Universal codomains to represent interval orders

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Summary

We introduce the notion of universal codomain for a class of binary relations. We say that a nonempty set Y endowed with a binary relation $\mathcal{R}_Y$ is a universal codomain for $\mathcal{C}$, if the representability of an element of $\mathcal{C}$ in some particular classical sense is equivalent to the existence of a map $F : X \to Y$ such that $x\mathcal{R}y \iff F(x)\mathcal{R}_Y F(y)$ $(x, y \in X)$. Given an interval order defined on a set, we study its representability by means of a single monotonic function that takes values on a suitable universal codomain.

Representability of interval-orders

A binary relation $\mathcal{R}$ on X is said to be an *interval order* if it is reflexive and

$$(x\mathcal{R}z) \text{ and } (y\mathcal{R}w) \implies (x\mathcal{R}w) \text{ or } (y\mathcal{R}z) \quad \text{for every } x, y, z, w \in X.$$

Notice that $\mathcal{R}$ is, in particular, complete, because for any $x, y \in X$ it follows by reflexivity that $(x\mathcal{R}x)$ and $(y\mathcal{R}y)$ and, by hypothesis, this implies that $(x\mathcal{R}y)$ or $(y\mathcal{R}x)$.

An interval order $\mathcal{R}$ is said to be *representable* if there exists a pair of functions $u, v : X \to \mathbb{R}$ such that $x\mathcal{R}y \iff u(x) \le v(y)$ $(x, y \in X)$.

Interval orders are perhaps the best class of ordered structures to build models of uncertainty or to represent and manipulate vague or imperfectly described pieces of knowledge. The representability of an interval order $\mathcal{R}$ defined on X by means of two functions $u, v : X \to \mathbb{R}$ leads us in a natural way to use *interval methods* to deal with this kind of orderings, namely an element $x \in X$ would be assigned an *interval* $[u(x), v(x)]$ (that may eventually collapse into a point if $u(x) = v(x)$) of real numbers.

Given an interval order $\mathcal{R}$ we shall associate two new binary relations $\mathcal{R}^*$ and $\mathcal{R}^{**}$, that are defined by $x\mathcal{R}^*y \iff (y\mathcal{P}z \implies x\mathcal{P}z \ (z \in X))$ and similarly $x\mathcal{R}^{**}y \iff (z\mathcal{P}x \implies z\mathcal{P}y \ (z \in X))$. We denote $x\mathcal{P}^*y \iff (y\mathcal{R}^*x)$, $x\mathcal{P}^{**}y \iff (y\mathcal{R}^{**}x)$, $x\mathcal{I}^*y \iff x\mathcal{R}^*y\mathcal{R}^*x$, $x\mathcal{R}^{**}y \iff (y\mathcal{P}^{**}x)$ and $x\mathcal{I}^{**}y \iff x\mathcal{R}^{**}y\mathcal{R}^{**}x$ $(x, y \in X)$. Now we may observe that $x\mathcal{P}^*y \iff x\mathcal{P}z\mathcal{R}y$ for some $z \in X$ $(x, y \in X)$, and, similarly, $x\mathcal{P}^{**}y \iff x\mathcal{R}z\mathcal{P}y$ for some $z \in X$ $(x, y \in X)$. It is straightforward to see that $\mathcal{R}$ is transitive (hence it is a total preorder) iff $\mathcal{R}, \mathcal{R}^*$, and $\mathcal{R}^{**}$ coincide.

It is already known that the representability of an interval order $\mathcal{R}$, through a pair of functions $u, v : X \to \mathbb{R}$ s is *equivalent* to the representability of $\mathcal{R}$ through *another* pair of functions $u', v' : X \to \mathbb{R}$ such that, in addition, u' and v' represent the associated total preorders $\mathcal{R}^{**}$ and $\mathcal{R}^*$, respectively.

Universal codomains

Let $\mathcal{C}$ be a particular class of binary relations defined on X (for instance, $\mathcal{C}$ could be a class of total preorders, or interval orders, or semiorders). We say that a nonempty set Y endowed with a binary relation $\mathcal{R}_Y$ is a *universal codomain* for $\mathcal{C}$ if the representability of an element of $\mathcal{C}$ in some particular classical sense (e.g.: the representability of interval orders $\mathcal{R}$ through a pair of real-valued functions is *equivalent* to the existence of a map $F : X \to Y$ such that $x\mathcal{R}y \iff F(x)\mathcal{R}_Y F(y)$ $(x, y \in X)$.

Let $\mathcal{R}$ be an interval order on X representable by means of a pair (u, v) of real-valued functions. Now we may identify each interval $[u(x), v(x)]$ with the point $(u(x), v(x))$ of the plane $\mathbb{R}^2$. *This suggests that some suitable subset of the plane $\mathbb{R}^2$ could be a universal codomain to represent interval orders.*

Let us introduce now another *motivation*, based upon a *standard construction*, that also suggests the use of suitable subsets of $\mathbb{R}^2$ to build universal codomains to represent some classes of binary relations (in particular, *interval orders*).

Proposition 1 *Let $\mathcal{R}$ be a binary relation on X and $\mathcal{G}_\mathcal{R} = \{(x_1, x_2) \in X \times X : x_1\mathcal{R}x_2\}$ denote the graph of $\mathcal{R}$. Endow now the graph $\mathcal{G}_\mathcal{R}$ with a new binary relation $\mathcal{R}(\mathcal{G})$ given by*

$$(x_1, x_2)\mathcal{R}(\mathcal{G})(y_1, y_2) \iff x_1\mathcal{R}y_2.$$

Then the following properties hold true:

(i) *The new relation $\mathcal{R}(\mathcal{G})$ is reflexive.*
(ii) *If $\mathcal{R}$ is an interval order, then so is $\mathcal{R}(\mathcal{G})$.*
(iii) *If $\mathcal{R}$ is an interval order, then $\mathcal{P}(\mathcal{G})$ is given by*
$$(x_1, x_2)\mathcal{P}(\mathcal{G})(y_1, y_2) \iff \neg(y_1\mathcal{R}x_2).$$
(iv) *If $\mathcal{R}$ is an interval order, then $\mathcal{I}(\mathcal{G})$ is given by*
$$(x_1, x_2)\mathcal{I}(\mathcal{G})(y_1, y_2) \iff (x_1\mathcal{R}y_2) \text{ and } (y_1\mathcal{R}x_2).$$
(v) *If $\mathcal{R}$ is an interval order, then $\mathcal{R}^*(\mathcal{G})$, $\mathcal{P}^*(\mathcal{G})$, $\mathcal{I}^*(\mathcal{G})$, $\mathcal{R}^{**}(\mathcal{G})$, $\mathcal{P}^{**}(\mathcal{G})$, $\mathcal{I}^{**}(\mathcal{G})$ satisfy that*

$$\begin{aligned} (x_1, x_2)\mathcal{R}^*(\mathcal{G})(y_1, y_2) &\iff x_2\mathcal{R}^*y_2, & (x_1, x_2)\mathcal{P}^*(\mathcal{G})(y_1, y_2) &\iff x_2\mathcal{P}^*y_2,\\ (x_1, x_2)\mathcal{I}^*(\mathcal{G})(y_1, y_2) &\iff x_2\mathcal{I}^*y_2, & (x_1, x_2)\mathcal{R}^{**}(\mathcal{G})(y_1, y_2) &\iff x_1\mathcal{R}^{**}y_1,\\ (x_1, x_2)\mathcal{P}^{**}(\mathcal{G})(y_1, y_2) &\iff x_1\mathcal{P}^{**}y_1, & (x_1, x_2)\mathcal{I}^{**}(\mathcal{G})(y_1, y_2) &\iff x_1\mathcal{I}^{**}y_1. \end{aligned}$$

Motivated by the construction above, we analyse now the following set to represent interval orders: Let $\mathcal{Y} = \mathcal{G}_\le = \{(a, b) \in \mathbb{R}^2 : a \le b\}$. Endow $\mathcal{Y}$ with the binary relation $\mathcal{R}_\mathcal{Y}$ given by

$$(a, b)\mathcal{R}_\mathcal{Y}(c, d) \iff a \le d \ ((a, b), (c, d) \in \mathcal{Y}).$$

Proposition 2

(i) *The binary relation $\mathcal{R}_\mathcal{Y}$ defined on $\mathcal{Y}$ is an interval order.*
(ii) *The interval order $\mathcal{R}_\mathcal{Y}$ is representable by means of a pair of real-valued functions.*
(iii) *The structure $(\mathcal{Y}, \mathcal{R}_\mathcal{Y})$ is a universal codomain to represent interval orders.*
(iv) *The structure $(\mathcal{Y}, \mathcal{R}_\mathcal{Y})$ is equivalent to the structure $(\mathcal{ST}, \mathcal{R}_I)$ of interval ordered symmetric triangular fuzzy numbers. (Hence, $(\mathcal{ST}, \mathcal{R}_I)$ is also a universal codomain to represent interval orders).*

Continuous representability and universal codomains

Let $\mathcal{R}$ an interval order on a topological space (X, τ). We say that $\mathcal{R}$ is *τ-continuously representable* if there exist *continuous* $u, v : (X, \tau) \longrightarrow (\mathbb{R}, \tau_u)$ such that $x\mathcal{R}y \iff u(x) \le v(y)$ $(x, y \in X)$.

It is quite natural to look for universal codomains that have an structure similar to that of the binary relations $\mathcal{R}$ belonging to the class $\mathcal{C}$ that we are trying to represent in those codomains. In the particular case of $(\mathcal{Y}, \mathcal{R}_\mathcal{Y})$, if we deal with representations of interval orders by means of a pair of continuous real-valued functions, we should endow $\mathcal{Y}$ with a topology $\tau_\mathcal{Y}$ such that the projections $p_1, p_2 : \mathcal{Y} \to \mathbb{R}$ become *continuous*. Thus, for every $a \in \mathbb{R}$ the subsets $p_i^{-1}(a, +\infty), p_i^{-1}(-\infty, a) \subseteq \mathcal{Y}$ $(i = 1, 2)$ must be $\tau_\mathcal{Y}$-open. This suggests to endow $\mathcal{Y}$ with the topology $\tau_\mathcal{Y}$ defined from the *subbasis*:
$\mathcal{S} = \{[(-\infty, a) \times \mathbb{R}] \cap \mathcal{Y}\} \bigcup \{[\mathbb{R} \times (-\infty, b)] \cap \mathcal{Y}\} \bigcup \{[(c, +\infty) \times \mathbb{R}] \cap \mathcal{Y}\} \bigcup \{[\mathbb{R} \times (d, +\infty)] \cap \mathcal{Y}\}$.
It is straightforward to see the topology $\tau_\mathcal{Y}$ defined by $\mathcal{S}$ is actually the usual Euclidean topology on $\mathcal{Y}$. Moreover, a glance to Proposition 1(v) and Proposition 2(ii) shows that, with respect to the Euclidean topology on $\mathcal{Y}$, the projections p_1, p_2 are continuous representations for, respectively, $\mathcal{R}^{**}_\mathcal{Y}$ and $\mathcal{R}^*_\mathcal{Y}$ relative to the interval order $\mathcal{R}_\mathcal{Y}$. This fact has the following consequence.

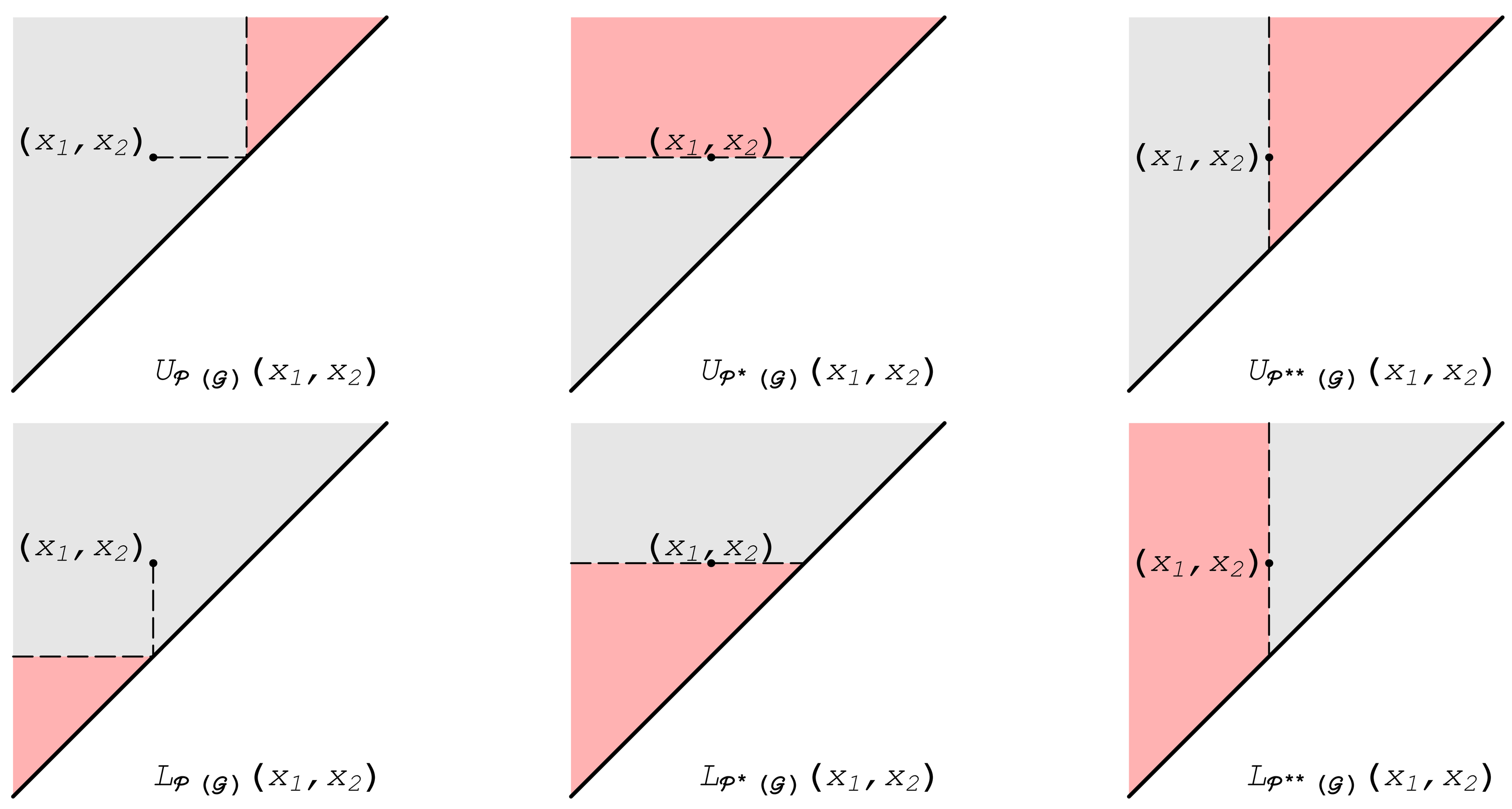
Proposition 3 *The structure $(\mathcal{Y}, \mathcal{R}_\mathcal{Y}, \tau_u)$ is a universal codomain to represent interval orders defined on a topological space (X, τ), by means of a pair of* continuous *functions $u, v : (X, \tau) \to (\mathbb{R}, \tau_u)$ such that $x\mathcal{R}y \iff u(x) \le v(y)$ $(x, y \in X)$ and, in addition, u represents $\mathcal{R}^{**}$ and v represents $\mathcal{R}^*$.*

Finally, we analyze now the relationship between the continuous representability of an interval order $\mathcal{R}$ on a topological space (X, τ), and the continuous representability of the interval order $\mathcal{R}(\mathcal{G})$ (see Proposition 1(ii)) defined on the graph $\mathcal{G}_\mathcal{R}$, on which we consider the topology $\tau_\mathcal{G}$ given by restriction of the topology $\tau \times \tau$ on $X \times X$. For every $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in \mathcal{G}_\mathcal{R}$, denote:
$U_{\mathcal{P}(\mathcal{G})}(x) = \{y : x\mathcal{P}(\mathcal{G})y\}$; $U_{\mathcal{P}^*(\mathcal{G})}(x) = \{y : x\mathcal{P}^*(\mathcal{G})y\}$; $U_{\mathcal{P}^{**}(\mathcal{G})}(x) = \{y : x\mathcal{P}^{**}(\mathcal{G})y\}$;
$L_{\mathcal{P}(\mathcal{G})}(x) = \{z : z\mathcal{P}(\mathcal{G})x\}$; $L_{\mathcal{P}^*(\mathcal{G})}(x) = \{z : z\mathcal{P}^*(\mathcal{G})x\}$; and $L_{\mathcal{P}^{**}(\mathcal{G})}(x) = \{z : z\mathcal{P}^{**}(\mathcal{G})x\}$.

Lemma 4 *Let $\mathcal{R}$ be an interval order on a set X. Let $x = (x_1, x_2) \in \mathcal{G}_\mathcal{R}$. Then:*

(i) $U_{\mathcal{P}(\mathcal{G})}(x) = (U_\mathcal{P}(x_2) \times X) \cap \mathcal{G}_\mathcal{R}$ (ii) $L_{\mathcal{P}(\mathcal{G})}(x) = (X \times L_\mathcal{P}(x_1)) \cap \mathcal{G}_\mathcal{R}$
(iii) $U_{\mathcal{P}^*(\mathcal{G})}(x) = (X \times U_{\mathcal{P}^*}(x_2)) \cap \mathcal{G}_\mathcal{R}$ (iv) $L_{\mathcal{P}^*(\mathcal{G})}(x) = (X \times L_{\mathcal{P}^*}(x_2)) \cap \mathcal{G}_\mathcal{R}$
(v) $U_{\mathcal{P}^{**}(\mathcal{G})}(x) = (U_{\mathcal{P}^{**}}(x_1) \times X) \cap \mathcal{G}_\mathcal{R}$ (vi) $L_{\mathcal{P}^{**}(\mathcal{G})}(x) = (L_{\mathcal{P}^{**}}(x_1) \times X) \cap \mathcal{G}_\mathcal{R}$.

In particular, in the case of the usual total order $\le$ defined on the real line $\mathbb{R}$, we have the following contour sets in $\mathcal{G}_\mathcal{R}$:



As an immediate consequence of Lemma 4 and the continuity of the projection maps, we obtain the following result. (Recall that a topology on X is said to be *natural* for $\mathcal{R}$ if all $U_\mathcal{P}(x) = \{y : x\mathcal{P}y\}$; $L_\mathcal{P}(x) = \{z : z\mathcal{P}x\}$; $U_{\mathcal{P}^*}(x) = \{y : x\mathcal{P}^*y\}$; $L_{\mathcal{P}^*}(x) = \{z : z\mathcal{P}^*x\}$; $U_{\mathcal{P}^{**}}(x) = \{y : x\mathcal{P}^{**}y\}$ and $L_{\mathcal{P}^{**}}(x) = \{z : z\mathcal{P}^{**}x\}$ are τ-open).

Proposition 5 *Let $\mathcal{R}$ be an interval order on (X, τ). Then τ is a natural topology with respect to $\mathcal{R}$ iff the topology $\tau_\mathcal{G}$ is a natural topology on the graph $\mathcal{G}_\mathcal{R}$ with respect to $\mathcal{R}(\mathcal{G})$.*

With respect to representability, the following result is in order now:

Proposition 6 *Let $\mathcal{R}$ be an interval order on X. Then $\mathcal{R}$ is representable through a pair of functions $u, v : X \to \mathbb{R}$ iff $\mathcal{R}(\mathcal{G})$ is representable through a pair of functions $U, V : \mathcal{G}_\mathcal{R} \longrightarrow \mathbb{R}$ such that $(x_1, x_2)\mathcal{R}(\mathcal{G})(y_1, y_2) \iff U(x_1, x_2) \le V(y_1, y_2)$ $((x_1, x_2), (y_1, y_2) \in \mathcal{G}_\mathcal{R})$.*

Suppose in addition that $\mathcal{R}$ on X is representable through a pair (u, v) such that u represents $\mathcal{R}^{**}$ and v represents $\mathcal{R}^*$. By Proposition 1(vi) we may observe that given $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{G}_\mathcal{R}$, it follows that $x\mathcal{R}^{**}(\mathcal{G})y \iff x_1\mathcal{R}^{**}y_1 \iff u(x_1) \le u(y_1) \iff u(p_1(x)) \le u(p_1(y))$. Thus $u \circ p_1$ represents $\mathcal{R}^{**}(\mathcal{G})$. Similarly we may prove that $v \circ p_2$ represents $\mathcal{R}^*(\mathcal{G})$. Conversely, if $\mathcal{R}(\mathcal{G})$ is representable through (U, V), then $U \circ e$ represents $\mathcal{R}^{**}$ and $V \circ e$ represents $\mathcal{R}^*$, where e denotes the evaluation map. For *continuous* representability the situation goes as follows, as a direct consequence of Proposition 5 and the continuity of the projections and the evaluation map:

Corollary 7 *Let $\mathcal{R}$ be an interval order on (X, τ). Then $\mathcal{R}$ is representable through a pair of continuous functions $u, v : (X, \tau) \longrightarrow (\mathbb{R}, \tau_u)$ iff $\mathcal{R}(\mathcal{G})$ is representable through a pair of continuous functions $U, V : (\mathcal{G}_\mathcal{R}, \tau_\mathcal{G}) \longrightarrow (\mathbb{R}, \tau_u)$.*

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