ELSEVIER

Fuzzy Sets and Systems 106 (1999) 275–284

FUZZY sets and systems

# Generalised filters 1

M.H. Burton[a,*], M. Muraleetharan[a], J. Gutiérrez García[b]

[a] *Department of Mathematics, Rhodes University, Grahamstown, 6140, South Africa*
[b] *Matematika Saila, Euskal Herriko Unibertsitatea, 48080, Bilbo, Spain*

Received March 1997; received in revised form August 1997

**Abstract**

The notion of a filter is generalised and the resulting mathematical object is called a generalised filter. These filters fit naturally into the context of generalised uniform spaces. The basic theory of generalised filters is established and the relationship between generalised filters and prefilters is discovered. 

*AMS classification:* 03E72; 04A20; 04A72; 18A22; 54A40; 54E15

*Keywords:* Filter; Ultrafilter; Prefilter; Prime prefilter; Saturated prefilter; Fuzzy filter; Uniformity; Fuzzy uniformity; Super uniformity

## 1. Introduction

Uniform spaces were introduced to form an appropriate abstract setting for the study of completeness. These spaces, which fall between metric spaces and topological spaces, have been extensively studied and the reader is referred to [25] for the basic theory of uniform spaces. The study of uniform space notions is facilitated by the notion of a *filter* and the basic theory of filters can also be found in [25]. For the record, let us just note that a uniformity on a set $X$ is subset $\mathbb{D} \subseteq 2^{X \times X}$ and a filter on $X$ is a subset $\mathbb{F} \subseteq 2^X$. In other words

$$\mathbb{D} \in 2^{2^{X \times X}}, \quad \mathbb{F} \in 2^{2^X}.$$

Since the notion of a *fuzzy set* was introduced by Zadeh [28], there has been an attempt to extend useful mathematical notions to this wider setting, replacing sets by fuzzy sets. A fuzzy set, $\mu$, on a set $X$ is simply a function $\mu : X \to I$, where $I$ denotes the unit interval. In [20], Lowen introduced the notion of a *fuzzy uniform space* and the fuzzy uniform space analogues of compactness, relative compactness, completeness, precompactness and boundedness have now been extended to the fuzzy setting [1–5, 9, 10, 21, 22]. The basic theory of function spaces has also been extended to this setting [6]. This was accomplished using the theory of *prefilters* [18, 19]. The reader is referred to [2], Section 2, [15], Section 2 for a summary of the basic theory of prefilters. Let us recall that a fuzzy uniformity on a set $X$ is a subset $\mathscr{D} \subseteq I^{X \times X}$ and a prefilter on $X$ is a subset $\mathscr{F} \subseteq I^X$. In other words,

$$\mathscr{D} \in 2^{I^{X \times X}}, \quad \mathscr{F} \in 2^{I^X}.$$

The reader is referred to [2, Section 3], for a summary of the basic theory of fuzzy uniform spaces.

* Corresponding author. E-mail: mamb@warthog.ru.ac.za.

0165-0114/99/$ – see front matter 
PII: S0165-0114(97)00262-5

Recently, in [14], the notion of a uniform space has been generalised even further. A *super uniformity*, $\delta$, on a set $X$ is defined to be a function

$$\delta : I^{X\times X} \to I,$$

satisfying certain conditions. In other words,

$$\delta \in I^{I^{X\times X}}.$$

As might be expected, a fundamental tool for investigating super uniform spaces is the analogue of a filter in this setting, which is called a *fuzzy filter*. A fuzzy filter, $\varphi$, on $X$ is a function $\varphi : I^X \to I$ satisfying certain conditions.

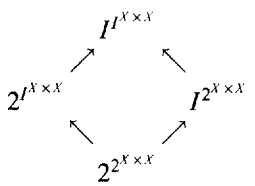
In other words,

$$\varphi \in I^{I^X}.$$

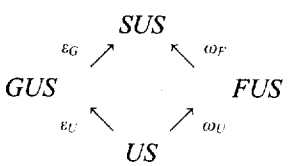
In [7], another extension of the notion of a uniform space was defined and investigated. A *generalised uniformity*, $d$, on a set $X$ is defined to be a function $d : 2^{X\times X} \to I$ satisfying certain natural conditions. In other words,

$$d \in I^{2^{X\times X}}.$$

There are natural injections between the sets as indicated




and it is shown in [15] that there are categorical embeddings between the categories: *US* (of uniform spaces), *FUS* (of fuzzy uniform spaces), *GUS* (of generalised uniform spaces) and *SUS* (of super uniform spaces) as indicated




Although prefilters were used in the investigation of generalised uniform spaces, it would seem that, in this context, the natural analogue of a filter on $X$ would be a function $f : 2^X \to I$. In other words, an element

$$f \in I^{2^X},$$

satisfying certain conditions. We show that such an analogue, which we shall call a generalised filter, does exist and we establish the basic theory of these generalised filters. We first recall some relevant definitions, notation and results.

## 2. Preliminaries

For the sake of fixing notation, we recall some basic definitions.

We shall let $I$ denote the closed unit interval $[0,1]$ and we let $I_0 \stackrel{\text{def}}{=} I\backslash\{0\} = (0,1]$, $I_1 = I\backslash\{1\} = [0,1)$.

We denote the characteristic function of a subset $A \subseteq X$ by $1_A$. If $\mu \in I^X$ we define

$$\mu^\alpha = \{x \in X : \mu(x) > \alpha\}.$$

If $X$ is a set then a *filter* on $X$ is a nonempty subset $\mathbb{F} \subseteq 2^X$ satisfying the following conditions:

F1 $\emptyset \notin \mathbb{F}$;

F2 $F, G \in \mathbb{F} \Rightarrow F \cap G \in \mathbb{F}$;

F3 $F \in \mathbb{F}$, $F \subseteq G \Rightarrow G \in \mathbb{F}$.

If $\mathbb{F}$ is nonempty and satisfies

FB1 $\emptyset \notin \mathbb{F}$;

FB2 $\forall F, G \in \mathbb{F}$, $\exists H \in \mathbb{F}$, $H \subseteq F \cap G$

we call $\mathbb{F}$ a *filter base* on $X$.

A *prefilter* on $X$ is a nonempty subset $\mathscr{F} \subseteq I^X$ which satisfies the following conditions:

PF1 $0 \notin \mathscr{F}$;

PF2 $\nu, \mu \in \mathscr{F} \Rightarrow \nu \wedge \mu \in \mathscr{F}$;

PF3 $\nu \in \mathscr{F}$, $\nu \leqslant \mu \Rightarrow \mu \in \mathscr{F}$.

If $\mathscr{F}$ is nonempty and satisfies

PFB1 $0 \notin \mathscr{F}$;

PFB2 $\forall \nu_1, \nu_2 \in \mathscr{F}$, $\exists \nu_3 \in \mathscr{F}$, $\nu_3 \leqslant \nu_1 \wedge \nu_2$

we call $\mathscr{F}$ a *prefilter base* on $X$.

Naturally, a filter is a filter base and a prefilter is a prefilter base.

If $X$ is a set and $\mathbb{F}$ is a family of subsets of $X$, we define

$$\langle \mathbb{F} \rangle \stackrel{\text{def}}{=} \{A \subseteq X : \exists F \in \mathbb{F},\ F \subseteq A\}.$$

Similarly, if $\mathscr{F} \subseteq I^X$ we define

$$\langle \mathscr{F} \rangle \stackrel{\text{def}}{=} \{\mu \in I^X : \exists \nu \in \mathscr{F},\ \nu \leqslant \mu\}.$$

If $\mathscr{F}$ is a prefilter base on $X$ we define the *characteristic*, $c(\mathscr{F})$, of $\mathscr{F}$ by

$$c(\mathscr{F}) \stackrel{\text{def}}{=} \inf_{\nu \in \mathscr{F}} \sup \nu.$$

If $\mathscr{F}$ is a prefilter base and $c(\mathscr{F}) > 0$, we define

$$\hat{\mathscr{F}} \stackrel{\text{def}}{=} \left\{ \sup_{\varepsilon \in I_0} (\mu_\varepsilon - \varepsilon) : (\mu_\varepsilon) \in \mathscr{F}^{I_0} \right\}$$

and call $\hat{\mathscr{F}}$ the *saturation* of $\mathscr{F}$. We say that $\mathscr{F}$ is *saturated* if $\hat{\mathscr{F}} = \mathscr{F}$.

Note that in case $c(\mathscr{F}) = c = 0$, if we define $\hat{\mathscr{F}}$ as before then we have $0 \in \hat{\mathscr{F}}$. We conclude that if $\mathscr{F}$ is saturated then $c(\mathscr{F}) > 0$.

If $\mathscr{F}$ is a prefilter base, $c(\mathscr{F}) = c > 0$ and $0 < \alpha \leqslant c$ then we define

$$\mathscr{F}^\alpha \stackrel{\text{def}}{=} \{\nu^\beta : \nu \in \mathscr{F},\ \beta < \alpha\}.$$

On the other hand, if $\mathbb{F}$ is a filter on $X$ and $\alpha \in (0, 1]$ then we define

$$\mathbb{F}^\alpha \stackrel{\text{def}}{=} \{\nu \in I^X : \forall \beta < \alpha,\ \nu^\beta \in \mathbb{F}\}.$$

The reader is referred to [19, 20, 18] for the basic notation and theory of prefilters as well as to [2, 15]. As regards notation, we follow the notation established in [2] which is the notation of Lowen, with some modifications.

We list some of the main results here. Firstly, concerning prefilter bases:

**Theorem 2.1.** *Let $X$ be a set and $\mathscr{F}$ a prefilter base on $X$, $c(\mathscr{F}) = c, 0 < \alpha \leqslant c$. Then*

1. *$\langle \mathscr{F} \rangle$ is a prefilter and $\mathscr{F} \subseteq \langle \mathscr{F} \rangle$;*
2. *$c(\mathscr{F}) = c(\langle \mathscr{F} \rangle)$;*
3. *$\mathscr{F}^\alpha$ is a filter base;*
4. *$\mathscr{F}^\alpha = \bigcup_{0<\beta<\alpha} \mathscr{F}^\beta$;*
5. *If $c > 0$ then $\hat{\mathscr{F}}$ is a prefilter base and $\mathscr{F} \subseteq \hat{\mathscr{F}}$;*
6. *If $c > 0$ then $c(\hat{\mathscr{F}}) = c(\mathscr{F})$;*
7. *If $c > 0$ then $\tilde{\mathscr{F}} \stackrel{\text{def}}{=} \langle \hat{\mathscr{F}} \rangle = \langle \widehat{\mathscr{F}} \rangle$ is a saturated prefilter.*

Secondly, concerning prefilters:

**Theorem 2.2.** *Let $X$ be a set and $\mathscr{F}$ a prefilter on $X$, $c(\mathscr{F}) = c > 0$, $0 < \alpha \leqslant c$. Then*

1. *$\tilde{\mathscr{F}} = \hat{\mathscr{F}}$ is a prefilter and $\mathscr{F} \subseteq \tilde{\mathscr{F}}$;*
2. *$\mathscr{F}^\alpha$ is a filter;*
3. *$\hat{\mathscr{F}} = \{\nu \in I^X : \forall \varepsilon > 0,\ \nu + \varepsilon \in \mathscr{F}\}$;*
4. *$c(\mathscr{F}) = \inf\{\alpha \in I : \alpha 1_X \in \mathscr{F}\}$*
   *$= \sup\{\alpha \in I : \alpha 1_X \notin \mathscr{F}\}$.*

The proofs of the results in the last two theorems can be found in the cited literature or supplied by the reader. The next result is crucial and we include a proof.

**Theorem 2.3.** *Let $\mathscr{F}$ be a prefilter, $c(\mathscr{F}) = c > 0$ and $0 < \alpha \leqslant c$. Then*

$$\mathscr{F}^\alpha = (\hat{\mathscr{F}})^\alpha.$$

**Proof.** Since $\mathscr{F} \subseteq \hat{\mathscr{F}}$, $\mathscr{F}^\alpha \subseteq (\hat{\mathscr{F}})^\alpha$. To show the reverse inclusion, let $F \in (\hat{\mathscr{F}})^\alpha$. Then there exists $\nu \in \hat{\mathscr{F}}$ and $\beta < \alpha$ such that $F = \nu^\beta$. Let $2\varepsilon = \alpha - \beta$ and $\gamma = \beta + \varepsilon$. Then $\mu \stackrel{\text{def}}{=} \nu + \varepsilon \in \mathscr{F}$ and $F = \nu^\beta = (\nu + \varepsilon)^{\beta+\varepsilon} = \mu^\gamma \in \mathscr{F}^\alpha$. □

**Corollary 2.4.** *Let $\mathscr{F}$ be a prefilter, $c(\mathscr{F}) = c > 0$ and $0 < \alpha \leqslant c$. Then*

$$\hat{\mathscr{F}} = \bigcap_{0<\alpha<c} (\mathscr{F}^\alpha)^\alpha.$$

**Proof.** $\hat{\mathscr{F}}$ is saturated and so, by ([15, Proposition 12]), we know that

$$\hat{\mathscr{F}} = \bigcap_{0<\alpha<c} (\hat{\mathscr{F}}^\alpha)^\alpha.$$

Now, the result follows from the previous theorem. □

## 3. Generalised filters

We call a nonzero function $f : 2^X \to I$ a *generalised filter* (or a *g-filter*) on $X$ iff

GF1 $f(\emptyset) = 0$;

GF2 $\forall A, B \subseteq X,\ f(A \cap B) \geqslant f(A) \wedge f(B)$;

GF3 $\forall A, B \subseteq X,\ A \subseteq B \Rightarrow f(A) \leqslant f(B)$.

Of course, the requirement that $f$ be nonzero is equivalent to requiring that $f(X) > 0$.

For $f: 2^X \to I$ and $A \subseteq X$, we define

$$\langle f\rangle(A) \stackrel{\text{def}}{=} \sup_{B \subseteq A} f(B).$$

If $f$ is nonzero and satisfies
GFB1 $f(\emptyset) = 0$;
GFB2 $\forall A, B \subseteq X,\ f(A) \wedge f(B) \leqslant \langle f\rangle(A \cap B)$,
we shall call $f$ a *generalised filter base* (or a *g-filter base*) on $X$.

Naturally, a g-filter is a g-filter base. Furthermore,

**Theorem 3.1.** *If $X$ is a set and $f$ is a g-filter base on $X$ then $\langle f\rangle$ is a g-filter.*

**Proof.** The conditions GF1 and GF3 are easily checked.

GF2: Let $A, B \subseteq X$. If $\langle f\rangle(A) \wedge \langle f\rangle(B) = 0$ then $\langle f\rangle(A) \wedge \langle f\rangle(B) \leqslant \langle f\rangle(A \cap B)$. So let $\alpha < \langle f\rangle(A) \wedge \langle f\rangle(B)$.

Then

$$\begin{aligned} \alpha &< \sup_{U \subseteq A} f(U) \wedge \sup_{V \subseteq B} f(V) \\ &\Rightarrow \exists U \subseteq A, \quad \exists V \subseteq B, \\ &\qquad \alpha < f(U) \wedge f(V) \leqslant \langle f\rangle(U \cap V) \\ &\Rightarrow \exists W \subseteq U \cap V \subseteq A \cap B, \ \alpha < f(W) \\ &\Rightarrow \alpha < \langle f\rangle(A \cap B). \end{aligned}$$

Thus, $\langle f\rangle(A) \wedge \langle f\rangle(B) \leqslant \langle f\rangle(A \cap B)$. □

If $f$ is a g-filter base on $X$, we define the *characteristic*, $c(f)$, of $f$ by

$$c(f) = \sup_{A \subseteq X} f(A).$$

It follows from definition that $c(f) > 0$.

Just as for prefilters, we have:

**Lemma 3.2.** *If $X$ is a set and $f$ is a g-filter base on $X$ then*

$$c(f) = c(\langle f\rangle).$$

**Proof.**

$$\begin{aligned} c(\langle f\rangle) &= \sup_{A \subseteq X} \langle f\rangle(A) = \sup_{A \subseteq X} \sup_{B \subseteq A} f(B) \\ &= \sup_{B \subseteq X} f(B) = c(f). \end{aligned}$$ □

The proof of the following lemma is straightforward and is left to the reader.

**Lemma 3.3.** *Let $f$ be a g-filter on $X$ and let $A, B \subseteq X$. Then*
1. $c(f) = f(X)$;
2. $f(A \cap B) = f(A) \wedge f(B)$.

If $f$ is a g-filter (base) on $X$ with $c(f) = c$ then for $0 \leqslant \alpha < c$, we define the (*upper*) *$\alpha$-level filter (base)*, $f^\alpha$, associated with $f$ by

$$f^\alpha \stackrel{\text{def}}{=} \{F \subseteq X:\ f(F) > \alpha\}$$

and for $0 < \alpha \leqslant c$, we define the (*lower*) *$\alpha$-level filter (base)*, $f_\alpha$, associated with $f$ by

$$f_\alpha \stackrel{\text{def}}{=} \{F \subseteq X:\ f(F) \geqslant \alpha\}.$$

**Theorem 3.4.** *If $f$ is a g-filter (base) on $X$ with $c(f) = c$ and:*
(a) *$0 \leqslant \alpha < c$ then $f^\alpha$ is a filter (base) on $X$;*
(b) *$0 < \alpha \leqslant c$ then $f_\alpha$ is a filter (base) on $X$.*

**Proof.**

(a) Let $f$ be a g-filter on $X$. $f(X) = c > \alpha \Rightarrow X \in f^\alpha$. Thus, $f^\alpha \neq \emptyset$.
If $F \in f^\alpha$ then $f(F) > \alpha \geqslant 0$, and hence, $F \neq \emptyset$.
If $A, B \in f^\alpha$ then $f(A) \wedge f(B) = f(A \cap B) > \alpha$, and hence, $A \cap B \in f^\alpha$.
Finally, if $A \in f^\alpha$ and $A \subseteq B$ then $f(B) \geqslant f(A) > \alpha$, and hence, $B \in f^\alpha$.

The proofs of the remaining three assertions are left to the reader. □

It is left as an easy exercise to show that the $\alpha$-level filters decrease as $\alpha$ increases and we record this as a lemma.

**Lemma 3.5.** *If $f$ is a g-filter (base) with $c(f) = c$ and $0 \leqslant \alpha \leqslant \beta < c$ then*

$$f_c \subseteq f^\beta \subseteq f^\alpha \subseteq f^0.$$

The notion of a g-filter is a strict extension of the notion of a filter in the sense that we can associate a g-filter with every filter and there are g-filters which are not merely copies of filters. More precisely:

**Theorem 3.6.** *Let $X$ be a set, let $F(X)$ denote the collection of all filters on $X$ and let $G(X)$ denote the collection of all g-filters on $X$. Let*

$$\psi : F(X) \to G(X), \quad \mathbb{F} \mapsto 1_{\mathbb{F}}.$$

*Then $\psi$ is injective but not surjective.*

**Proof.** The proof that $1_{\mathbb{F}}$ is a g-filter is left as an exercise.

To see that $\psi$ is not surjective it is sufficient to find a g-filter whose characteristic value is different from 1. □

We note the following examples of g-filters, leaving the checking to the reader.

**Examples 3.7.**

(a) Let $X = \{1,2,3\}$ and define $f$ by

$$f(F) = 0 \quad \text{if } 1 \notin F,$$
$$f(\{1\}) = f(\{1,3\}) = \tfrac{1}{4},$$
$$f(\{1,2\}) = \tfrac{3}{4},$$
$$f(\{1,2,3\}) = 1.$$

(b) Let $U_n \overset{\text{def}}{=} \{m \in \mathbb{N} : m \geqslant n\}$ and define $g$ by

$$g(F) \overset{\text{def}}{=} \begin{cases} 1/n & \text{if } F = U_n, \\ 0 & \text{otherwise} \end{cases}$$

and let $f = \langle g \rangle$.

(c) Let

$$g_{\alpha,a}(F) \overset{\text{def}}{=} \begin{cases} \alpha & \text{if } F = \{a\}, \\ 0 & \text{otherwise} \end{cases}$$

and let $f_{\alpha,a} = \langle g_{\alpha,a} \rangle$.

This g-filter has the special property that, for all $A, B \subseteq X$

$$f_{\alpha,a}(A \cup B) = f_{\alpha,a}(A) \vee f_{\alpha,a}(B).$$

## 4. G-filters from prefilters

Let $\mathscr{F}$ be a prefilter on $X$ with $c(\mathscr{F}) = c > 0$. For $F \subseteq X$ define

$$S_{\mathscr{F}}(F) \overset{\text{def}}{=} \{\alpha \in (0,c] : F \in \mathscr{F}^{\alpha}\}.$$

**Lemma 4.1.** *Let $\mathscr{F}$ be a prefilter with $c(\mathscr{F}) = c > 0$. Then $S_{\mathscr{F}}(F) = \emptyset$ or $S_{\mathscr{F}}(F)$ is an interval of form $(\beta, c]$.*

**Proof.** If $S_{\mathscr{F}}(F) \neq \emptyset$ then there exists some $\alpha \in S_{\mathscr{F}}(F)$.

If $\alpha \leqslant \gamma \leqslant c$ then $F \in \mathscr{F}^{\alpha} \subseteq \mathscr{F}^{\gamma}$ and so $\gamma \in S_{\mathscr{F}}(F)$.

Since

$$\mathscr{F}^{\alpha} = \bigcup_{0<\beta<\alpha} \mathscr{F}^{\beta},$$

we have

$$\begin{aligned} \alpha \in S_{\mathscr{F}}(F) &\Rightarrow F \in \mathscr{F}^{\alpha} = \bigcup_{0<\beta<\alpha} \mathscr{F}^{\beta} \\ &\Rightarrow \exists \beta < \alpha,\ F \in \mathscr{F}^{\beta} \\ &\Rightarrow \exists \beta < \alpha,\ \beta \in S_{\mathscr{F}}(F). \end{aligned}$$ □

This lemma allows us to define, for $F \subseteq X$

$$f_{\mathscr{F}}(F) \overset{\text{def}}{=} \begin{cases} c - \inf S_{\mathscr{F}}(F) & \text{if } S_{\mathscr{F}}(F) \neq \emptyset, \\ 0 & \text{if } S_{\mathscr{F}}(F) = \emptyset. \end{cases}$$

We now need to check that the object defined above is indeed a g-filter.

**Theorem 4.2.** *If $\mathscr{F}$ is a prefilter with $c(\mathscr{F}) = c > 0$ then $f_{\mathscr{F}}$ is a g-filter.*

**Proof.**

(a)

$$\begin{aligned} \forall \alpha \leqslant c(\mathscr{F}),\ \emptyset \notin \mathscr{F}^{\alpha} &\Rightarrow \forall \alpha \leqslant c(\mathscr{F}),\ \alpha \notin S_{\mathscr{F}}(\emptyset) \\ &\Rightarrow S_{\mathscr{F}}(\emptyset) = \emptyset \\ &\Rightarrow f_{\mathscr{F}}(\emptyset) = 0. \end{aligned}$$

(b) Let $A, B \subseteq X$. Then

$$\begin{aligned} \alpha < f_{\mathscr{F}}(A) \wedge f_{\mathscr{F}}(B) &\Rightarrow \inf S_{\mathscr{F}}(A) < c - \alpha \text{ and } \inf S_{\mathscr{F}}(B) < c - \alpha \\ &\Rightarrow c - \alpha \in S_{\mathscr{F}}(A) \text{ and } c - \alpha \in S_{\mathscr{F}}(B) \\ &\Rightarrow A, B \in \mathscr{F}^{c-\alpha} \\ &\Rightarrow A \cap B \in \mathscr{F}^{c-\alpha} \\ &\Rightarrow c - \alpha \in S_{\mathscr{F}}(A \cap B) \\ &\Rightarrow \inf S_{\mathscr{F}}(A \cap B) \leqslant c - \alpha \\ &\Rightarrow c - \inf S_{\mathscr{F}}(A \cap B) = f_{\mathscr{F}}(A \cap B) \geqslant \alpha. \end{aligned}$$

Thus,

$$f_{\mathscr{F}}(A\cap B)\geqslant f_{\mathscr{F}}(A)\wedge f_{\mathscr{F}}(B).$$

(c) Let $A\subseteq B\subseteq X$. Then

$$\begin{aligned} f_{\mathscr{F}}(A)>\alpha &\Rightarrow\ c-\inf S_{\mathscr{F}}(A)>\alpha\\ &\Rightarrow\ \inf S_{\mathscr{F}}(A)<c-\alpha\\ &\Rightarrow\ c-\alpha\in S_{\mathscr{F}}(A)\\ &\Rightarrow\ A\in\mathscr{F}^{c-\alpha}\\ &\Rightarrow\ B\in\mathscr{F}^{c-\alpha}\\ &\Rightarrow\ c-\alpha\in S_{\mathscr{F}}(B)\\ &\Rightarrow\ \inf S_{\mathscr{F}}(B)\leqslant c-\alpha\\ &\Rightarrow\ c-\inf S_{\mathscr{F}}(B)=f_{\mathscr{F}}(B)\geqslant\alpha. \end{aligned}$$

It follows that $f_{\mathscr{F}}(A)\leqslant f_{\mathscr{F}}(B)$. □

If $\mathscr{F}$ is a prefilter and $\mu\in I^X$ we follow [18] and define the *characteristic set of* $\mathscr{F}$ *with respect to* $\mu$, denoted $\mathscr{C}^{\mu}(\mathscr{F})$, by

$$\begin{aligned}\mathscr{C}^{\mu}(\mathscr{F})&=\{\alpha\in I:\ \forall v\in\mathscr{F},\ \exists x\in X,\ v(x)>\mu(x)+\alpha\}\\&=\{\alpha\in I:\ \mu+\alpha\notin\mathscr{F}\}.\end{aligned}$$

The number $c^{\mu}(\mathscr{F})\overset{\text{def}}{=}\sup\mathscr{C}^{\mu}(\mathscr{F})$ will be called the *characteristic value* of $\mathscr{F}$ with respect to $\mu$. If $F\subseteq X$ we write $c^F(\mathscr{F})$ instead of the more cumbersome $c^{1_F}(\mathscr{F})$.

**Theorem 4.3.** *If $\mathscr{F}$ is a prefilter with $c(\mathscr{F})=c>0$ and $F\subseteq X$ then $f_{\mathscr{F}}(F)=c-c^F(\mathscr{F})$.*

**Proof.** We just have to prove that $c^F(\mathscr{F})=\sup\mathscr{C}^F(\mathscr{F})=\inf S_{\mathscr{F}}(F)$.

As usual, for $F\subseteq X$, $1_F+\alpha\overset{\text{def}}{=}(1_F+\alpha 1_X)\wedge 1$ and it is easy to see that $1_F+\alpha=\alpha 1_X\vee 1_F$. Now,

$$\begin{aligned}&\inf S_{\mathscr{F}}(F)<\alpha\\ &\Rightarrow\ \alpha\in S_{\mathscr{F}}(F)\\ &\Rightarrow\ \exists\mu\in\mathscr{F},\ \exists\beta<\alpha,\ \mu^{\beta}=F\\ &\Rightarrow\ \exists\mu\in\mathscr{F},\ \exists\beta<\alpha,\ \mu\leqslant\beta 1_X\vee 1_{\mu^{\beta}}=1_F+\beta\\ &\Rightarrow\ \exists\beta<\alpha,\ 1_F+\beta\in\mathscr{F}\\ &\Rightarrow\ \alpha\notin\mathscr{C}^F(\mathscr{F})\\ &\Rightarrow\ c^F(\mathscr{F})\leqslant\alpha.\end{aligned}$$

Therefore, $c^F(\mathscr{F})\leqslant\inf S_{\mathscr{F}}(F)$.

Furthermore,

$$\begin{aligned}&c^F(\mathscr{F})<\alpha\\ &\Rightarrow\ \alpha\notin\mathscr{C}^F(\mathscr{F})\\ &\Rightarrow\ 1_F+\alpha=\alpha 1_X\vee 1_F\in\mathscr{F}\\ &\Rightarrow\ \forall\beta>\alpha,\ (\alpha 1_X\vee 1_F)^{\alpha}=F\in\mathscr{F}^{\beta}\\ &\Rightarrow\ \forall\beta>\alpha,\ \beta\in S_{\mathscr{F}}(F)\\ &\Rightarrow\ \inf S_{\mathscr{F}}(F)\leqslant\alpha.\end{aligned}$$

Thus, $c^F(\mathscr{F})=\inf S_{\mathscr{F}}(F)$. □

## 5. Prefilters from g-filters

Our next task is to show that a g-filter gives rise to a prefilter. However, we first discover the connection between the characteristic of a prefilter and the g-filter that it generates.

**Lemma 5.1.** *If $\mathscr{F}$ is a prefilter on $X$ with $c(\mathscr{F})=c>0$ then:*

$$c(f_{\mathscr{F}})=c(\mathscr{F}).$$

**Proof.** Let $c=c(\mathscr{F})$. Then

$$c(f_{\mathscr{F}})=f_{\mathscr{F}}(X)=c-\inf S_{\mathscr{F}}(X).$$

Now,

$$\begin{aligned}\forall\alpha\leqslant c,\ X\in\mathscr{F}^{\alpha}&\Rightarrow\ \forall\alpha\leqslant c,\ \alpha\in S_{\mathscr{F}}(X)\\&\Rightarrow\ \inf S_{\mathscr{F}}(X)=0\\&\Rightarrow\ c(f_{\mathscr{F}})=f_{\mathscr{F}}(X)=c.\end{aligned}$$ □

For a g-filter $f$ with $c(f)=c$ we define

$$\mathscr{F}_f\overset{\text{def}}{=}\{v\in I^X:\ \forall 0<\alpha\leqslant c,\ \forall\beta<\alpha,\ v^{\beta}\in f^{c-\alpha}\}.$$

Of course, we need to check that this does produce a prefilter.

**Theorem 5.2.** *If $f$ is a g-filter then $\mathscr{F}_f$ is a prefilter.*

**Proof.**

(a) We observe that

$$\forall\beta\leqslant c(f),\ 0^{\beta}=\{x\in X:\ 0(x)>\beta\}=\emptyset.$$

It follows that

$$\forall 0<\alpha\leqslant c,\ \forall\beta<\alpha,\ 0^{\beta}\notin f^{c-\alpha}$$

and this means that $0\notin\mathscr{F}_f$.
On the other hand, since $c=c(f)=f(X)>0$ we have

$$\forall 0<\alpha\leqslant c,\ \forall\beta<\alpha,\ (1_X)^{\beta}=X\in f^{c-\alpha}$$

and so $1_X\in\mathscr{F}_f$.

(b) Let $\mu,\nu\in\mathscr{F}_f$. It follows from

$$\forall\beta<c(f),\ \nu^{\beta}\cap\mu^{\beta}=(\nu\wedge\mu)^{\beta}$$

that $\nu\wedge\mu\in\mathscr{F}_f$.

(c) Let $\nu\in\mathscr{F}$ and $\nu\leqslant\mu$. Let $0<\alpha\leqslant c$, $\beta<\alpha$. Then $\nu^{\beta}\in f^{c-\alpha}$. Since $\nu^{\beta}\subseteq\mu^{\beta}$ we have

$$f(\mu^{\beta})\geqslant f(\nu^{\beta})\geqslant c-\alpha.$$

Thus, $\mu\in\mathscr{F}_f$. □

The following lemma simplifies some of the work later on.

**Lemma 5.3.** *If $f$ is a g-filter with $c(f)=c>0$ then*

$$\mathscr{F}_f=\{\nu\in I^X:\ \forall 0\leqslant\gamma<c,\ \nu^{\gamma}\in f_{c-\gamma}\}.$$

**Proof.** Let us define

$$\mathscr{G}\overset{\text{def}}{=}\{\nu\in I^X:\ \forall 0\leqslant\gamma<c,\ \nu^{\gamma}\in f_{c-\gamma}\}.$$

Let $\nu\in\mathscr{F}_f$. To show that $\nu\in\mathscr{G}$ let $0\leqslant\gamma<c$. Choose $\alpha$ such that $\gamma<\alpha<c$. Then $\nu^{\gamma}\in f^{c-\alpha}$. Since $\alpha$ is arbitrary, we have

$$\forall\alpha\in(\gamma,c),\ f(\nu^{\gamma})>c-\alpha$$

and hence, $f(\nu^{\gamma})\geqslant c-\gamma$. In other words, $\nu^{\gamma}\in f_{c-\gamma}$.

Conversely, let $\nu\in\mathscr{G}$. To show that $\nu\in\mathscr{F}_f$ let $0<\alpha\leqslant c$, $0\leqslant\beta<\alpha$. Then we have $0\leqslant\beta<c$. Thus, $\nu^{\beta}\in f_{c-\beta}$ and so $f(\nu^{\beta})\geqslant c-\beta>c-\alpha$. Therefore, $\nu^{\beta}\in f^{c-\alpha}$. □

The correlation between g-filters and prefilters is not completely straightforward. In fact, as we shall see, the prefilter associated with a g-filter is rather special.

**Theorem 5.4.** *If $f$ is a g-filter then the associated prefilter $\mathscr{F}_f$ is saturated.*

**Proof.** Suppose that

$$\forall\varepsilon>0,\ \nu+\varepsilon\in\mathscr{F}_f.$$

We show that $\nu\in\mathscr{F}_f$. To this end, we let $\alpha\leqslant c(f)$ and $\beta<\alpha$ and show that $\nu^{\beta}\in f^{c-\alpha}$.

Choose $\gamma$ such that $\beta<\gamma<\alpha$ and let $\varepsilon=\gamma-\beta$. Then, since $\nu+\varepsilon\in\mathscr{F}_f$ we have

$$(\nu+\varepsilon)^{\gamma}=(\nu+\varepsilon)^{\beta+\varepsilon}=\nu^{\beta}\in f^{c-\alpha}.\qquad\square$$

We saw, in Lemma 5.1, the connection between the characteristic of a prefilter and the g-filter that it generates. Let us now find the connection between a g-filter and the prefilter that it generates.

**Theorem 5.5.** *Let $f$ be a g-filter on $X$. Then*

$$c(\mathscr{F}_f)=c(f).$$

**Proof.** Let $c(f)=c$. Then

$$\begin{aligned}&\forall\nu\in\mathscr{F}_f,\ \forall 0\leqslant\beta<\alpha\leqslant c,\ \nu^{\beta}\in f^{c-\alpha}\\ &\Rightarrow\ \forall\nu\in\mathscr{F}_f,\ \forall 0\leqslant\beta<\alpha\leqslant c,\ \nu^{\beta}\neq\emptyset\\ &\Rightarrow\ \forall\nu\in\mathscr{F}_f,\ \forall 0\leqslant\beta<\alpha\leqslant c,\ \sup\nu>\beta\\ &\Rightarrow\ \forall\nu\in\mathscr{F}_f,\ \sup\nu\geqslant c\\ &\Rightarrow\ \inf_{\nu\in\mathscr{F}_f}\sup\nu=c(\mathscr{F}_f)\geqslant c.\end{aligned}$$

On the other hand,

$$\forall\alpha\leqslant c,\ \forall\beta<\alpha,\ (c1_X)^{\beta}=X\in f^{c-\alpha}$$

and so $c1_X\in\mathscr{F}_f$. Thus,

$$c(\mathscr{F}_f)=\inf_{\nu\in\mathscr{F}_f}\sup\nu\leqslant\sup c1_X=c.\qquad\square$$

The use of $\alpha$-level theorems has proved to be very useful in various situations. See, e.g. [26, 27, 17]. We therefore, investigate the $\alpha$-levels of g-filters.

**Lemma 5.6.** *Let $f$ be a g-filter with $c(f)=c$ and let $\alpha\in(0,c]$. Then*

$$\mathscr{F}_f^{\alpha}=f^{c-\alpha}.$$

**Proof.** Let $F\in(\mathscr{F}_f)^{\alpha}$. Then there exists $\nu\in\mathscr{F}_f$, $\beta<\alpha$ such that $F=\nu^{\beta}$. Since $\nu\in\mathscr{F}_f$, we have $\nu^{\beta}=F\in f^{c-\alpha}$.

Conversely, if $F \in f^{c-\alpha}$ then $f(F) \stackrel{\text{def}}{=} t > c-\alpha$. Let $v = (c-t)1_X \vee 1_F$. We intend to invoke Lemma 5.3 to show that $v \in \mathscr{F}_f$. To this end, let $0 \leqslant \gamma < c$.

If $\gamma \in [c-t, c)$ then $v^\gamma = F$ and so $f(v^\gamma) = f(F) = t \geqslant c-\gamma$.

If $\gamma \in [0, c-t)$ then $v^\gamma = X$ and so $f(v^\gamma) = f(X) = c \geqslant c-\gamma$.

We therefore have $v^\gamma \in f_{c-\gamma}$ for all $\gamma \in [0,c)$ and so $v \in \mathscr{F}_f$ and $F = v^{c-t}$ with $c-t < \alpha$. Thus, $F \in \mathscr{F}_f^\alpha$. □

**Lemma 5.7.** *If $\mathscr{F}$ is a prefilter on $X$ with $c(\mathscr{F}) = c > 0$ then, for $\alpha \in [0,c)$*

$$(f_{\mathscr{F}})^\alpha = \mathscr{F}^{c-\alpha}.$$

**Proof.**

$$\begin{aligned} A \in (f_{\mathscr{F}})^\alpha &\Leftrightarrow f_{\mathscr{F}}(A) = c - \inf S_{\mathscr{F}}(A) > \alpha \\ &\Leftrightarrow \inf S_{\mathscr{F}}(A) < c-\alpha \\ &\Leftrightarrow c-\alpha \in S_{\mathscr{F}}(A) \\ &\Leftrightarrow A \in \mathscr{F}^{c-\alpha}. \end{aligned}$$ □

We now establish the g-filter analogue of Theorem 2.1.4.

**Lemma 5.8.** *If $f$ is a g-filter with $c(f) = c$ and $0 \leqslant \alpha < c$ then*

$$f^\alpha = \bigcup_{c > \beta > \alpha} f^\beta.$$

**Proof.** If $F \in f^\alpha$ choose $\beta$ such that $\alpha < \beta < f(F)$. Then $F \in f^\beta$ and so $F \in \bigcup_{\beta > \alpha} f^\beta$.

Conversely, let $F \in \bigcup_{c > \beta > \alpha} f^\beta$. Then $F \in f^\beta$ for some $\beta > \alpha$. Thus, $f(F) > \beta > \alpha$ and hence, $F \in f^\alpha$. □

**Corollary 5.9.** *If $f$ is a g-filter with $c(f) = c$ and $0 < \alpha \leqslant c$ then*

$$f^{c-\alpha} = \bigcup_{0 < \beta < \alpha} f^{c-\beta}.$$

In [2] we saw that saturated prefilters are specified by their $\alpha$-level filters. We show that a similar situation pertains for g-filters.

**Lemma 5.10.** *If $f$ and $g$ are g-filters with $c(f) \neq c(g)$ then $f \neq g$.*

**Lemma 5.11.** *Let $f$ and $g$ be g-filters with $c(f) = c(g) = c$. Then*

$$\forall \alpha < c,\ f^\alpha = g^\alpha \ \Leftrightarrow\ f = g.$$

**Proof.**

$$\begin{aligned} \forall \alpha < c,\ f^\alpha = g^\alpha &\Leftrightarrow \forall \alpha < c,\ \forall A \subseteq X,\ A \in f^\alpha \Leftrightarrow A \in g^\alpha \\ &\Leftrightarrow \forall A \subseteq X,\ \forall \alpha \leqslant c, \\ &\quad (f(A) > \alpha \Leftrightarrow g(A) > \alpha) \\ &\Leftrightarrow \forall A \subseteq X,\ f(A) = g(A) \\ &\Leftrightarrow f = g. \end{aligned}$$ □

We have seen that to each g-filter there corresponds a saturated prefilter and, conversely, to each prefilter there corresponds a g-filter. This inspires the following theorem:

**Theorem 5.12.** *Let*

$$\mathscr{S}(X) \stackrel{\text{def}}{=} \{\mathscr{F} \in 2^{I^X} : \mathscr{F} \text{ is a saturated prefilter on } X\},$$

$$\mathscr{G}(X) \stackrel{\text{def}}{=} \{f \in I^{2^X} : f \text{ is a g-filter on } X\}.$$

*Then*

$$\psi : \mathscr{S}(X) \to \mathscr{G}(X), \quad \mathscr{F} \mapsto f_{\mathscr{F}}$$

*is a bijection.*

**Proof.** We first show that $\psi$ is injective. To this end, let $\mathscr{F}, \mathscr{G} \in \mathscr{S}(X)$ with $\mathscr{F} \neq \mathscr{G}$.

If $c(\mathscr{F}) \neq c(\mathscr{G})$ then

$$\begin{aligned} f_{\mathscr{F}}(X) &= c(\mathscr{F}) - \inf\{\alpha \leqslant c(\mathscr{F}) : X \in \mathscr{F}^\alpha\} \\ &= c(\mathscr{F}) - 0 \\ &\neq c(\mathscr{G}) \\ &= f_{\mathscr{G}}(X) \end{aligned}$$

and so $f_{\mathscr{F}} \neq f_{\mathscr{G}}$.

If $c(\mathscr{F}) = c(\mathscr{G}) = c$ then

$$\exists \alpha \leqslant c, \quad \mathscr{F}^\alpha \neq \mathscr{G}^\alpha.$$

This follows from the fact that *saturated* prefilters are completely determined by their $\alpha$-level filters ([1, Theorem 2; 15, Theorem 11]). Suppose that

$F \in \mathscr{F}^\alpha \setminus \mathscr{G}^\alpha$. Then $\alpha \in S_{\mathscr{F}}(F) \setminus S_{\mathscr{G}}(F)$. Thus, $\inf S_{\mathscr{F}}(F) < \alpha$ and $\alpha \leqslant \inf S_{\mathscr{G}}(F)$. Thus,

$$f_{\mathscr{F}}(F) = c - \inf S_{\mathscr{F}}(F) > c - \alpha \geqslant c - \inf S_{\mathscr{G}}(F) = f_{\mathscr{G}}(F).$$

So, once again, $f_{\mathscr{F}} \neq f_{\mathscr{G}}$.

In order to show that $\psi$ is surjective, let $f \in \mathscr{G}(X)$ and let $c(\mathscr{F}) = c$. Then

$$\mathscr{F}_f = \{\nu \in I^X : \forall \alpha \leqslant c,\ \forall \beta < \alpha,\ \nu^\beta \in f^{c-\alpha}\}.$$

Now, appealing to Lemmas 5.6 and 5.7, we have

$$\forall \alpha \in [0, c),\ (f_{\mathscr{F}_f})^\alpha = (\mathscr{F}_f)^{c-\alpha} = f^{c-(c-\alpha)} = f^\alpha.$$

It therefore follows from Lemma 5.11 that

$$\psi(\mathscr{F}_f) = f_{\mathscr{F}_f} = f. \quad \square$$

We extract the following corollaries:

**Corollary 5.13.** *If $f$ is a g-filter on $X$ then*

$$f_{\mathscr{F}_f} = f.$$

**Corollary 5.14.** *If $\mathscr{F}$ is a saturated prefilter on $X$ then*

$$\mathscr{F}_{f_{\mathscr{F}}} = \mathscr{F}.$$

**Proof.** Let $\psi : \mathscr{S} \to \mathscr{G}$ as in the theorem. Then

$$\psi(\mathscr{F}) = f_{\mathscr{F}}$$

and

$$\psi(\mathscr{F}_{f_{\mathscr{F}}}) = f_{(\mathscr{F}_{f_{\mathscr{F}}})} = f_{\mathscr{F}}.$$

Thus, it follows from the injectivity of $\psi$ that $\mathscr{F} = \mathscr{F}_{f_{\mathscr{F}}}$. $\square$

We have developed the g-filter analogues of various prefilter notions and it is natural, therefore, to seek a g-filter analogue of the saturation operator. In other words, if $f$ is a g-filter, we seek a definition of $\hat{f}$, the *saturation* of $f$, which is consistent with the theory which we have developed thus far. We would require, among other things, that the saturation of the g-filter associated with a prefilter is the g-filter associated with the saturation of the prefilter. In symbols

$$\widehat{f_{\mathscr{F}}} = f_{\hat{\mathscr{F}}}.$$

However, we have the following lemma:

**Lemma 5.15.** *If $\mathscr{F}$ is a prefilter then*

$$f_{\mathscr{F}} = f_{\hat{\mathscr{F}}}.$$

**Proof.** For $F \subseteq X$:

$$\begin{aligned} f_{\hat{\mathscr{F}}}(F) &= c(\hat{\mathscr{F}}) - \inf\{\alpha : F \in (\hat{\mathscr{F}})^\alpha\} \\ &= c(\mathscr{F}) - \inf\{\alpha : F \in (\mathscr{F})^\alpha\} \\ &= f_{\mathscr{F}}(F). \quad \square \end{aligned}$$

Thus, for a prefilter $\mathscr{F}$

$$\widehat{f_{\mathscr{F}}} = f_{\mathscr{F}}.$$

The most natural definition of $\hat{f}$ which accomplishes this is the simple

$$\hat{f} \overset{\text{def}}{=} f.$$

In this sense, g-filters are already saturated. This explains why, in [7], the definition of a generalised uniformity did not include a saturation condition analagous to FU2 in Definition 2.1 of [20]. The situation is also illustrated by the following theorem which extends Theorem 5.4.

**Theorem 5.16.** *If $\mathscr{F}$ is a prefilter then*

$$\mathscr{F}_{f_{\mathscr{F}}} = \hat{\mathscr{F}}.$$

**Proof.** From Theorem 5.4 we know that $\mathscr{F}_{f_{\mathscr{F}}}$ is a saturated prefilter and so, according to [15, Theorem 11], we must show that

$$\forall \alpha \leqslant c(\hat{\mathscr{F}}) = c(\mathscr{F}),\ (\mathscr{F}_{f_{\mathscr{F}}})^\alpha = (\hat{\mathscr{F}})^\alpha.$$

Now,

$$(\mathscr{F}_{f_{\mathscr{F}}})^\alpha = (f_{\mathscr{F}})^{c-\alpha} = \mathscr{F}^\alpha = \hat{\mathscr{F}}^\alpha. \quad \square$$

From this last result we obtain the following characterisation of the saturation of a prefilter.

**Corollary 5.17.** *If $\mathscr{F}$ is a prefilter with $c(\mathscr{F}) = c > 0$ then*

$$\hat{\mathscr{F}} = \{\nu \in I^X : \forall 0 < \alpha \leqslant c,\ \forall \beta < \alpha,\ \nu^\beta \in \mathscr{F}^\alpha\}.$$

**Proof.**

$$\begin{aligned} v \in \hat{\mathscr{F}} &\Leftrightarrow v \in \mathscr{F}_{f_{\mathscr{F}}} \\ &\Leftrightarrow \forall 0 < \alpha \leqslant c,\ \forall \beta < \alpha, \\ &\quad v^{\beta} \in (f_{\mathscr{F}})^{c-\alpha} = \mathscr{F}^{\alpha}. \end{aligned}$$ □

This concludes our study of the basic properties of generalised filters. In [8], we investigate prime g-filters and the behaviour of images and preimages, of g-filters.

## Acknowledgements

We would like to thank the Rhodes University Research Council and the Foundation for Research Development for their generous financial support. We also gratefully acknowledge the helpful comments of Prof. M.A. de Prada Vicente.

## References

[1] M.H. Burton, The relationship between a fuzzy uniformity and its family of α-level uniformities, Fuzzy Sets and Systems 54 (3) (1993) 311–315.
[2] M.H. Burton, Cauchy filters and prefilters, Fuzzy Sets and Systems 54 (3) (1993) 317–331.
[3] M.H. Burton, Completeness in fuzzy uniform spaces, Quaestiones Math. 16 (1) (1993) 13–36.
[4] M.H. Burton, Precompactness in fuzzy uniform spaces, Quaestiones Math. 16 (1) (1993) 37–49.
[5] M.H. Burton, Boundedness in uniform spaces and fuzzy uniform spaces, Fuzzy Sets and Systems 58 (2) (1993) 195–207.
[6] M.H. Burton, The fuzzy uniformisation of function spaces, Quaestiones Math. 20 (3) (1997), to appear.
[7] M.H. Burton, M.A. De Prada Vicente, J. Gutiérrez García, Generalised uniform spaces, J. Fuzzy Math. 4 (2) (1996) 363–380.
[8] M.H. Burton, M. Muraleetharan, J. Gutiérrez García, Generalised filters 2, submitted.
[9] J.J. Chadwick, A generalised form of compactness in fuzzy topological spaces, J. Math. Anal. Appl. 162 (1) (1991) 92–110.
[10] J.J. Chadwick, relative compactness and compactness of general subsets of an I-topological space, Quaestiones Math. 14 (4) (1991) 491–507.
[11] M.A. de Prada Vicente, M. Saralegui Aranguren, Fuzzy filters, J. Math. Anal. Appl. 129 (1988) 560–568.
[12] P. Eklund, W. Gähler, Fuzzy filters, functors and convergence, in: Applications of Category Theory to Fuzzy Sets, ch. IV, Kluwer, Dordrecht, 1992, 109–136.
[13] J. Gutiérrez García, M.A. de Prada Vicente, On a fuzzy Smirnov compactification, J. Fuzzy Math. 2 (4) (1995) 793–808.
[14] J. Gutiérrez García, M.A. de Prada Vicente, Super uniform spaces, Quaestiones Math. 20 (3) (1997), to appear.
[15] J. Gutiérrez García, M.A. de Prada Vicente, M.H. Burton, Embeddings into the category of super uniform spaces, Quaestiones Math. 20 (3) (1997), to appear.
[16] J. Gutiérrez García, M.A. de Prada Vicente, A. Šostak, Even fuzzy topologies and related structures, Quaestiones Math. 20 (3) (1997), to appear.
[17] A. Kandil, K.A. Hashem, N.N. Morsi, A level-topologies criterion for Lowen fuzzy uniformizability, Fuzzy Sets and Systems 62 (1994) 211–226.
[18] R. Lowen, Convergence in fuzzy topological spaces, General Topology and its Applications 10 (1979) 147–160.
[19] R. Lowen, Fuzzy neighbourhood spaces, Fuzzy Sets and Systems 7 (1980) 165–189.
[20] R. Lowen, Fuzzy uniform spaces, J. Math. Anal. Appl. 82 (1981) 370–385.
[21] R. Lowen, P. Wuyts, Completeness, compactness and precompactness in fuzzy uniform spaces: Part 1, J. Math. Anal. Appl. 90 (2) (1982) 563–581.
[22] R. Lowen, P. Wuyts, Completeness, compactness and precompactness in fuzzy uniform spaces: Part 2, J. Math. Anal. Appl. 92 (2) (1983) 342–371.
[23] M. Macho Stadler, M.A. de Prada Vicente, *t*-Prefilter theory, Fuzzy Sets and Systems 38 (1990) 115–124.
[24] M. Macho Stadler, M.A. de Prada Vicente, Fuzzy convergence versus convergence, Bull. Cal. Math. Soc. 85 (1993) 437–444.
[25] S. Willard, General Topology, Addison-Wesley, Reading, MA, 1970.
[26] P. Wuyts, On the determination of fuzzy topological spaces and fuzzy neighbourhood spaces by their level-topologies, Fuzzy Sets and Systems 12 (1984) 71–85.
[27] P. Wuyts, On level-topologies and maximality of fuzzy topological spaces, Fuzzy Sets and Systems 79 (1996) 337–339.
[28] L.A. Zadeh, Fuzzy sets, Inform and Control 8 (1965) 338–353.