Monotone normality from a pointfree point of view

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University of the Basque Country, UPV/EHU

Almería, 25 de junio de 2014

Monotone normality, quasi-metrizable spaces and the role of the T_1 axiom

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- Joint work with:
Iraide Mardones Pérez, *María Ángeles de Prada Vicente*,
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  Not even T_1!

Historical account

- The notion of monotone normality was introduced in 1966 by Borges an named in 1970 by Zenor as a strengthening of normality and is probably what you would guess if asked to define "normal in a monotone way".

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- The notion appeared in the context of generalizations of metrizability. (Probably this is the reason why monotonically normal spaces are usually assumed to be T_1, hence Hausdorff. Note that normal spaces are not necessarily Hausdorff!)
- Every metrizable and every linearly ordered space is monotonically normal. (So monotone normality is not a strange condition.
  In fact, it can be argued that if a space can be "explicitly" and "constructively" shown to be normal, then it is probably monotonically normal.

What is monotone normality?

Monotonically normal space

From Wikipedia, the free encyclopedia

In mathematics, a **monotonically normal space** is a particular kind of normal space, with some special characteristics, and is such that it is hereditarily normal, and any two separated subsets are strongly separated. They are defined in terms of a monotone normality operator.

A T_1 topological space $(X, \mathcal{T})$ is said to be *monotonically normal* if the following condition holds:

For every $x \in G$, where G is open, there is an open set $\mu(x, G)$ such that

1. $x \in \mu(x, G) \subseteq G$
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There are some equivalent criteria of monotone normality.

But this is not what one would guess if asked to define "normal in a monotone way"!!

Equivalent definitions [edit]

Definition 2 [edit]

A space X is called monotonically normal if it is T_1 and for each pair of disjoint closed subsets A, B there is an open set $G(A, B)$ with the properties

1. $A \subseteq G(A,B) \subseteq G(A,B)^- \subseteq X \setminus B$ and
2. $G(A,B) \subseteq G(A',B')$, whenever $A \subseteq A'$ and $B' \subseteq B$.

This operator G is called **monotone normality operator**.

Note that if G is a monotone normality operator, then $\tilde{G}$ defined by $\tilde{G}(A,B) = G(A,B) \setminus G(B,A)^-$ is also a monotone normality operator; and $\tilde{G}$ satisfies

$$\tilde{G}(A,B) \cap \tilde{G}(B,A) = \emptyset$$

For this reason we some time take the monotone normality operator so as to satisfy the above requirement; and that facilitates the proof of some theorems and of the equivalence of the definitions as well.

Properties [edit]

An important example of these spaces would be, assuming Axiom of Choice, the linearly ordered spaces; however, it really needs axiom of choice for an arbitrary linear order to be normal (see van Douwen's paper). Any generalised metric is monotonically normal even without choice. An important property of monotonically normal spaces is that any two separated subsets are strongly separated there. Monotone normality is hereditary property and a monotonically normal space is always normal by the first condition of the second equivalent definition.

We list up some of the properties :

1. A closed map preserves monotone normality.
2. A monotonically normal space is hereditarily collectionwise normal.
3. Elastic spaces are monotonically normal.

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From Wikipedia: $T_4 \equiv$ normal $+\ T_1$

Definitions [edit]

A topological space *X* is a **normal space** if, given any disjoint closed sets *E* and *F*, there are open neighbourhoods *U* of *E* and *V* of *F* that are also disjoint. More intuitively, this condition says that *E* and *F* can be separated by neighbourhoods.

A **T_4 space** is a T_1 space *X* that is normal; this is equivalent to *X* being Hausdorff and normal.

A **completely normal space** or a **hereditarily normal space** is a topological space *X* such that every subspace of *X* with subspace topology is a normal space. It turns out that *X* is completely normal if and only if every two separated sets can be separated by neighbourhoods.

A **completely T_4 space**, or **T_5 space** is a completely normal Hausdorff topological space *X*; equivalently, every subspace of *X* must be a T_4 space.

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- Metrizable spaces are monotonically normal (and T_1). What about quasi-metrizable spaces?
- Normality is a well-stablished topic in Pointfree Topology. What about monotone normality?
  Certainly this must be done avoiding the T_1 axiom, a "very point-dependent axiom".

What is meant by a monotonization of a topological property?

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- Monotone countable paracompactness
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metric spaces

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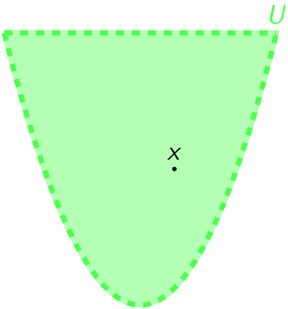

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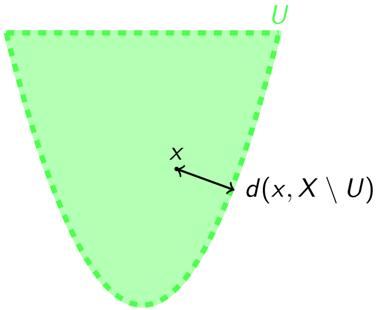

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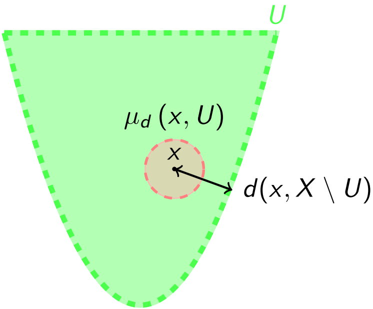

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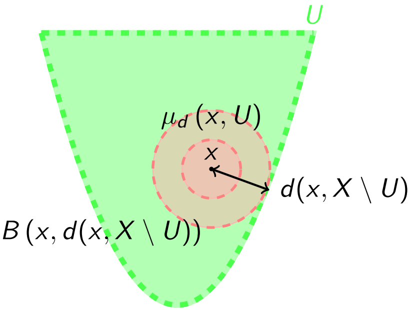

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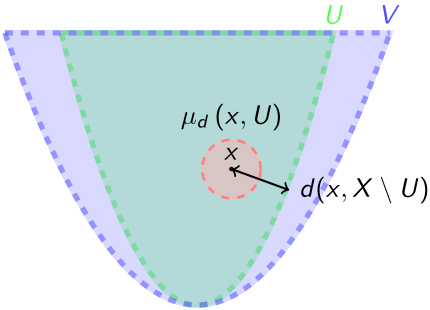

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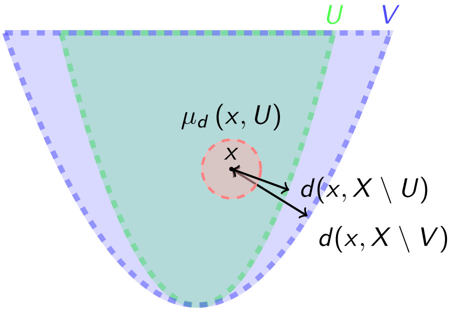

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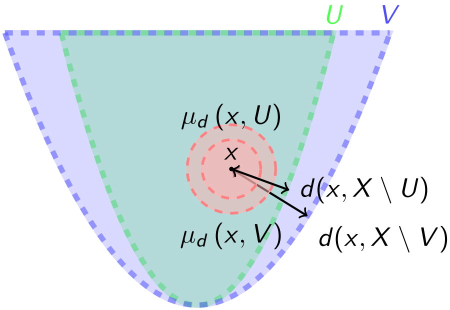
Metrics spaces are regular

(A space is regular if for each open set U and each $x \in U$ there exists an open set $\mu(x, U)$ such that $x \in \mu(x, U) \subseteq \overline{\mu(x, U)} \subseteq U$.)

Given a metric space (X, d) and open set $U \subseteq X$ and $x \in U$...
since $X \setminus U$ is closed and $x \notin X \setminus U$we have that $d(x, X \setminus U) > 0$...
we take $\mu_d(x, U) = B\left(x, \frac{d(x,X\setminus U)}{2}\right)$...
and then $x \in \mu_d(x, U) \subseteq \overline{\mu_d(x, U)} \subseteq \overline{B}\left(x, \frac{d(x,X\setminus U)}{2}\right) \subseteq B(x, d(x, X \setminus U)) \subseteq U$.



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Metrics spaces are regular in a "monotone way"

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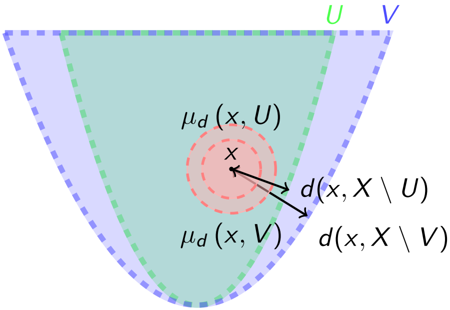

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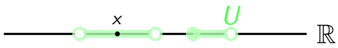

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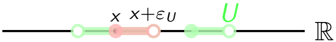

$\mu(x, U) = [x, x + \varepsilon_U)$

where ε_U is the biggest ε such that $[x, x + \varepsilon) \subseteq U$

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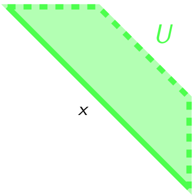

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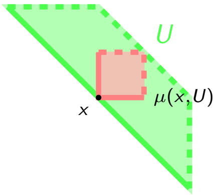

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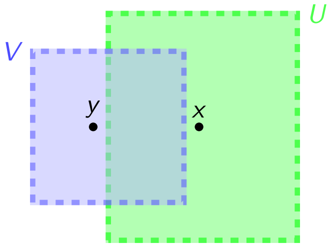

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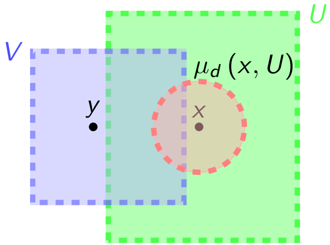

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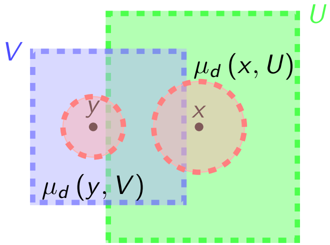

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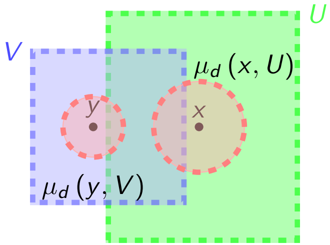

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Equivalently,

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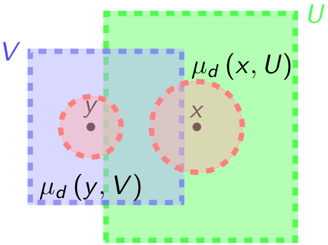

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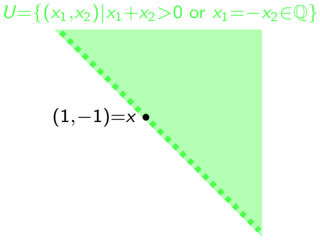

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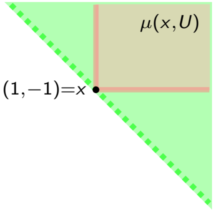

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What is monotone normality? strong monotone regularity

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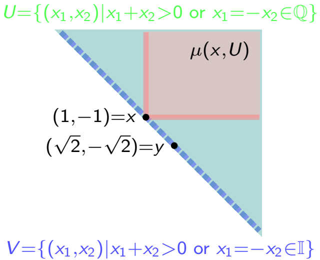

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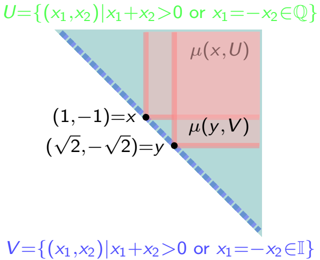

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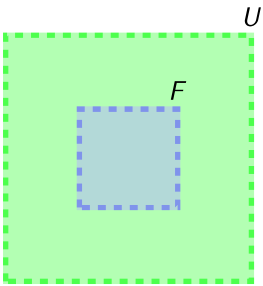

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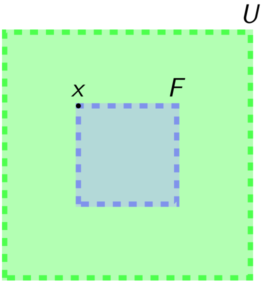

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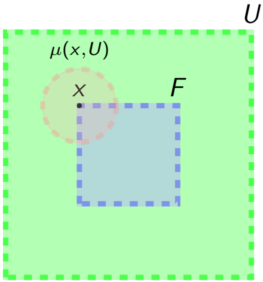

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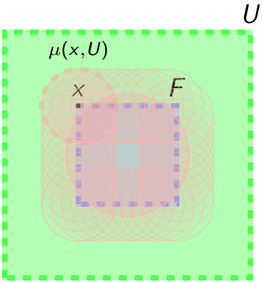

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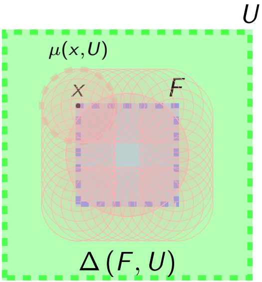

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Consequently the two definitions are equivalent, but only for T_1 spaces!

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(For T_1 spaces they are all equivalent.)

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  Suppose A is a closed subspace of a monotonically normal space X. Then there is a function $\Phi_A \colon C(A, [0,1]) \to C(X, [0,1])$ such that:

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Example

Let $(X, o(X))$ be an a rbitrary space and $Y = X \cup \{\infty\}$ with $\infty \notin X$ the one-point extension of X with topology $o(Y) = o(X) \cup \{Y\}$.

- $(Y, o(Y))$ is trivially monotonically normal (but not T_1).
- The subspace topology on X is $o(X)$.

If $(X, o(X))$ fails to be monotonically normal we have the desired counterexample.

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  What about the monotonically normal analogue of the Tietze-Urysohn theorem?

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  Hence it is natural to try to study which quasi-metrizable spaces are monotonically normal.

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  If time permits I will present some ideas at the end of the talk...

Monotone normality without T_1 axiom

Every topological space X induces, in a natural way, a partial order $\leq$ on X (called the specialization order) defined by $y \leq x \iff y \in \overline{\{x\}}$.

For each $x \in X$ we shall also denote $\downarrow x = \{y \in X \mid y \leq x\} = \overline{\{x\}}$.

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Theorem (Characterization of MN without T_1)

Let X be a topological space. The following are equivalent:

(1) *X is monotonically normal;*

(2) *There is an assignment of an open set $\mu(x, U)$ to each pair (x, U) such that U is an open neighborhood of $\downarrow x$, in such a way that*

- (i) $\downarrow x \subseteq \mu(x, U) \subseteq \overline{\mu(x, U)} \subseteq U$;
- (ii) *if* $x \leq y$ *and* $U \subseteq V$, *then* $\mu(x, U) \subseteq \mu(y, V)$.
- (iii) *if* $\mu(x, U) \cap \mu(y, V) \neq \varnothing$ *then either* $x \in V$ *or* $y \in U$.

J.G.G., I. Mardones-Pérez and M.A. de Prada Vicente, *Monotone normality free of T_1 axiom*, Acta Math. Hungar. (2009).

As a corollary of the previous characterization, and in connection with hereditary monotone normality we have the following:

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Theorem

Given a space X we denote $UL(X) = \{(f, g) \in USC(X) \times LSC(X, L) \mid f \leq g\}$.

A space X is monotonically normal if and only if there exists an order-preserving function $\Lambda\colon UL(X) \to C(X)$ such that $f \leq \Lambda(f, g) \leq g$ for any $(f, g) \in UL(X)$.

T. Kubiak, *Monotone insertion of continuous functions*, Q & A in General Topology (1995).

It must be emphasized here that T. Kubiak was the first in studying monotone normality for non T_1 spaces. The result previous result is valid for non T_1 spaces!

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Combining this theorem with the previous result we obtain the following:

Theorem

A space X is monotonically normal if and only if for each closed $A \subseteq X$ there exists a function $\Phi_A \colon C(A,[0,1]) \to C(X,[0,1])$ such that:

(1) *for each $f \in C(A,[0,1])$, $\Phi_A(f)$ extends f;*

(2) *if $f,g \in C(A,[0,1])$ and $f \leq g$ in A, then $\Phi_A(f) \leq \Phi_A(g)$ in X;*

(3) *If $A_1 \subseteq A_2$ are closed and $f_i \colon C(A_i,[0,1])$ are such that $f_{2|A_1} \geq f_1$ and $f_2(x) = 1$ for any $x \in A_2 \setminus A_1$, then $\Phi_{A_2}(f_2) \geq \Phi_{A_1}(f_1)$.*

(4) *If $A_1 \subseteq A_2$ are closed and $f_i \colon C(A_i,[0,1])$ are such that $f_{2|A_1} \leq f_1$ and $f_2(x) = 0$ for any $x \in A_2 \setminus A_1$, then $\Phi_{A_2}(f_2) \leq \Phi_{A_1}(f_1)$.*

Quasi-metrizable spaces

Let X be a non-empty set. A map $d\colon X \times X \to [0, +\infty)$ is a quasi-metric if the following two conditions hold for all $x, y, z \in X$:

(QM1) $d(x, y) = d(y, x) = 0$ if and only if $x = y$;

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Every quasi-metric d generates a T_0 topology τ_d which has as a base the family of d-balls $\{B_d(x, \varepsilon) \mid x \in X, \varepsilon > 0\}$, where

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The specialization order $\leq_d$ on X is given by

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$$d(x,y) = \begin{cases} \sup\{y_1 - x_1, y_2 - x_2\} \wedge 1, & \text{if } x_1 \leq y_1 \text{ and } x_2 \leq y_2; \\ 1, & \text{otherwise.} \end{cases}$$

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In this sense it could be mentioned, citing from:

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If the quasi-metric space is T_1 we have the following characterization:

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Let (X, d) be a T_1 quasi-metric space. The following are equivalent:

(1) (X, τ_d) *is monotonically normal;*

(2) *There exists a map* $h\colon X \times (0, +\infty) \to (0, +\infty)$ *such that:*

- (h1) $0 < h(x, \varepsilon) \leq \varepsilon$;
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Let (X, d) be a T_1 quasi-metric space and $k \in (0, 1]$ such that:

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J.G.G., S. Romaguera and J.M. Sánchez-Álvarez, *Quasi-metrics and monotone normality*, Topology Appl. (2011).

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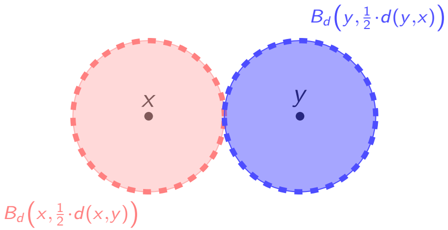

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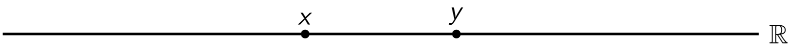

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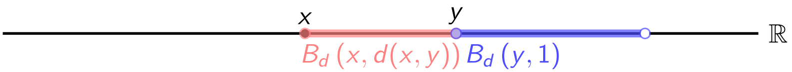

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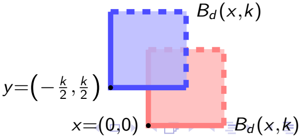
Note that in the case of the Sorgenfrey plane, for each $k \in (0, 1]$ one can choose $x = (0, 0)$ and $y = (-\frac{k}{2}, \frac{k}{2})$, then $d(x, y) = 1$ and so

$$B_d(x, k \cdot d(x, y)) \cap B_d(y, k \cdot d(y, x)) \neq \varnothing.$$



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In this case the previous proposition is, once again, nothing but the well known fact that metrizable spaces are monotonically normal.

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- The complexity (quasi-metric) space $(\mathcal{C}, d_{\mathcal{C}})$.
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A space X is said to be:

- subfit if for each $U \in o(X)$ and $x \in U$ there exists $y \in \overline{\{x\}}$ with $\overline{\{y\}} \subseteq U$.
- weakly regular if for each $U \in o(X)$ and $x \in U$, $\overline{\{x\}} \subseteq U$.

Lemma

Let X be a T_0 normal space. Then:

$$X \text{ is } T_2 \iff X \text{ is } T_1 \iff X \text{ is weakly regular} \iff X \text{ is subfit.}$$

Proposition

Let X be a subfit topological space. The following are equivalent:

(1) X is monotonically normal.

(2) X has a Borges operator.

J.G.G., J. Picado and M.A. de Prada Vicente, *Monotone normality and stratifiability from a pointfree point of view*, Topology Appl. (2014).

Monotone normality in Pointfree Topology

A space X is subfit if and only if

given $U, V \in o(X)$ s.t. $U \not\subseteq V$ there exists W with $U \cup W = X \neq V \vee W$.

$\ldots$

$\ldots$

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Thank you!