and the cohomology of the orbit space $M/\mathcal{F}$. Since this space may be somewhat pathological from the cohomological point of view, we shall use the *basic cohomology*, which has proven to be a rich and adapted invariant for the study of $M/\mathcal{F}$. When the action is periodic, we have indeed a circle action, and the mentioned relation is given by the classical *Gysin sequence*. In [**7**], the Gysin sequence has also been constructed for isometric flows, i.e., actions $\mathbb{R} \times (M, \mu) \longrightarrow (M, \mu)$ preserving a riemannian metric μ on M. We denote by $\Omega^*(M/\mathcal{F})$ the complex of *basic forms* and by $H^*(M/\mathcal{F})$ its cohomology. The Gysin sequence obtained is:

(1.1)

$$\cdots \to H^i(M/\mathcal{F}) \longrightarrow H^i(M) \longrightarrow H^{i-1}(M/\mathcal{F}) \xrightarrow{\wedge[d\chi]} H^{i+1}(M/\mathcal{F}) \to \dots$$

where the connecting morphism is multiplication by the *Euler class* $[d\chi] \in H^2(M/\mathcal{F})$, being χ the characteristic form of the flow. The main tool used to get this is the quasi-isomorphism (i.e., isomorphism in cohomology):

$$(\Omega^*(M), d) \simeq (\Omega^*(M/\mathcal{F}) \oplus \Omega^{*-1}(M/\mathcal{F}), D) \tag{1.2}$$

where the differential D is given by $D(\alpha, \beta) = (d\alpha + d\chi \wedge \beta, -d\beta)$. In the proof, it is crucial that the closure of the 1-parameter group $\{\Psi_t\}_t$ in the group of diffeomorphisms of M is compact.

2000 *Mathematics Subject Classification.* Primary 58A12, 37C85; Secondary 57R30.

Key words and phrases. Basic cohomology, Riemannian flows.

The author wishes to thank Prof. Saralegi for his helpful discussions and suggestions. This research has been supported by Gobierno Vasco-Eusko Jaurlaritza and UPV 127.310 EA005/99.

In this paper we extend the scope of the Gysin sequence to the case of riemannian flows. A flow $\mathcal{F}$ on M is *riemannian* if there exists a riemannian metric μ which is *bundle-like*, i.e., if the orthogonal component of μ is invariant (note that in the isometric case, the entire metric must be invariant). There are two main differences with the isometric case. In one hand, $d\chi$ is basic for an isometric flow, but in the riemannian case, we have a decomposition $d\chi = \mathfrak{e} + \kappa \wedge \chi$, where $\mathfrak{e}$ and κ are, respectively, the *Euler form* and the *mean curvature form* of the flow. We can suppose these two forms to be basic for a suitable bundle-like metric μ (see [**5**]). On the other hand, the closure of $\{\Psi_t\}_t$ is no longer necessarily compact. So, we need a different approach, using strongly the local structure of the flow, to get a decomposition like (1.2). The quasi-isomorphism we get is:

$$(\Omega^*(M), d) \simeq (\Omega^*(M/\mathcal{F}) \oplus \Omega^{*-1}(M/\mathcal{F}), D), \tag{1.3}$$

with a new differential $D(\alpha, \beta) = (d\alpha + \mathfrak{e} \wedge \beta, -d\beta + \kappa \wedge \beta)$, where the new term $\kappa \wedge \beta$ appears, showing the difference with the isometric case. The Gysin sequence derived canonically is:

$$\cdots \to H^i(M/\mathcal{F}) \longrightarrow H^i(M) \longrightarrow H^{i-1}_{\kappa}(M/\mathcal{F}) \xrightarrow{\wedge[\mathfrak{e}]} H^{i+1}(M/\mathcal{F}) \to \dots$$

where $H^*_{\kappa}(M/\mathcal{F})$ denotes the *twisted cohomology*, and the connecting morphism is, up to sign, multiplication by the *Euler class* $[\mathfrak{e}] \in H^2_{-\kappa}(M/\mathcal{F})$. We also give a geometric characterization of the vanishing of the Euler class similar to the one given in [**11**]. The vanishing of the Euler class is equivalent to the presence of a foliation transverse to the flow, with an additional condition, stated in Theorem 4.2. These results have been announced in [**9**]. Here, we also describe an example of a non-isometric riemannian flow with nonzero Euler class.

2. Riemannian flows

In this section, we review some well known facts about riemannian flows. Recall that a flow is a 1-dimensional oriented foliation. A flow $\mathcal{F}$ is *riemannian* if there exists a holonomy invariant riemannian metric μ. Such a metric is said to be *bundle-like*. We can choose a smooth nonsingular vector field X defining $\mathcal{F}$ such that $\mu(X, X) = 1$. We call the 1-form $\chi = i_X \mu$ the *characteristic form* of the flow, where i_X denotes the contraction by X. The *mean curvature form* of $\mathcal{F}$ is defined by $\kappa = L_X \chi$, where L_X is the Lie derivative respect to X. These two forms depend on the flow $\mathcal{F}$ and the metric μ. A form $\omega \in \Omega^*(M)$ is *basic* if for every vector field V tangent to $\mathcal{F}$, we have $i_V \omega = i_V d\omega = 0$. We denote the complex of basic forms as $\Omega^*(M/\mathcal{F})$, and its cohomology by $H^*(M/\mathcal{F})$.

It has been shown in [**5**] that we can choose a bundle-like metric μ such that κ is basic. In this conditions, κ is closed, and defines a class $[\kappa] \in H^1(M/\mathcal{F})$, which does not depend on the metric μ (see [**1**]), and that vanishes if and only if the flow is isometric. We shall call this invariant of $\mathcal{F}$ the *Álvarez class*. We have the decomposition:

$$d\chi = \mathfrak{e} + \chi \wedge \kappa. \tag{2.1}$$

The 2-form $\mathfrak{e}$ thus defined is called the *Euler form*, and satisfies $d\mathfrak{e} = -\kappa \wedge \mathfrak{e}$. We denote by $H^*_{\kappa}(M/\mathcal{F})$ the *twisted cohomology*, which is the dual of the basic cohomology (see [**8**]), i.e. $H^i(M/\mathcal{F}) \cong H^{n-i}_{\kappa}(M/\mathcal{F})$ for every $i \geq 0$. The twisted cohomology can be described as the cohomology of the basic complex endowed with the *κ-twisted differential* $d_\kappa\omega = d\omega - \kappa\wedge\omega$. We also define the *$(-\kappa)$-twisted differential* $d_{-\kappa}\omega = d\omega + \kappa\wedge\omega$.

The local structure of a riemannian flow has been described by Carrière in [**4**]. Let L be a leaf of the flow. Then, the closure of L is diffeomorphic to a torus $\mathbb{T}^{k+1}$, and there exists a neighbourhood U of $\overline{L}$ such that $\mathcal{F}$ restricted to U is conjugated to the flow obtained by suspension of a diffeomorphism (T,R) of $\mathbb{T}^k\times\mathbb{D}^{n-k}$, where T is an irrational traslation of $\mathbb{T}^k$ and R a rotation of $\mathbb{R}^{n-k}$. We shall refer to such a neighbourhood $U\cong\mathbb{T}^{k+1}\times\mathbb{D}^{n-k}$ as a *Carrière neighbourhood*. If we consider the metric induced in U by the flat metric of $\mathbb{T}^{k+1}\times\mathbb{D}^{n-k}$, both the mean curvature form and the Euler form vanish. Hence, the flow restricted to U is isometric.

3. Gysin sequence

The *short Gysin sequence* is the following sequence of differential complexes:

$$0\longrightarrow\Omega^*(M/\mathcal{F})\xrightarrow{\ \iota\ }\Omega^*(M)\xrightarrow{\ \rho\ }\Omega^*(M)/\Omega^*(M/\mathcal{F})\longrightarrow 0 \tag{3.1}$$

where ρ is the proyection induced by the inclusion ι. The cohomology of the complex $\Omega^*(M)/\Omega^*(M/\mathcal{F})$ is called the *$\mathcal{F}$-relative de Rham cohomology*, and some of its properties have been studied in [**10**]. Our aim is to describe it in terms of basic forms. To achieve this, we use the following:

Proposition 3.1. *In the above conditions, we have the quasi-isomorphism:*

$$\begin{array}{rcl}\Phi:\ (\Omega^*(M/\mathcal{F})\oplus\Omega^{*-1}(M/\mathcal{F}),D) & \longrightarrow & (\Omega^*(M),d)\\ (\alpha,\beta) & \longmapsto & \alpha+\chi\wedge\beta\end{array} \tag{3.2}$$

where the differential D is defined by $D(\alpha,\beta)=(d\alpha+\mathfrak{e}\wedge\beta,-d\beta+\kappa\wedge\beta)$.

Proof. Denote by $\overline{\mathcal{F}}$ the foliation induced by the closures of the leaves of $\mathcal{F}$. We can consider a basis of the stratified manifold $M/\overline{\mathcal{F}}$ consisting of projections of Carrière neighbourhoods. Applying a Mayer-Vietoris argument (see [**3**], p.289) to $M/\overline{\mathcal{F}}$, it suffices to prove the result for a Carrière neighbourhood U, and this holds due to the isometric case. □

From 3.2 and 3.1, we obtain that the $\mathcal{F}$-relative cohomology and the twisted cohomology are quasi-isomorphic. This leads us to the *long Gysin sequence*:

Theorem 3.2 (Gysin sequence). *Let $\mathcal{F}$ be a riemannian foliation on the closed manifold M, and choose a metric μ with basic mean curvature form. Then, we have the following long exact sequence:*

$$\cdots\to H^i(M/\mathcal{F})\longrightarrow H^i(M)\longrightarrow H^{i-1}_\kappa(M/\mathcal{F})\xrightarrow{\wedge[\mathfrak{e}]}H^{i+1}(M/\mathcal{F})\to\dots \tag{3.3}$$

where the connecting morphism is, up to sign, multiplication by the Euler class *$[\mathfrak{e}]\in H^2_{-\kappa}(M/\mathcal{F})$.*

In the case of an isometric flow, we can choose a metric such that the mean curvature form vanishes. So, (3.3) generalizes (1.1).

Remark 3.3. The long Gysin sequence does not depend on the metric μ.

Remark 3.4. From the quasi-isomorphism (3.2) and the duality result of [**8**], we obtain that the terms $E_2^{\cdot,1}$ and $E_2^{\cdot,0}$ in the spectral sequence of a riemannian flow are dual. This has been proven with more generality in [**2**] using analytic techniques, different from our geometric approach.

4. Vanishing of the Euler class

The Euler form $\mathfrak{e}$ vanishes if and only if the orthogonal complement of $\mathcal{F}$ is involutive. These equivalent conditions have been weakened in [**11**] for isometric flows. In that case, the Euler class $[d\chi] \in H^2(M/\mathcal{F})$ is an invariant of the foliation, and its vanishing is equivalent to the presence of a fibration transverse to $\mathcal{F}$. For a riemannian flow, we prove that the Euler class $[\mathfrak{e}] \in H^2_{-\kappa}(M/\mathcal{F})$ does not depend on the metric in the sense of the following:

Proposition 4.1. *Let μ_1 and μ_2 be two bundle-like metrics with basic mean curvature forms κ_1 and κ_2. Consider the canonical (up to a multiplicative positive constant) isomorphism:*

$$\varphi : \ H^2_{-\kappa_2}(M/\mathcal{F}) \longrightarrow H^2_{-\kappa_1}(M/\mathcal{F}) \tag{4.1}$$

given by $\varphi([\omega]) = [e^f \omega]$, where $df = \kappa_1 - \kappa_2$. Then, $\varphi([\mathfrak{e}_2])$ and $[\mathfrak{e}_1]$ are proportional. In particular, the vanishing of the Euler class does not depend on μ, but just on $\mathcal{F}$.

Proof. Proceeding as in Proposition 3.1, we get the *twisted Gysin sequence*:

$$\longrightarrow H^i_{-\kappa}(M/\mathcal{F}) \longrightarrow H^i_{-\kappa}(M) \longrightarrow H^{i-1}(M/\mathcal{F}) \xrightarrow{\wedge[\mathfrak{e}]} H^{i+1}_{-\kappa}(M/\mathcal{F}) \longrightarrow$$

for every basic κ. The isomorphism (4.1) induces a chain morphism between the twisted Gysin sequences for κ_1 and κ_2, yielding the result. □

Let $\mathcal{G}$ be a foliation transverse to a nonsingular vector field X defining $\mathcal{F}$. There is a unique 1-form ω determined by $\ker \omega = T\mathcal{G}$ and $i_X\omega = 1$. We mean by a *torsion* of $\mathcal{G}$ respect to X a 1-form τ such that $d\omega = \omega \wedge \tau$. Note that if a torsion is basic, for $\mathcal{F}$, then it must be $L_X\omega$.

Theorem 4.2. *For a riemannian flow, the following conditions are equivalent:*

(1) *the Euler class vanishes;*
(2) *there exists a foliation $\mathcal{G}$ transverse to $\mathcal{F}$, with basic torsion.*

Proof. Assume (1) and take $\gamma \in \Omega^1(M/\mathcal{F})$ such that $d_{-\kappa}\gamma = \mathfrak{e}$. Then, $\omega = \chi - \gamma$ is integrable, defining a foliation $\mathcal{G}$ that satisfies (2). For the converse, pick a bundle-like metric μ and construct $\nu = \omega \otimes \omega + \mu_{\mathcal{G}}$. Then, ν is bundle-like, κ_ν is basic, and $\mathfrak{e}_\nu = 0$. □

Remark 4.3. The vanishing of the Euler class of $\mathcal{F}$ implies the vanishing of the Godbillon-Vey class of the transverse foliation $\mathcal{G}$ described in Theorem 4.2.

5. An example

The vanishing of the Euler class and the Álvarez class are independent. A trivial bundle and the Hopf fibration are immediate examples of isometric flows with zero and nonzero Euler class, respectively. In [**4**], Carrière exhibits a non-isometric riemannian flow whose Euler class is zero. In this section, we describe a riemannian flow where neither the Álvarez class or the Euler class vanish.

Consider the matrix $B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in SL(2,\mathbb{Z})$, and its two real eigenvalues λ_1 and λ_2, which satisfy $0 < \lambda_1 < 1 < \lambda_2$ and $\lambda_1\lambda_2 = 1$. Denote by $\{(a_1, b_1), (a_2, b_2)\}$ an orthonormal basis of eigenvectors. The matrixes of $SL(4,\mathbb{Z})$ given by:

$$A = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} Id_2 & Id_2 \\ 0 & Id_2 \end{pmatrix},$$

define automorphisms of the torus $\mathbb{T}^4$. We define M^6 as the suspension of the representation $\rho : \mathbb{Z}^2 \longrightarrow \mathrm{Diff}(\mathbb{T}^4)$ given by $\rho(k,l) = A^k \circ I^l$. More precisely, M^6 is the orbit space of the action $\Upsilon : \mathbb{Z}^2 \times (\mathbb{T}^4 \times \mathbb{R}^2) \longrightarrow (\mathbb{T}^4 \times \mathbb{R}^2)$, given by:

$$\Upsilon((k,l),[y_1,y_2,z_1,z_2],(t,x)) = (\rho(k,l)([y_1,y_2,z_1,z_2]),(x+k,t+l)).$$

Notice that there is a fibration $\pi : M^6 \longrightarrow \mathbb{T}^2$ with fiber $\mathbb{T}^4$. This manifold was introduced in another context in [**6**], where some of its properties are studied. A parallelization of M^6 is given by:

$$\begin{array}{ll} X = \dfrac{\partial}{\partial t}, \quad T = \dfrac{\partial}{\partial x}, \quad Y_i = \lambda_i^t \Big(a_i \dfrac{\partial}{\partial y_1} + b_i \dfrac{\partial}{\partial y_2}\Big) & \\ Z_i = x\lambda_i^t\Big(a_i \dfrac{\partial}{\partial y_1} + b_i \dfrac{\partial}{\partial y_2}\Big) + \lambda_i^t\Big(a_i \dfrac{\partial}{\partial z_1} + b_i \dfrac{\partial}{\partial z_2}\Big) & i = 1,2. \end{array} \tag{5.1}$$

denote by $\{\alpha,\beta,\gamma_1,\gamma_2,\delta_1,\delta_2\}$ the dual basis of 1-forms. Let $\mathcal{F}$ be the flow defined by Y_1, and μ the metric such that (5.1) is an orthonormal parallelism. One can check that $L_{Y_1}\mu_Q = 0$, and so, μ is bundle-like. From (5.1), we obtain for $\mathcal{F}$:

$$\chi = \gamma_1, \quad \kappa = (\log \lambda_1)\alpha, \quad \mathfrak{e} = -\beta \wedge \delta_1. \tag{5.2}$$

Notice that both κ and $\mathfrak{e}$ are basic.

Proposition 5.1. *The Álvarez class of $\mathcal{F}$ is not zero.*

Proof. Since $\iota^* : H^1(M^6/\mathcal{F}) \longrightarrow H^1(M^6)$ is injective, it suffices to prove that $[\alpha] \in H^1(M^6)$ is not zero. This holds because $[\alpha] = \pi^*([\zeta])$, where ζ is one of the generators of $H^1(\mathbb{T}^2)$. □

Proposition 5.2. *The Euler class of $\mathcal{F}$ is not zero.*

Proof. The cohomology of $(\Omega^*(M^6/\mathcal{F}), d_{-\kappa})$ can be computed using π-basic functions as coefficients. Suppose that there exists a form $\omega \in \Omega^1(M^6/\mathcal{F})$ such that $d_{-\kappa}\omega = \mathfrak{e}$. Using (5.2) we get the equations $X(f) = 1$ and $T(f) = 0$ on $\mathbb{T}^2$, being $f = i_{Z_1}\omega$. This is a contradiction. □

References

[1] J.A. Álvarez, *The basic component of the mean curvature form of riemannian foliations*, Annals of Global Analysis and Geometry **10** (1992), 179–194.

[2] ———, *Duality in the spectral sequence of riemannian foliations*, Amer. J. Math. **111** (1989), 905–925.

[3] G.E. Bredon, *Topology and Geometry*, Springer Verlag (1993).

[4] Y. Carrière, *Flots riemanniens*, Astérisque **116** (1984), 31–52

[5] D. Domínguez, *Finiteness and tautness for riemannian foliations*, Amer. J. Math. **120** (1998), 1237–1276.

[6] M. Fernández, M. de León, M. Saralegi, *A six dimensional compact symplectic solvmanifold without Kähler structures*, Osaka J. Math. **33** (1996), 19–35.

[7] F. Kamber, P. Tondeur, *Duality theorems for foliations*, Astérisque **116** (1984), 458–471.

[8] ———, *Foliations and Metrics*, Birkhäusser Progr. Math. **32** (1983), 103–152.

[9] J.I. Royo Prieto, *The Euler class for riemannian flows*, to appear in C. Rend. Acad. Sci.

[10] K. Rummler, *Quelques notions simples en géometrie riemannienne et leurs applications aux feuilletages compacts*, Comment. Math. Helv. **54** (1979), 224–239.

[11] M. Saralegi, *The Euler class for isometric flows*, Pitman Research Notes in Math. **131** (1985), 220-227.

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