Hutton $[\mathbf{0}, \mathbf{1}]$-(quasi-)uniformities induced by fuzzy (quasi-)metric spaces

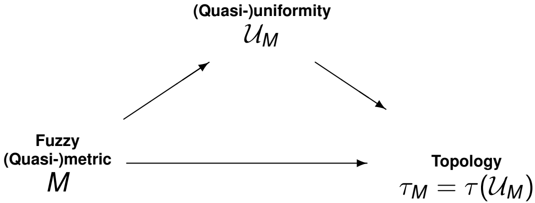
J. Gutiérrez García and M.A. de Prada Vicente

Universidad del País Vasco-Euskal Herriko Unibertsitatea

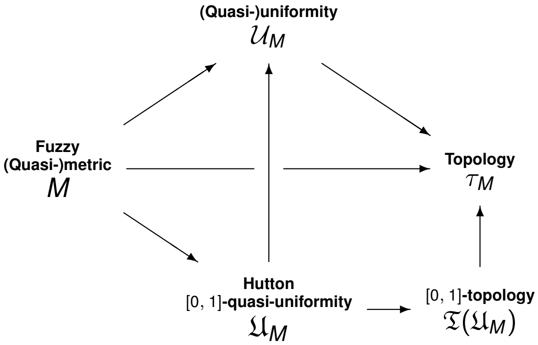
Almería, 7 de junio de 2005

CIMMA
CONGRESO INTERNACIONAL MEDITERRÁNEO DE MATEMÁTICAS ALMERÍA 2005

Original diagram



Our diagram


(Quasi-)uniformity
$\mathcal{U}_M$
Fuzzy
(Quasi-)metric
M
Topology
τ_M
Hutton
$[0, 1]$-quasi-uniformity
$\mathfrak{U}_M$
$[0, 1]$-topology
$\mathfrak{T}(\mathfrak{U}_M)$

Probabilistic metrics

A *Menger probabilistic metric space* is a set X and a mapping $\mathfrak{F}$ from $X \times X$ to the set of all nonnegative probability distribution functions with the following properties (we shall denote $\mathfrak{F}(x, y) = \mathfrak{F}_{xy}$): (for all $x, y, z \in X$ and $r, s \geq 0$)

(PM1) $\mathfrak{F}_{xy}(r) = 1$ for all $r > 0$ if and only if $x = y$;

(PM2) $\mathfrak{F}_{xy} = \mathfrak{F}_{yx}$;

(PM3) $\mathfrak{F}_{xz}(r + s) \geq \mathfrak{F}_{xy}(r) * \mathfrak{F}_{yz}(s)$.

where $* : [0, 1] \times [0, 1] \to [0, 1]$ is a t-norm.

Strong uniformity and strong-topology (I)

The *strong uniformity* is defined through the uniform basis

$$\mathcal{U} = \{U(\varepsilon) \,:\, \varepsilon > 0\},$$

where $U(\varepsilon) = \{(x, y) \in X \times X \,:\, \mathfrak{F}_{x,y}(\varepsilon) > 1 - \varepsilon\}$.

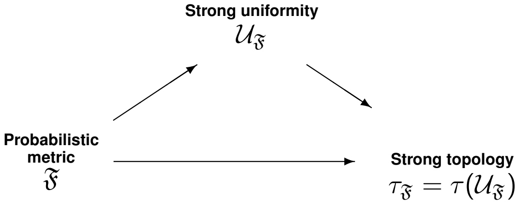
The *strong topology* or (ε, λ)*-topology* is the topology induced by the strong uniformity, i.e. it is defined through the following neighbourhood base:

$$\mathcal{N}_x = \{N_x(\varepsilon, \lambda) \,:\, \varepsilon > 0, \lambda \in (0, 1]\},$$

where $N_x(\varepsilon, \lambda) = \{y \in X \,:\, \mathfrak{F}_{x,y}(\varepsilon) > 1 - \lambda\}$.

CIMMA
CONGRESO INTERNACIONAL MEDITERRÁNEO DE MATEMÁTICAS ALMERÍA 2005

Strong uniformity and strong topology (II)



There are two properties essential to obtain a topological space derived from a probabilistic metric space:

(i) $\mathfrak{F}_{x,y}(t) > r \implies \exists t' < t$ such that $\mathfrak{F}_{x,y}(t') > r$
(it follows from left-continuity of the distribution function);

(ii) $\forall r \in [0, 1) \quad \exists t < 1$ such that $t * t \geq r$
(it follows from $\sup_{t<1} t * t = 1$;
a consequence of $*$ being left- continuous).

Fuzzy (quasi-)metric (I)

In the nineties, George and Veeramani introduced a notion of *fuzzy metric* as follows:

Given a continuous t-norm $*$, a fuzzy metric M on a set X is a fuzzy set in $X \times X \times (0, +\infty)$ satisfying the following conditions: (for $x, y, z \in X$ and all $r, s > 0$)

(FM1) $M(x, y, r) > 0$;

(FM2) $M(x, y, r) = 1$ for all $r > 0$ if and only if $x = y$;

(FM3) $M(x, y, r) = M(y, x, r)$;

(FM4) $M(x, z, r + s) \geq M(x, y, r) * M(y, z, s)$;

(FM5) $M(x, y, \cdot)$ is continuous.

By dropping axiom (FM3), Gregori and Romaguera defined and studied the notion of a *fuzzy quasi-metric space*.

Fuzzy (quasi-)metric (II)

This notion is proved to be closely related to that of probabilistic metric spaces. The technical support for this relation is the *exponential law*, which allows to consider a function $M : X \times X \times (0, +\infty) \to [0, 1]$ as a function $\mathfrak{F} : X \times X \to [0, 1]^{(0,+\infty)}$, defined by $\mathfrak{F}(x, y)(t) = \mathfrak{F}_{xy}(t) = M(x, y, t)$.

It deserves to be mentioned here that the difference between fuzzy metric spaces and probabilistic metric spaces is that $\mathfrak{F}_{xy}$ is strictly positive in $(0, \infty)$ and continuous (not only left-continuous) and the condition $\lim_{t\to\infty} \mathfrak{F}_{xy}(t) = 1$ is not necessarily satisfied.

t-norms

Let us recall that a binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a *(left-)continuous t-norm* provided that it satisfies the following conditions:

(i) $*$ is associative and commutative;

(ii) $*$ is (left-)continuous;

(iii) $a * 1 = a$ for every $a \in [0,1]$;

(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.

Note that left-continuity of $*$ implies that $*$ is distributive over arbitrary sups, i.e. for $\alpha \in [0,1]$ and $\{\beta_i\}_{i \in J} \subset [0,1]$

$$\alpha * \Big(\bigvee_{i \in J} \beta_i \Big) = \bigvee_{i \in J} (\alpha * \beta_i).$$

CIMMA
CONGRESO INTERNACIONAL MEDITERRÁNEO DE MATEMÁTICAS ALMERÍA 2005

Strictly two-sided, commutative quantales

A *strictly two-sided, commutative quantale* (or a an *integral, commutative cl-monoid*) is a triple $(L, \leq, *)$ such that:

- $(L, \leq)$ is a complete lattice.
- $(L, *)$ is a commutative monoid such that the universal upper (resp. lower) bound $\top$ (resp. $\bot$) acts as unit (resp. zero) element.
- $*$ is distributive over arbitrary joins in $(L, \leq)$, i.e.

  $$\alpha * \Big(\bigvee_{i \in J} \beta_i\Big) = \bigvee_{i \in J} (\alpha * \beta_i) \quad \text{for all } \alpha \in L \text{ and } \{\beta_i\}_{i \in J} \subset L,$$

  where J stands for any index set.

Residuation

Every commutative quantale $(L, \leq, *)$ is *residuated* - i.e. there exist a binary operation $\rightarrow$ on L satisfying the following axiom

$$\alpha * \gamma \leq \beta \iff \gamma \leq \alpha \overset{*}{\rightarrow} \beta$$

for $\alpha, \beta, \gamma \in L$.

In particular the *implication* $\overset{*}{\rightarrow}$ is given by

$$\alpha \overset{*}{\rightarrow} \beta = \bigvee \{\gamma \in L : \alpha * \gamma \leq \beta\}$$

for all $\alpha, \beta \in L$.

Fundamental t-norms

In the case of the fundamental continuous t-norms $\wedge$, Prod and T_m (defined as $T_m(\alpha,\beta)=\max\{\alpha+\beta-1,0\}$ for each $\alpha,\beta\in L$), the corresponding implications are defined, respectively, as

$$\alpha \xrightarrow{\wedge} \beta = \begin{cases} \beta, & \text{if } \alpha>\beta; \\ 1, & \text{if } \alpha\le\beta; \end{cases} \qquad \alpha \xrightarrow{\text{Prod}} \beta = \begin{cases} \frac{\beta}{\alpha}, & \text{if } \alpha>\beta; \\ 1, & \text{if } \alpha\le\beta; \end{cases}$$

and

$$\alpha \xrightarrow{T_m} \beta = \begin{cases} \beta-\alpha+1, & \text{if } \alpha>\beta; \\ 1, & \text{if } \alpha\le\beta; \end{cases}$$

CONGRESO INTERNACIONAL MEDITERRÁNEO DE MATEMÁTICAS ALMERÍA 2005

Enlarging and arbitrary join-preserving maps from L^X into L^X

Let X be a set and $(L, \leq)$ a complete lattice. We denote by $\mathcal{H}_L(X)$ the collection of all enlarging and *arbitrary* join-preserving mappings from L^X into L^X, i.e. $\mathcal{H}_L(X)$ is that subset $(L^X)^{L^X}$ whose members W satisfy for each $a \in L^X$ and $\{a_i\}_{i \in J} \subset L^X$:

(W1) $W(a) \geq a$ (Enlarging)

(W2) $W\left(\bigvee_{i \in J} a_i\right) = \bigvee_{i \in J} W(a_i)$ (Join-preserving)
and $W(1_\varnothing) = 1_\varnothing$.

Note that if $L = \mathbf{2} = \{0, 1\}$, $\mathcal{H}_L(X)$ can be identified with the collection of all subsets of $X \times X$ containing the diagonal.

Hutton L-uniformities

Let $(L, \leq, ')$ be a complete lattice. A *Hutton L-quasi-uniformity* on X is a nonempty subset $\mathcal{U}$ of $\mathcal{H}_L(X)$ such that

(HU1) if $U \in \mathcal{U}$, $U \leq V$ and $V \in \mathcal{H}_L(X)$ then $V \in \mathcal{U}$;

(HU2) if $U, V \in \mathcal{U}$, there exists $W \in \mathcal{U}$ such that $W \leq U$ and $W \leq V$;

(HU3) if $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \leq U$ (where $\circ$ denotes the usual composition of functions).

A Hutton L-quasi-uniformity is called a *Hutton L-uniformity* if it additionally satisfies:

(HU4) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$.

Induced Hutton $[\mathbf{0}, \mathbf{1}]$(-quasi)-uniformity (I)

Construction: Let $(X, M, *)$ be a fuzzy quasi-metric space, $\varepsilon \in (0, 1]$ and $t > 0$ and define $W^M_{\varepsilon,t} : [0, 1]^X \to [0, 1]^X$ as

$$W^M_{\varepsilon,t}(\alpha * 1_{\{x\}})(y) = \alpha * \big((1 - \varepsilon) \to M(x, y, t)\big)$$

for each $x \in X$ and $\alpha \in (0, 1]$ and

$$W^M_{\varepsilon,t}(a) = \bigvee_{x \in X} W^M_{\varepsilon,t}\big(a(x) * 1_{\{x\}}\big)$$

for each $a \in [0, 1]^X$.
(Where by $\alpha * 1_{\{x\}} \in [0, 1]^X$ we denote the mapping defined as α in x and 0 otherwise).

CIMMA CONGRESO INTERNACIONAL MEDITERRÁNEO DE MATEMÁTICAS ALMERÍA 2005

Induced Hutton $[\mathbf{0}, \mathbf{1}]$-quasi-uniformity (II)

Result: The family $\mathfrak{B}_M = \left\{ W^M_{\varepsilon,t} : \varepsilon \in (0,1], t > 0 \right\}$ is a base for a Hutton $[0,1]$-quasi-uniformity on X.

We shall denote by $\mathfrak{U}_M$ the quasi-uniformity generated by $\mathfrak{B}_M$ and call it the *Hutton $[0,1]$-quasi-uniformity induced by M*.

Moreover, in the particular case $* = T_m$, we have that

$$\left(W^M_{\varepsilon,t}\right)^{-1} = W^M_{\varepsilon,t} \quad \text{for each } \varepsilon \in (0,1] \text{ and } t > 0 \text{ and}$$

consequently, if (X, M, T_m) is a fuzzy metric space, then $\mathfrak{U}_M$ is a Hutton $[0,1]$-uniformity

Lowen and Katsaras functors

The study of the relation between classical and fuzzy structures of topological nature was initiated by Lowen. He introduced the well-known adjoint functors

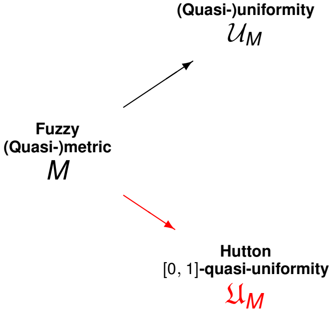
$$\omega : \textbf{TOP} \to [0,1]\text{-}\textbf{TOP} \qquad \text{and} \qquad \iota : [0,1]\text{-}\textbf{TOP} \to \textbf{TOP}.$$

In what respects to $[0,1]$-(quasi-)uniform spaces (in the sense of Hutton), it was Katsaras who explicit the relation between the category **(Q)UNIF** of (quasi-)uniform spaces and that of Hutton $[0,1]$-(quasi-)uniform spaces, $[0,1]$-**(Q)UNIF**. He defined the adjoint functors

$$\Phi : \textbf{(Q)UNIF} \to [0,1]\text{-}\textbf{(Q)UNIF} \text{ and } \Psi : [0,1]\text{-}\textbf{(Q)UNIF} \to \textbf{(Q)UNIF}.$$

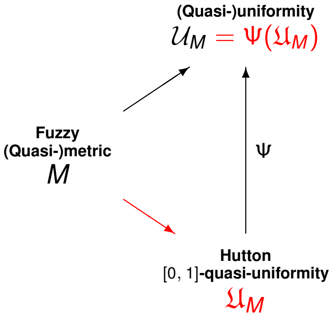
Commutativity with respect to Katsaras' functor

Result: Given a fuzzy (quasi-)metric space $(X, M, *)$, the uniformity $\mathcal{U}_M$ is precisely the image under Katsaras' functor of $\mathfrak{U}_M$, i.e. we have the following commutative diagram:



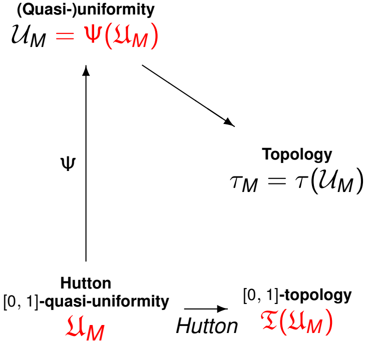
Commutativity with respect to Katsaras' functor

Result: Given a fuzzy (quasi-)metric space $(X, M, *)$, the uniformity $\mathcal{U}_M$ is precisely the image under Katsaras' functor of $\mathfrak{U}_M$, i.e. we have the following commutative diagram:



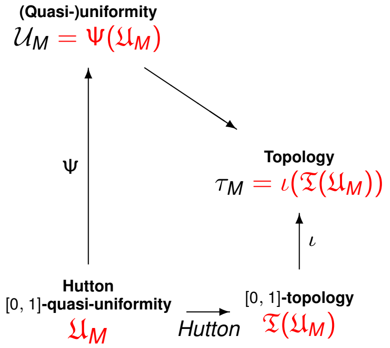
Commutativity with respect to Lowen's functor

Result: Given a fuzzy (quasi-)metric space $(X, M, *)$, the topology τ_M is precisely the image under Lowen's functor of the $[0,1]$-topology induced by $\mathfrak{U}_M$, i.e. we have the following commutative diagram:

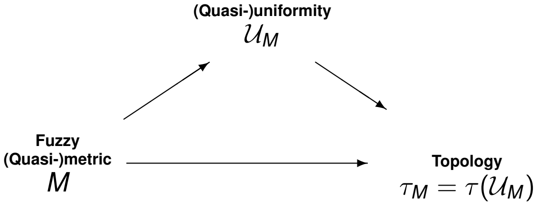


Commutativity with respect to Lowen's functor

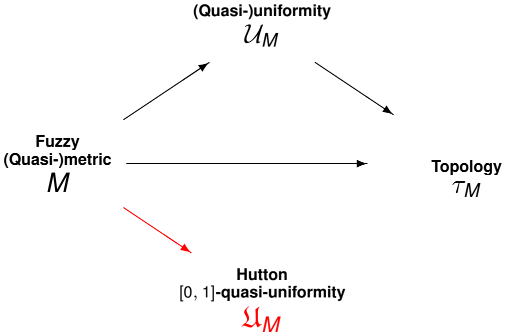
Result: Given a fuzzy (quasi-)metric space $(X, M, *)$, the topology τ_M is precisely the image under Lowen's functor of the $[0,1]$-topology induced by $\mathfrak{U}_M$, i.e. we have the following commutative diagram:



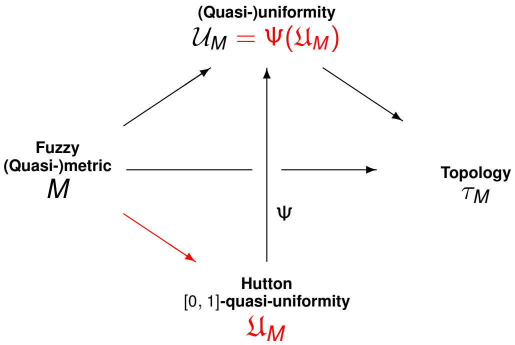
Original diagram



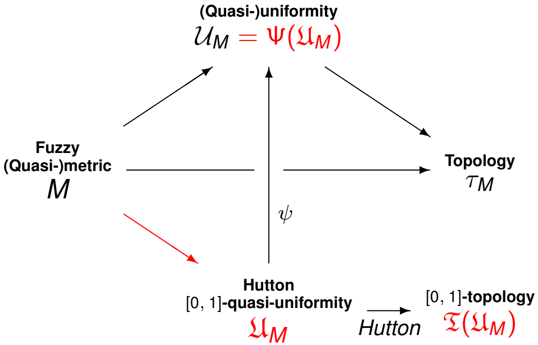
Our construction



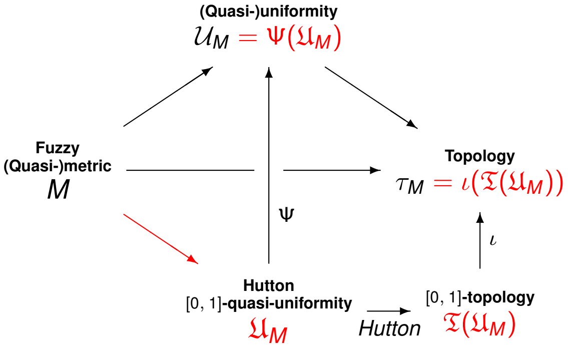
Commutativity with Katsaras' functor



Commutativity with Lowen's functor (I)



Commutativity with Lowen's functor (II)



Final diagram