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Generalised filters 2

M.H. Burton[a,*], M. Muraleetharan[a], J. Gutiérrez García[b]

[a] Department of Mathematics, Rhodes University, Grahamstown, 6140, South Africa
[b] Matematika Saila, Euskal Herriko Unibertsitatea, 48080, Bilbo, Spain

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Abstract

We continue the study, started in [8], of generalised filters. Prime prefilters have played a central role in the theory of (Lowen) fuzzy uniform spaces and Lowen discovered a characterisation of the set of all minimal prime prefilters finer than a given prefilter in terms of ultrafilters. We define the notion of a prime generalised filter and describe the set of all minimal prime g-filters finer than a given g-filter in terms of ultrafilters. The relationship between prime prefilters and prime g-filters is revealed. The behaviour of the images and preimages of g-filters are investigated. © 1999 Elsevier Science B.V. All rights reserved.

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1. Prime g-filters

In [9, 10] the theory of compact subsets of a topological space is lifted into the fuzzy setting. This was achieved with the aid of *prime* prefilters and the reader is referred to these papers for a succinct theory of prime prefilters. Prime prefilters also play a major role in [2–5] where the theories of: Cauchy filters, complete, precompact and bounded subsets of a uniform space are lifted to the fuzzy setting.

We are led therefore to seek a suitable definition of a prime g-filter which ties in with the theory of prime prefilters.

We call a g-filter f on X *prime* if

$$\forall A, B \subseteq X, \quad f(A \cup B) = f(A) \vee f(B).$$

In [18], Lowen develops the theory of prime prefilters. We quote two really useful results, in terms of the notation introduced in [2], from that paper.

Theorem 1.1 (Lowen). *Let $\mathscr{F}$ be a prefilter on X and let*

$$\mathscr{P}(\mathscr{F}) \stackrel{\text{def}}{=} \{\mathscr{G} \in I^X \colon \mathscr{G} \text{ is a prime prefilter and } \mathscr{F} \subseteq \mathscr{G}\}.$$

Then $\mathscr{P}(\mathscr{F})$ has minimal elements.

Theorem 1.2 (Lowen). *Let $\mathscr{F}$ be a prefilter on a set X and let*

$$\mathscr{P}_m(\mathscr{F}) \stackrel{\text{def}}{=} \{\mathscr{G} \in \mathscr{P}(\mathscr{F}) \colon \mathscr{G} \text{ is minimal}\}.$$

Then

$$\mathscr{P}_m(\mathscr{F}) = \{\mathscr{F} \vee \mathbb{F}_1 \colon \mathbb{F} \text{ is an ultrafilter, } \mathscr{F}_0 \subseteq \mathbb{F}\}.$$

* Corresponding author. E-mail: mamb@warthog.ru.ac.za.

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This last theorem, which characterises the minimal prime prefilters finer than a given prefilter, has found a number of applications. With this in mind, we attempt to construct a similar theory of prime g-filters.

We first find the connection between prime g-filters and ultrafilters.

Lemma 1.3. *Let $\mathbb{F}$ be a filter on X and let $0<\alpha\leqslant 1$. Then*

$\mathbb{F}$ *is an ultrafilter* $\Leftrightarrow$ $\alpha 1_{\mathbb{F}}$ *is a prime g-filter.*

Proof. ($\Rightarrow$) Let $\mathbb{F}$ be an ultrafilter. If $A\cup B\in\mathbb{F}$ then $\alpha 1_{\mathbb{F}}(A\cup B)=\alpha$. Furthermore, since $\mathbb{F}$ is an ultrafilter, $A\in\mathbb{F}$ or $B\in\mathbb{F}$. Thus $\alpha 1_{\mathbb{F}}(A)\vee\alpha 1_{\mathbb{F}}(B)=\alpha=\alpha 1_{\mathbb{F}}(A\cup B)$.

If $A\cup B\notin\mathbb{F}$ then $\alpha 1_{\mathbb{F}}(A\cup B)=0$. Since $\mathbb{F}$ is a filter, $A\notin\mathbb{F}$ and $B\notin\mathbb{F}$ and, hence, $\alpha 1_{\mathbb{F}}(A)\vee\alpha 1_{\mathbb{F}}(B)=0=\alpha 1_{\mathbb{F}}(A\cup B)$.

($\Leftarrow$) Let $\alpha 1_{\mathbb{F}}$ be prime and let $A\cup B\in\mathbb{F}$. Then $\alpha 1_{\mathbb{F}}(A\cup B)=\alpha=\alpha 1_{\mathbb{F}}(A)\vee\alpha 1_{\mathbb{F}}(B)$.

Therefore, $\alpha 1_{\mathbb{F}}(A)=\alpha$ or $\alpha 1_{\mathbb{F}}(B)=\alpha$ and so $A\in\mathbb{F}$ or $B\in\mathbb{F}$. Thus $\mathbb{F}$ is an ultrafilter. $\square$

Theorem 1.4. *Let f be a g-filter with $c(f)=c$. Then*

f *is a prime* $\Leftrightarrow$ f_c *is an ultrafilter.*

Proof. ($\Rightarrow$)

$$\begin{aligned} A\cup B\in f_c &\Leftrightarrow f(A\cup B)=f(A)\vee f(B)=c\\ &\Leftrightarrow f(A)=c \text{ or } f(B)=c\\ &\Leftrightarrow A\in f_c \text{ or } B\in f_c. \end{aligned}$$

($\Leftarrow$) If $\alpha<c$ then

$$\begin{aligned} \alpha<f(A\cup B) &\Rightarrow A\cup B\in f^{\alpha}=f_c\\ &\Rightarrow f(A\cup B)=c \text{ and } A\in f_c \text{ or } B\in f_c\\ &\Rightarrow f(A)\vee f(B)=c=f(A\cup B). \qquad\square \end{aligned}$$

Corollary 1.5. *If f is a prime g-filter with $c(f)=c$ then $f^{\alpha}=f^0=f_c$ for each $\alpha\in[0,c)$.*

Proof. We have $f_c\subseteq f^0$ and f_c is an ultrafilter. Thus for $\alpha\in[0,c)$ we have $f_c=f^{\alpha}=f^0$. $\square$

The reader can check that if $A\subseteq X$, $\alpha>0$ and $\mathbb{F}=\langle\{A\}\rangle$ then

$\alpha 1_{\mathbb{F}}$ is a prime $\Leftrightarrow$ A is a singleton.

If $\mathbb{F}$ is a filter then we define

$$\mathbb{P}(\mathbb{F})\overset{\text{def}}{=}\{\mathbb{K}\colon \mathbb{F}\subseteq\mathbb{K},\ \mathbb{K} \text{ is an ultrafilter}\}.$$

We now investigate the situation with regard to prime g-filters finer than a given g-filter.

Lemma 1.6. *If f is a g-filter, $\alpha\geqslant c=c(f)$ and $\mathbb{F}\in\mathbb{P}(f^0)$ then $\alpha 1_{\mathbb{F}}$ is a prime g-filter with $f\leqslant\alpha 1_{\mathbb{F}}$.*

Proof. It follows from Lemma 1.3 that $\alpha 1_{\mathbb{F}}$ is a prime. Furthermore, if $A\subseteq X$ with $f(A)>0$ then $A\in f^0\subseteq\mathbb{F}$ and so $\alpha 1_{\mathbb{F}}(A)=\alpha\geqslant c=f(X)\geqslant f(A)$. $\square$

Theorem 1.7. *If f is a prime g-filter with $c(f)=c$ and $\mathbb{F}=f_c$ then $f=c1_{\mathbb{F}}$.*

Proof. Let $A\subseteq X$. If $f(A)>0$ then $A\in f^0=f_c=\mathbb{F}$ and hence $f(A)=c=c1_{\mathbb{F}}(A)$. If $f(A)=0$ then $A\notin\mathbb{F}$ and so $f(A)=0=c1_{\mathbb{F}}(A)$. $\square$

Thus, the prime g-filters are precisely those g-filters of the form $\alpha 1_{\mathbb{F}}$ with $\mathbb{F}$ an ultrafilter. If f is a g-filter on X, let

$$\mathscr{P}(f)\overset{\text{def}}{=}\{g\colon g \text{ is a prime g-filter and } f\leqslant g\}.$$

We now aim for the g-filter equivalent of Lowen's Theorem 1.2.

Theorem 1.8. *If f is a g-filter with $c(f)=c$ then*

$$\mathscr{P}(f)=\{\alpha 1_{\mathbb{F}}\colon \mathbb{F}\in\mathbb{P}(f^0),\ \alpha\geqslant c\}.$$

Proof. Let $g\in\mathscr{P}(f)$ with $c(g)=\alpha$ and $\mathbb{F}=g_{\alpha}$. Then, by Theorem 1.7, $g=\alpha 1_{\mathbb{F}}$ with $\mathbb{F}$ an ultrafilter. Furthermore, since $f\leqslant g$, we have $c(f)\leqslant\alpha=c(g)$ and $\mathbb{F}\supseteq f^0$.

Conversely, if $g=\alpha 1_{\mathbb{F}}$ then, by Lemma 1.6, $g\in\mathscr{P}(f)$. $\square$

For a g-filter f let us define

$$\mathscr{P}_m(f)\overset{\text{def}}{=}\{g\colon g \text{ is a minimal prime g-filter and } f\leqslant g\}.$$

It is now an easy matter to obtain a characterisation of the minimal prime g-filters which are finer than a given g-filter.

Corollary 1.9. *If f is a g-filter with $c(f)=c$ then*

$$\mathscr{P}_m(f)=\{c1_{\mathbb{F}}: \mathbb{F}\in\mathbb{P}(f^0)\}.$$

Proof. Let $g\in\mathscr{P}_m(f)$. Then $g=\alpha 1_{\mathbb{F}}$ for some $\alpha\geqslant c$ and some $\mathbb{F}\in\mathbb{P}(f^0)$. If $\alpha>c$ then we can choose β such that $c<\beta<\alpha$ and then $h=\beta 1_{\mathbb{F}}\in\mathscr{P}(f)$ with $h\leqslant g$ and $h\neq g$ which contradicts the minimality of g. □

Our next task is to find the relationship between prime prefilters and prime g-filters. We first need the following lemma.

Lemma 1.10. *Let $(L,\leqslant)$ be a totally ordered set and let $(X,\preccurlyeq)$ be a partially ordered set. Let*

$$\varphi,\psi:(L,\leqslant)\to(X,\preccurlyeq)$$

be decreasing functions in the sense that

$$\forall\alpha,\beta\in L,\ (\alpha\leqslant\beta\Rightarrow\varphi(\beta)\preccurlyeq\varphi(\alpha),\ \psi(\beta)\preccurlyeq\psi(\alpha)).$$

Let $F\subseteq X$ have the property

$$\forall x,\ (x\in F,\ x\preccurlyeq y,\ \Rightarrow y\in F).$$

Then

$$\forall\alpha\in L,\ (\varphi(\alpha)\in F \text{ or } \psi(\alpha)\in F)$$
$$\Leftrightarrow\ (\forall\alpha\in L,\varphi(\alpha)\in F) \text{ or } (\forall\alpha\in L,\psi(\alpha)\in F).$$

Proof. We only have to show the forward implication so suppose that there exists $\alpha\in L$ such that $\varphi(\alpha)\notin F$. We must show that $\psi(\beta)\in F$ for each $\beta\in L$. Now,

$$\varphi(\alpha)\notin F\ \Rightarrow\ \psi(\alpha)\in F.$$

Thus, if $\beta\leqslant\alpha$ then

$$\psi(\alpha)\preccurlyeq\psi(\beta)\ \Rightarrow\ \psi(\beta)\in F.$$

On the other hand, if $\alpha<\beta$ then

$$\varphi(\beta)\preccurlyeq\varphi(\alpha)\ \Rightarrow\ \varphi(\beta)\notin F \text{ (otherwise } \varphi(\alpha)\in F)$$
$$\Rightarrow\ \psi(\beta)\in F. \quad □$$

Corollary 1.11. *Let $I\subseteq\mathbb{R}$ be an interval, X a set and let $\varphi,\psi:I\to\wp(X)$ be functions with the property such that*

$$\forall\alpha,\beta\in I,\ (\alpha\leqslant\beta\Rightarrow\varphi(\beta)\subseteq\varphi(\alpha),\ \psi(\beta)\subseteq\psi(\alpha)).$$

and let $\mathbb{F}$ be a filter on X. Then

$$\forall\alpha\in I,\ (\varphi(\alpha)\in\mathbb{F} \text{ or } \psi(\alpha)\in\mathbb{F})$$
$$\Leftrightarrow(\forall\alpha\in I,\varphi(\alpha)\in\mathbb{F}) \text{ or } (\forall\in I,\psi(\alpha)\in\mathbb{F}).$$

Theorem 1.12. *Let f be a prime g-filter on a set X with $c(f)=c$. Then $\mathscr{F}_f$ is also a prime.*

Proof. Let $\mu\vee\nu\in\mathscr{F}_f$. Then, according to Lemma 5.3 of [8], Theorem 1.4 and Corollary 1.5,

$$\forall\gamma\in[0,c),\ (\mu\vee\nu)^{\gamma}=\mu^{\gamma}\cup\nu^{\gamma}\in f_{c-\gamma}=f_c\overset{\text{def}}{=}\mathbb{F}$$

with $\mathbb{F}$ an ultrafilter on X. We therefore have

$$\forall\gamma\in[0,c),\ (\mu^{\gamma}\in\mathbb{F} \text{ or } \nu^{\gamma}\in\mathbb{F}).$$

We now invoke Corollary 1.11 and claim that

$$(\forall\gamma\in[0,c),\mu^{\gamma}\in\mathbb{F}) \quad\text{or}\quad (\forall\gamma\in[0,c),\nu^{\gamma}\in\mathbb{F}).$$

This, together with Lemma 5.3 of [8], shows that $\mu\in\mathbb{F}$ or $\nu\in\mathbb{F}$. □

Theorem 1.13. *Let $\mathscr{F}$ be a prime prefilter on a set X with $c(\mathscr{F})=c$. Then $f_{\mathscr{F}}$ is also a prime.*

Proof. We need to show that $f_{\mathscr{F}}(A\cup B)\leqslant f_{\mathscr{F}}(A)\vee f_{\mathscr{F}}(B)$ for $A,B\subseteq X$.

To this end let $0<\alpha<f_{\mathscr{F}}(A\cup B)$. Then

$$\alpha<c-\inf S_{\mathscr{F}}(A\cup B)$$
$$\Leftrightarrow\ A\cup B\in\mathscr{F}^{c-\alpha}=\mathscr{F}_0$$
$$\Leftrightarrow\ A\in\mathscr{F}_0 \text{ or } B\in\mathscr{F}_0$$
$$\text{(since } \mathscr{F}_0 \text{ is an ultrafilter)}$$
$$\Rightarrow\ \inf S_{\mathscr{F}}(A)\leqslant c-\alpha \text{ or } \inf S_{\mathscr{F}}(B)\leqslant c-\alpha$$
$$\Rightarrow\ f_{\mathscr{F}}(A)\geqslant\alpha \text{ or } f_{\mathscr{F}}(B)\geqslant\alpha$$
$$\Rightarrow\ f_{\mathscr{F}}(A)\vee f_{\mathscr{F}}(B)\geqslant\alpha.$$

Since α is arbitrary, we are done. □

Corollary 1.14. *If f is a g-filter and $\mathscr{F}$ is a prefilter then*

$$f \text{ is prime} \Leftrightarrow \mathscr{F}_f \text{ is prime},$$

$$\mathscr{F} \text{ is prime} \Leftrightarrow f_{\mathscr{F}} \text{ is prime}.$$

Proof. The proof follows immediately from Corollaries 5.13 and 5.14 in [8], 1.12 and 1.13. □

2. Images and preimages

If $h: X \to Y$ is a function and $f \in I^{2^X}$ is a g-filter base on X then we define the *direct image of* f, denoted $h(f)$, by

$$h(f): 2^Y \to I, B \mapsto h(f)(B)$$

$$\stackrel{\text{def}}{=} \begin{cases} \sup_{h(A)=B} f(A) & \text{if } h(A)=B \text{ for some } A \subseteq X, \\ 0 & \text{otherwise.} \end{cases}$$

We will show that the theory generated by this definition extends the corresponding theory of images of filters and filter bases.

Theorem 2.1. *If $h: X \to Y$ is a function and f is a g-filter base on X then $h(f)$ is a g-filter base on Y.*

Proof.
(i) Since f is non-zero, there exists $A \subseteq X$ such that $f(A) > 0$ and so $h(f)(h(A)) > 0$. In other words, $h(f)$ is non-zero.
(ii) $h(f)(\emptyset) = \sup_{h(A)=\emptyset} f(A) = f(\emptyset) = 0$.
(iii) If $B_1, B_2 \subseteq Y$ then

$$\alpha < h(f)(B_1) \wedge h(f)(B_2)$$
$$\Rightarrow \exists A_1, A_2 \subseteq X: h(A_1) = B_1, h(A_2) = B_2 \text{ and } \alpha < f(A_1) \wedge f(A_2)$$
$$\Rightarrow \exists A_3 \subseteq A_1 \cap A_2: \alpha < f(A_3)$$
$$\Rightarrow \exists B_3 := h(A_3) \subseteq B_1 \cap B_2: \alpha < h(f)(B_3)$$
$$\Rightarrow \langle h(f)\rangle(B_1 \cap B_2) = \sup_{B_3 \subseteq B_1 \cap B_2} h(f)(B_3) > \alpha.$$

Thus, $h(f)(B_1) \wedge h(f)(B_2) \leqslant \langle h(f)\rangle(B_1 \cap B_2)$. □

Theorem 2.2. *If $h: X \to Y$ is a function, f is a g-filter base on X and $\langle h(f)\rangle$ denotes the g-filter generated by the g-filter base $h(f)$ then:*
(1) *if f is a g-filter then $\langle h(f)\rangle(B) = f(h^{-1}[B])$ for each $B \subseteq Y$;*
(2) *if f is a prime g-filter then $\langle h(f)\rangle$ is a prime g-filter;*
(3) $\langle h(f)\rangle = \langle h(\langle f\rangle)\rangle$.

Proof. (1) It is clear that

$$\langle h(f)\rangle(B) = \sup_{B' \subseteq B} h(f)(B') = \sup_{B' \subseteq B} \sup_{h(A)=B'} f(A)$$
$$= \sup_{h(A) \subseteq B} f(A)$$

and, since $h(h^{-1}[B]) \subseteq B$, we have $f(h^{-1}[B]) \leqslant \langle h(f)\rangle(B)$.

The reverse inequality follows from the fact that if $h(A) \subseteq B$ then $A \subseteq h^{-1}[h(A)] \subseteq h^{-1}[B]$ and, since f is a g-filter, we have $f(A) \leqslant f(h^{-1}[B])$.

(2) Let $B_1, B_2 \subseteq Y$. Then

$$\langle h(f)\rangle(B_1 \cup B_2) = f(h^{-1}[B_1 \cup B_2])$$
$$= f(h^{-1}[B_1]) \vee f(h^{-1}[B_2])$$
$$= \langle h(f)\rangle(B_1) \vee \langle h(f)\rangle(B_2).$$

Thus $\langle h(f)\rangle$ is prime.

(3)

$$\langle h(\langle f\rangle)\rangle(B) = \sup_{h(A) \subseteq B} \langle f\rangle(A)$$
$$= \sup_{h(A) \subseteq B} \sup_{A' \subseteq A} f(A') = \sup_{h(A') \subseteq B} f(A')$$
$$= \langle h(f)\rangle(B). \quad \square$$

If $h: X \to Y$ is a function and $g \in I^{2^Y}$ is a g-filter base on Y then we define the *preimage of* g, denoted $h^{-1}(g)$, by

$$h^{-1}(g): 2^X \to I, \quad A \mapsto h^{-1}(g)(A)$$

$$\stackrel{\text{def}}{=} \begin{cases} \sup_{h^{-1}[B]=A} g(B) & \text{if } h^{-1}[B] = A \text{ for some } B \subseteq Y, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.3. *If $h: X \to Y$ is a function and g is a g-filter base on Y then $h^{-1}(g)$ is a g-filter base on X if and only if $g(B) = 0$ for all $B \subseteq Y$ such that $h^{-1}[B] = \emptyset$.*

Proof. ($\Rightarrow$) Since $h^{-1}(g)$ is a g-filter base on X we have $0 = h^{-1}(g)(\emptyset) = \sup_{h^{-1}[B]=\emptyset} g(B)$ and, so, for all $B \subseteq Y$ such that $h^{-1}[B] = \emptyset$, we have $g(B) = 0$.

($\Leftarrow$) (i) Since g is non-zero, there exists $B \subseteq Y$ such that $g(B) > 0$ and, so, $h^{-1}(g)[h^{-1}[B]] = \sup_{h^{-1}[B']=h^{-1}[B]} g(B') \geqslant g(B) > 0$.
In other words, $h^{-1}(g)$ is non-zero.
(ii) $h^{-1}(g)(\emptyset) = \sup_{h^{-1}[B]=\emptyset} g(B) = 0$.

(iii) If $A_1, A_2 \subseteq X$ then

$$\begin{aligned}
&\alpha < h^{-1}(g)(A_1) \wedge h^{-1}(g)(A_2) \\
&\Rightarrow \exists B_1, B_2 \subseteq Y\colon h^{-1}[B_1] = A_1, h^{-1}[B_2] = A_2 \\
&\qquad \text{and } \alpha < g(B_1) \wedge g(B_2) \\
&\Rightarrow \exists B_3 \subseteq B_1 \cap B_2\colon \alpha < g(B_3) \\
&\Rightarrow \exists A_3 = h^{-1}[B_3] \subseteq A_1 \cap A_2\colon \alpha < h^{-1}(g)(A_3) \\
&\Rightarrow \langle h^{-1}(g)\rangle(A_1 \cap A_2) \\
&\quad = \sup_{A_3 \subseteq A_1 \cap A_2} h^{-1}(g)(A_3) > \alpha.
\end{aligned}$$

Thus, $h^{-1}(g)(A_1) \wedge h^{-1}(g)(A_2) \leqslant \langle h^{-1}(g)\rangle(A_1 \cap A_2)$. □

Theorem 2.4. *If $h: X \to Y$ is a function and g is a g-filter base on Y then*:
(1) *if h is surjective then $h^{-1}(g)$ is a g-filter base*;
(2) *if g is a g-filter, $g(B) = 0$ for all $B \subseteq Y$ such that $h^{-1}[B] = \emptyset$ and h is injective, then $h^{-1}(g)$ is a g-filter.*
(3) *if g is prime g-filter, $g(B) = 0$ for all $B \subseteq Y$ such that $h^{-1}[B] = \emptyset$ and h is injective then $h^{-1}(g)$ is a prime g-filter.*

Proof. (1) Since h is surjective we have $h^{-1}[B] = \emptyset$ if and only if $B = \emptyset$ and so, for all $B \subseteq Y$ such that $h^{-1}[B] = \emptyset$, $g(B) = 0$.

(2) We just have to prove that, for each $A \subseteq X$,

$$\begin{aligned}
\langle h^{-1}(g)\rangle(A) &= \sup_{A' \subseteq A} h^{-1}(g)(A') \\
&= \sup_{A' \subseteq A} \sup_{h^{-1}[B] = A'} g(B) \\
&= \sup_{h^{-1}[B] \subseteq A} g(B) \leqslant h^{-1}(g)(A).
\end{aligned}$$

If $\alpha < \langle h^{-1}(g)\rangle(A)$ then there exists $B \subseteq Y$ such that $h^{-1}[B] \subseteq A$ and $\alpha < g(B)$. We consider $B' = h(A) \cup B$. Since h is injective we have $h^{-1}[B'] = h^{-1}[h(A)] \cup h^{-1}[B] = A$.

On the other hand, since g is a g-filter, we have $g(B') \geqslant g(B) > \alpha$ and so $h^{-1}(g)(A) > \alpha$.

(3) Let $A_1, A_2 \subseteq X$. If $\alpha < h^{-1}(g)(A_1 \cup A_2)$ then there exists $B \subseteq Y$ such that $h^{-1}[B] \subseteq A_1 \cup A_2$ and $\alpha < g(B)$.

We consider $B_i = (h(A_i) \cap B) \cup (B - h(X))$ for $i = 1, 2$. Since h is injective we have

$$\begin{aligned}
h^{-1}[B_i] &= h^{-1}[h(A_i) \cap B] \\
&= h^{-1}[h(A_i)] \cap h^{-1}[B] \\
&= A_i \cap h^{-1}[B] \subseteq A_i.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
B_1 \cup B_2 &= (h(A_1 \cup A_2) \cap B) \cup (B - h(X)) \\
&= (h(h^{-1}[B]) \cup (B - h(X)) = B
\end{aligned}$$

and, since g is a prime g-filter, we have either $g(B_1) > \alpha$ or $g(B_2) > \alpha$. Therefore, either $h^{-1}(g)(A_1) > \alpha$ or $h^{-1}(g)(A_2) > \alpha$ and so $h^{-1}(g)$ is prime. □

We turn our attention to the correspondence between prefilters associated with g-filters and g-filters associated with prefilters. Before we begin, we state the following lemma and leave the simple proof to the reader.

Lemma 2.5. *Let $h: X \to Y$ be a function, $\alpha \in I_1$, $\mu \in I^X$ and $\nu \in I^Y$ then*:
(1) $h(h^{-1}[\nu]) = \nu \wedge 1_{h(X)}$;
(2) $h^{-1}(h(\mu)) \geqslant \mu$;
(3) $(h(\mu))^\alpha = h(\mu^\alpha)$;
(4) $(h^{-1}[\nu])^\alpha = h^{-1}[\nu^\alpha]$.

We recall that if $h: X \to Y$, $\mathscr{F}$ a prefilter base on X and $\mathscr{G}$ a prefilter base on Y, we define

$$h(\mathscr{F}) \stackrel{\text{def}}{=} \{h(\mu)\colon \mu \in \mathscr{F}\},$$

$$h^{-1}(\mathscr{G}) \stackrel{\text{def}}{=} \{h^{-1}[\nu]\colon \nu \in \mathscr{G}\}.$$

The following lemma, some of which appears in [2, Lemma 2.11], concerns images and preimages of prefilters.

Lemma 2.6. *If $h: X \to Y$, $\mathscr{F}$ a prefilter base on X and $\mathscr{G}$ a prefilter base on Y then*:
(1) *$h(\mathscr{F})$ is a prefilter base on Y*;
(2) *if $\mathscr{F}$ is a prefilter then $\langle h(\mathscr{F})\rangle = \{\nu \in I^Y\colon h^{-1}[\nu] \in \mathscr{F}\}$*;
(3) *if $\mathscr{F}$ is a prime prefilter then $\langle h(\mathscr{F})\rangle$ is prime*;
(4) $\langle h(\mathscr{F})\rangle = \langle h[\langle \mathscr{F}\rangle]\rangle$;

(5) *if* $h^{-1}[\nu]\neq 0$ *for each* $\nu\in\mathscr{G}$ *then* $h^{-1}(\mathscr{G})$ *is a prefilter base on* X;
(6) *if* h *is surjective then* $h^{-1}(\mathscr{G})$ *is a prefilter base on* X;
(7) *if* $\mathscr{G}$ *is a prefilter,* $h^{-1}[\nu]\neq 0$ *for each* $\nu\in\mathscr{G}$ *and* h *is injective then* $h^{-1}(\mathscr{G})$ *is a prefilter on* X.

Theorem 2.7. *Let* $h: X\to Y$ *be a function and* $\mathscr{F}$ *a prefilter on* X *with* $c(\mathscr{F})=c>0$, *then*

$$\langle h(f_{\mathscr{F}})\rangle = f_{\langle h(\mathscr{F})\rangle}.$$

Proof. We just have to prove that $(\langle h(f_{\mathscr{F}})\rangle)^{\alpha} = (f_{\langle h(\mathscr{F})\rangle})^{\alpha} = (\langle h(\mathscr{F})\rangle)^{c-\alpha}$ for each $0\leqslant\alpha<c$. Now

$$\begin{aligned}(\langle h(f_{\mathscr{F}})\rangle)^{\alpha} &= \{B\subseteq Y: \langle h(f_{\mathscr{F}})\rangle(B)>\alpha\}\\ &= \{B\subseteq Y: f_{\mathscr{F}}(h^{-1}[B])>\alpha\}\\ &= \{B\subseteq Y: h^{-1}[B]\in(f_{\mathscr{F}})^{\alpha}=\mathscr{F}^{c-\alpha}\}.\end{aligned}$$

If $B\subseteq Y$ and $h^{-1}[B]\in\mathscr{F}^{c-\alpha}$, there exists $\mu\in\mathscr{F}$ and $\beta<c-\alpha$ such that $\mu^{\beta}=h^{-1}[B]$. Since $\mu\in\mathscr{F}$ we have $h(\mu)\in\langle h(\mathscr{F})\rangle$ and $h(\mu^{\beta})=(h(\mu))^{\beta}\in(\langle h(\mathscr{F})\rangle)^{c-\alpha}$.

On the other hand, $B\supseteq h(h^{-1}[B])=h(\mu^{\beta})$ and so $B\in(\langle h(\mathscr{F})\rangle)^{c-\alpha}$. Therefore, $B\in(f_{\langle h(\mathscr{F})\rangle})^{\alpha}$.

Conversely, if $B\subseteq Y$ and $B\in(\langle h(\mathscr{F})\rangle)^{c-\alpha}$, there exists $\nu\in\langle h(\mathscr{F})\rangle$ and $\beta<c-\alpha$ such that $\nu^{\beta}=B$.

Since $\nu\in\langle h(\mathscr{F})\rangle$, there exists $\mu\in\mathscr{F}$ such that $h(\mu)\leqslant\nu$. Thus we have $\mu\leqslant h^{-1}[h(\mu)]\leqslant h^{-1}[\nu]$ and so $h^{-1}[\nu]\in\mathscr{F}$.

Now $h^{-1}[B]=h^{-1}[\nu^{\beta}]=(h^{-1}[\nu])^{\beta}\in\mathscr{F}^{c-\alpha}$ and it follows that $B\in(\langle h(f)\rangle)^{\alpha}$. □

Theorem 2.8. *Let* $h: X\to Y$ *be a function and* f *a* g*-filter on* X *with* $c(f)=c>0$, *then*

$$\langle h(\mathscr{F}_f)\rangle = \mathscr{F}_{\langle h(f)\rangle}.$$

Proof. We just have to prove that $(\langle h(\mathscr{F}_f)\rangle)^{\alpha} = (\mathscr{F}_{\langle h(f)\rangle})^{\alpha} = (\langle h(f)\rangle)^{c-\alpha}$ for each $0\leqslant\alpha<c$.

Now $(\langle h(\mathscr{F}_f)\rangle)^{\alpha}=\{B\subseteq Y:\ \exists\nu\in\langle h(\mathscr{F}_f)\rangle,\ \exists\beta<\alpha$ such that $\nu^{\beta}=B\}$ and

$$\begin{aligned}(\mathscr{F}_{\langle h(f)\rangle})^{\alpha} &= (\langle h(f)\rangle)^{c-\alpha}\\ &= \{B\subseteq Y: \langle h(f)\rangle(B)>c-\alpha\}\\ &= \{B\subseteq Y: \exists A\subseteq X \text{ such that}\\ &\qquad h(A)\subseteq(B) \text{ and } f(A)>c-\alpha\}.\end{aligned}$$

Let $B=\nu^{\beta}$ with $\nu\in\langle h(\mathscr{F}_f)\rangle$ and $\beta<\alpha$. Then there exists $\mu\in\mathscr{F}_f$ such that $h(\mu)\leqslant\nu$. Therefore, $h(\mu^{\beta})=h(\mu)^{\beta}\subseteq\nu^{\beta}=B$. Now we have $A=\mu^{\beta}\subseteq X$, $h(A)\subseteq B$ and $f(A)=f(\mu^{\beta})>c-\alpha$ and hence $B\in(\mathscr{F}_{\langle h(f)\rangle})^{\alpha}$.

Conversely, let $B\subseteq Y$ have the property that there exists $A\subseteq X$ with $h(A)\subseteq B$ and $f(A)\overset{\text{def}}{=}t>c-\alpha$. Let $\mu=(c-t)1_X\vee 1_A$. We intend to invoke Lemma 5.3 of [8] to show that $\mu\in\mathscr{F}_f$. To this end, let $0\leqslant\gamma<c$.

If $\gamma\in[c-t,c)$ then $\mu^{\gamma}=A$ and so $f(\mu^{\gamma})=f(A)=t\geqslant c-\gamma$.

If $\gamma\in[0,c-t)$ then $\mu^{\gamma}=X$ and so $f(\mu^{\gamma})=f(X)=c\geqslant c-\gamma$.

We therefore have $\mu^{\gamma}\in f_{c-\gamma}$ for all $\gamma\in[0,c)$ and so $\mu\in\mathscr{F}_f$. Therefore, $h(\mu)\in h(\mathscr{F}_f)$ and $(h(\mu))^{c-t}=h(\mu^{c-t})=h(A)\in(\langle h(\mathscr{F}_f)\rangle)^{\alpha}$. Finally, since $h(A)\subseteq B$, we also have $B\in(\langle h(\mathscr{F}_f)\rangle)^{\alpha}$. □

Theorem 2.9. *Let* $h: X\to Y$ *be a function and* $\mathscr{G}$ *a prefilter on* Y *with* $c(\mathscr{G})=c>0$, *then*

$$\langle h^{-1}(f_{\mathscr{G}})\rangle = f_{\langle h^{-1}(\mathscr{G})\rangle}.$$

Proof. We just have to prove that $(\langle h^{-1}(f_{\mathscr{G}})\rangle)^{\alpha} = (f_{\langle h^{-1}(\mathscr{G})\rangle})^{\alpha} = (\langle h^{-1}(\mathscr{G})\rangle)^{c-\alpha}$ for each $0\leqslant\alpha<c$.

Now

$$\begin{aligned}(\langle h^{-1}(f_{\mathscr{G}})\rangle)^{\alpha} &= \{A\subseteq X: \exists B\subseteq Y \text{ such that}\\ &\qquad h^{-1}[B]\subseteq A \text{ and } f_{\mathscr{G}}(B)>\alpha\}\\ &= \{A\subseteq X: \exists B\subseteq Y \text{ such that}\\ &\qquad h^{-1}[B]\subseteq A \text{ and } B\in\mathscr{G}^{c-\alpha}\}.\end{aligned}$$

So let $B\subseteq Y$ with $h^{-1}[B]\subseteq A$ and $B\in\mathscr{G}^{c-\alpha}$. Then there exist $\nu\in\mathscr{G}$ and $\beta<c-\alpha$ such that $B=\nu^{\beta}$. Therefore, $h^{-1}[\nu]\in h^{-1}(\mathscr{G})$ and, since $h^{-1}[B]=h^{-1}[\nu^{\beta}]=(h^{-1}[\nu])^{\beta}\in(\langle h^{-1}(\mathscr{G})\rangle)^{c-\alpha}$ and $h^{-1}[B]\subseteq A$, we have $A\in(\langle h^{-1}(\mathscr{G})\rangle)^{c-\alpha}$.

Conversely, let $A=\mu^{\beta}$ with $\mu\in\langle h^{-1}(\mathscr{G})\rangle$ and $\beta<c-\alpha$. Then there exists $\nu\in\mathscr{G}$ such that $h^{-1}[\nu]\leqslant\mu$. Therefore, $h^{-1}[\nu^{\beta}]=(h^{-1}[\nu])^{\beta}\subseteq\mu^{\beta}=A$. Now we have $B=\nu^{\beta}\subseteq Y$, $h^{-1}[B]\subseteq A$ and $B\in\mathscr{G}^{c-\alpha}$ and so $A\in(\langle h^{-1}(f_{\mathscr{G}})\rangle)^{\alpha}$. □

Theorem 2.10. *Let* $h: X\to Y$ *be a function and* g *a* g*-filter on* Y *with* $c(g)=c>0$, *then*:

$$\langle h^{-1}(\mathscr{F}_g)\rangle = \mathscr{F}_{\langle h^{-1}(g)\rangle}.$$

Proof. We just have to prove that $(\langle h^{-1}(\mathscr{F}_g)\rangle)^\alpha = (\mathscr{F}_{\langle h^{-1}(g)\rangle})^\alpha = (\langle h^{-1}(g)\rangle)^{c-\alpha}$ for each $0 \leqslant \alpha < c$.

Now $(\langle h^{-1}(\mathscr{F}_g)\rangle)^\alpha = \{A \subseteq X : \exists \mu \in \langle h^{-1}(\mathscr{F}_g)\rangle, \exists \beta < \alpha$ such that $\mu^\beta = A\}$ and

$$\begin{aligned}(\mathscr{F}_{\langle h^{-1}(g)\rangle})^\alpha &= (\langle h^{-1}(g)\rangle)^{c-\alpha} \\ &= \{A \subseteq X : \exists A' \subseteq A \text{ such that } \\ &\quad h^{-1}(g)(A') > c - \alpha\} \\ &= \{A \subseteq X : \exists B \subseteq Y \text{ such that } \\ &\quad h^{-1}[B] \subseteq A \text{ and } g(B) > c - \alpha\}.\end{aligned}$$

So let $A = \mu^b$ with $\mu \in \langle h^{-1}(\mathscr{F}_g)\rangle$ and $\beta < \alpha$. Then, there exists $v \in \mathscr{F}_g$ such that $h^{-1}[v] \leqslant \mu$. Therefore, $h^{-1}[v^\beta] = (h^{-1}[v])^\beta \subseteq \mu^\beta = A$. Now, we have $B = v^\beta \subseteq Y$, $h^{-1}[B] \subseteq A$ and $g(B) = g(v^\beta) > c - \alpha$. Thus $A \in (\mathscr{F}_{\langle h^{-1}(g)\rangle})^\alpha$.

Conversely, let $B \subseteq Y$ with $h^{-1}[B] \subseteq A$ and $g(B) \stackrel{\text{def}}{=} t > c - \alpha$. Let $v = (c-t)1_Y \vee 1_B$. We intend to invoke Lemma 5.3 from [8] to show that $v \in \mathscr{F}_g$. To this end, let $0 \leqslant \gamma < c$.

If $\gamma \in [c-t, c)$ then $v^\gamma = B$ and so $g(v^\gamma) = g(B) = t \geqslant c - \gamma$.

If $\gamma \in [0, c-t)$ then $v^\gamma = Y$ and so $g(v^\gamma) = g(Y) = c \geqslant c - \gamma$.

We therefore have $v^\gamma \in g_{c-\gamma}$ for all $\gamma \in [0, c)$ and so $v \in \mathscr{F}_g$. Therefore, $h^{-1}[v] \in h^{-1}(\mathscr{F}_g)$ and $(h^{-1}[v])^{c-t} = h^{-1}[v^{c-t}] = h^{-1}[B] \in (\langle h^{-1}(\mathscr{F}_g)\rangle)^\alpha$. Finally, since $h^{-1}[B] \subseteq A$, we also have $A \in (\langle h^{-1}(\mathscr{F}_g)\rangle)^\alpha$. $\square$

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