Continuous Extension of Pairwise Disjoint Families of Real-valued Functions and Hedgehog-valued Functions

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Outline

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   - The Problem That We Studied

2. Our Results/Contribution
   - The hedgehog and hedgehog-valued functions. Restatement of the problem.
   - Main Results

Frantz's original problem

Extension of pairwise disjoint families of real-valued functions.

Problem

Let A be a closed subset of a normal space X and let $\{f_i : A \to \mathbb{R}\}_{i\in I}$ be a family of real-valued continuous and pairwise disjoint functions (i.e. $f_i \cdot f_j = 0$ for each $i \neq j$).

Do there exist pairwise disjoint continuous extensions $\left\{\widehat{f_i} : A \to \mathbb{R}\right\}_{i\in I}$ of the respective f_i over all of X?

M. Frantz.
Controlling Tietze-Urysohn extensions.
Pacific J. Math., 169 (1995), 53-73.

Frantz's first answer.

In the paper mentioned above Frantz gives the following partial answer:

- The answer is affirmative for a finite family.
- The answer is again affirmative in the case of a countable collection.
- For the case of an arbitrary infinite collection he doesn't know the answer.
- Arbitrary collections can be extended in the case when X is a metric space.

Barov and Dijkstra's contribution.

Later on S. Barov and J. Dijstra continued working on the same problem and obtained the following results:

- Every countable collection of pairwise disjoint continuous functions on a closed subset of a normal space has a pairwise disjoint extension.
- The pairwise disjoint extension property need not hold for uncountable collection of functions. [By using the Bing's example].

S. Barov and J. Dijkstra.
On boundary avoiding selections and some extension theorems.
Pacific J. Math., 203 (2002), no. 1, 79–87.

When can pairwise disjoint families of continuous real-valued functions be continuously extended?

In view of the previous results it is natural to ask the following:

Problem

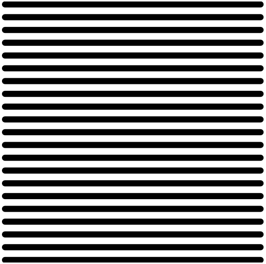
Which spaces do have the pairwise disjoint extension property?

Of course, a further problem would be to obtain a Tietze type theorem characterizing that class of spaces.

In what follows we will present this class of spaces and prove the announced Tietze type theorem.

The hedgehog

Let κ be some cardinal an I be a set with $|I| = \kappa$. Let $\sim$ be the equivalence relation on $X = [0,1] \times I$ obtained by identifying all $(0, i)$ with $i \in I$.



The hedgehog

Let κ be some cardinal an I be a set with $|I| = \kappa$. Let $\sim$ be the equivalence relation on $X = [0,1] \times I$ obtained by identifying all $(0,i)$ with $i \in I$.

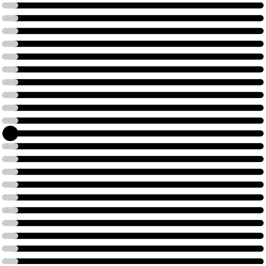
The hedgehog with κ spines $J(\kappa)$ is the set of equivalence classes $X/\sim$.

We will make also use of the projections

$$\{\Pi_i : J(\kappa) \to [0,1]\}_{i \in I}$$

defined by

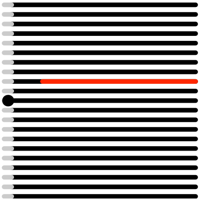
$$\Pi_i((t,j)) = \begin{cases} t, & \text{if } j = i; \\ 0, & \text{if } j \neq i. \end{cases}$$

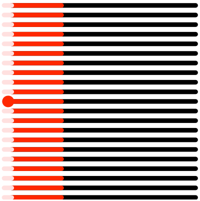


The metric hedgehog

The metric hedgehog with κ spines consists on the space $J(\kappa)$ endowed with the metric d defined, for each $(t,i),(s,j) \in J(\kappa)$, by

$$d\big((t,i),(s,j)\big) = \begin{cases} |t-s|, & \text{if } j = i; \\ t+s, & \text{if } j \neq i. \end{cases}$$





The compact hedgehog

A partial order on $J(\kappa)$ can be naturally defined as follows:

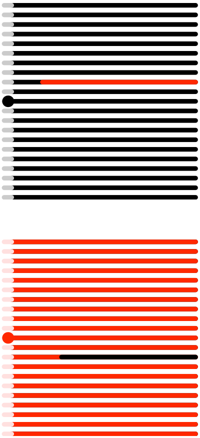
$$(t,i) \leq (s,j) \quad \text{if } (t,i) = \overline{0} \text{ or } i = j \text{ and } t \leq s.$$

$J(\kappa)$ with the order defined above is a complete-continuous semilattice.

The compact hedgehog

We can consider the Lawson topology $\lambda(J(\kappa))$. The space $(J(\kappa), \lambda(J(\kappa)))$ is denoted $\Lambda J(\kappa)$.

We will refer to it as the compact hedgehog to distinguish it from the metric.



The compact hedgehog

It is not difficult to see that the hedgehog is order-isomorphic to the axes of the cube $[0,1]^I$ (with $|I| = \kappa$):

$$J(\kappa) \simeq \bigcup_{i \in I} \left\{ \varphi \in [0,1]^I : \varphi(j) = 0 \quad \forall j \neq i \right\}$$

The compact hedgehog is homeomorphic to the axes of the cube viewed as a (closed) subspace of the Tychonov cube $\left([0,1]^I, \tau_{Tych}\right)$.

Hedgehog-valued functions

A hedgehog-valued function $f : X \to J(\kappa)$ is completely determined by the pairwise disjoint family of functions $\{\Pi_i \circ f : X \to [0,1]\}_{i \in I}$.

We can also prove the following result which will be crucial in our work:

Result (1)

Let $f : X \to J(\kappa)$. Then $f : X \to \Lambda J(\kappa)$ is continuous iff $\Pi_\kappa \circ f : X \to [0,1]$ is continuous and also $\Pi_i \circ f : X \to ([0,1], \tau_u)$ is continuous for each $i \in I$.

Restatement of the problem

After the previous result we can now restate the original problem as a problem of extension of hedgehog-valued functions:

Problem

Let A be a closed subset of a normal space X and let $f : A \to \Lambda J(\kappa)$ be a hedgehog-valued continuous function.

Does there exist a continuous extension $f : X \to \Lambda J(\kappa)$ over all of X?

Main Results

In what follows we will present our results in the same order we have obtained them.

Result (2)

If for each closed subset A any continuous f $: A \to \Lambda J(\kappa)$ *can continuously be extended, then the space X must necessarily be* κ*-collectionwise normal.*

Recall that the counterexample provided by Barov and Dijkstra was based in Bing's example, i.e. they considered a normal space which is not collectionwise normal.

Main Results

Result (3)

The converse implication is false.

Example (1)

The Tychonov cube $[0, 1]^{\mathbb{R}}$ is collectionwise normal.

However, if we think of the compact hedgehog $J(c)$ as a (closed) subspace of $[0, 1]^{\mathbb{R}}$, then the identity map on $J(c)$ cannot be extended to the whole $[0, 1]^{\mathbb{R}}$.

This statement would be equivalent to $J(c)$ being a retract of $[0, 1]^{\mathbb{R}}$.

But this is not possible since $J(c)$ is not separable.

Main Results

Result (4)

If X is hereditarily κ-collectionwise normal, then for each closed subset A, any continuous $f : A \to \Lambda J(\kappa)$ can be continuously extended.

Recall that this result improves Frantz's result for metric spaces.

This result has already been obtained by Yamazaki in:

K. Yamazaki.
Controlling extensions of functions and C-embedding.
Topology Proceedings, 26 (2001), 1-19.

Main Results

Finally we tried to find the characterization of the class of spaces for which the extension result could be ensured. We consider this as our real Main Result.

In order to simplify its formulation, we recall the following definition:

Definition

A subspace A is κ-T_z-embedded in X if every disjoint family (of cardinality at most κ) of cozero sets of A may be extended disjointly to a family of cozero-sets of X.

C.E. Aull.
Extendability and Expandability.
Boll. U. M. I., (6), 5-A, (1986) 129-135.

Main Results

The Tietze type result we obtain is the following:

Result (Main Result)

The following are equivalent:

(1) *X is normal and every closed $A \subset X$ is κ-T_z-embedded;*

(2) *for A closed in X, every continuous function $f : A \to \Lambda J(\kappa)$ can be extended to a continuous one over X;*

(3) *for A closed in X, every collection $\{f_i\}_{i \in I}$, with $|I| = \kappa$, of pairwise disjoint real-valued continuous functions on A, can be continuously extended over X*

Main Results

But there was still the question of whether this class of spaces was precisely the class of hereditarily κ-collectionwise normal spaces.

We are indebted to Prof. Juhász who, in a personal communication, provided us with the following:

Result (Juhász and Szentmiklossy)

Under ♣ principle, one can modify Ostaszewski's construction to get a locally compact T_2 S-space X that is not normal. Then the one-point compactification of X is T_4 but not T_5.

Main Results

Example

If we assume Juhász ♣ principle, there exists a hereditarily separable and normal space X which fails to be hereditarily κ-collectionwise normal.

Let A be a closed subspace in X and $f : A \to \Lambda J(\kappa)$ a continuous function.

Since A is separable and normality is equivalent to ω-collectionwise normality, it follows that f can be extended to a continuous function over X.

Consequently X is normal and every closed $A \subset X$ is κ-T_z-embedded