Proceedings of the Workshop in Applied Topology WiAT'09, pp. 29 – 38

From the real numbers to domains

María Angeles de Prada Vicente

Departamento de Matemáticas, Universidad País Vasco (UPV-EHU), Leioa, Spain
(mariangeles.deprada@ehu.es)

Abstract

The purpose of this talk is to identify those mathematical features of the order of the real numbers that are essential when dealing with real–valued functions; abstract and axiomatize these features, and show how lattice theory (domain theory) can provide an appropriate framework for general constructions in the theory of lattice-valued functions. Our attention will be focussed on the generation, insertion and extension of functions satisfying some type of continuity. As an application, some well-known classes of topological spaces will be identified.

1. Introduction

Starting with real-valued functions, we will go to lattice-valued functions in a generalization which pretends to keep many interesting results and applications, mainly in what concerns to generation, insertion and extension of such functions.

All the results presented here are part of a joint research with Tomasz Kubiak and Javier Gutiérrez García and have already been published (cf. [6, 7, 8]).

Among the related questions we are going to deal with, are those order and topological aspects which provide conditions for generating, inserting and extending lattice-valued functions. On the way, some restrictions will be imposed, if necessary, either on the functions themselves or on their domains.

Our model will be real-valued functions together with the known results about the above mentioned topics: generation, insertion and extension. Capturing the essence of the model will serve us to select a class of lattices as codomain that will provide the expected results.

A first look to real-valued functions led M.H. Stone in 1949 ([12]) to realise that they are completely determined by the collection of sets $\{[f \prec t] := \{x \in X : f(x) \prec t\} : t \in \mathbb{R}\}$ where $\prec \in \{\leq, \geq, <, >\}$. Indeed, given $f : X \to \mathbb{R}$, we have

$$\begin{aligned} f(x) &= \inf\{t \in \mathbb{R} : x \in [f < t]\} &&= \inf\{t \in \mathbb{R} : x \in [f \leq t]\} \\ &= \sup\{t \in \mathbb{R} : x \in [f > t]\} &&= \sup\{t \in \mathbb{R} : x \in [f \geq t]\}. \end{aligned}$$

The index set $\mathbb{R}$ can be substituted by any dense subset (particular interest will have the countable dense subset $\mathbb{Q}$).

Stone called the previous collection the spectral family of the function, because of the resemblance with the spectral family for a self-adjoint operator in Hilbert spaces.

It is natural to try to isolate those properties needed to determine the function or the aspects of topological/order nature. Once we abstract those properties, if well done, we should be able to recover functions as well as their order and topological-type properties.

This is what Stone did in [12], developing a method which substituted successfully real-valued functions for real-indexed (or yet rational-indexed) families of subsets of the domain. Simple properties of functions are reflected by simple properties of their spectral families, and when the domain is a topological space there are connections between topological properties of the function and topological properties of its spectral family.

2. Lattices

The power of Stone's method relies in elementary, but deep, properties of the usual order of the reals, such as interpolation (strict in the case of the strictly less than relation) and approximation (both $\bigvee$-approximation and $\bigwedge$-approximation). In fact, most of the proofs in which real valued functions are involved depend on these two properties.

So, our objective will be to introduce in a complete lattice L new relations stronger than the lattice order, and select the lattices which have with respect to those relations, the two properties (interpolation and approximation) needed for most of the interesting results for real-valued functions. Let us mention two well know classes of lattices satisfying the above conditions: continuous lattices with the way-below relation and completely distributive lattices with the wedge-below relation (or well inside relation). An advantage of the second ones is their self-duality, advantage not shared by continuous lattices.

Definition 1. Given a complete lattice L and $a, b \in L$, we write

(1) $a \ll b$ (a *way-below* b) if for directed subsets $D \subset L$ the relation $b \leq \bigvee D$ implies the existence of $d \in D$ with $a \leq d$.
(2) $a \lhd b$ (a *wedge-below* b) if for arbitrary subsets $C \subset L$ the relation $b \leq \bigvee C$ implies the existence of $c \in C$ with $a \leq c$.
(3) $a \blacktriangleleft b$ (*wedge-above*) if for arbitrary subsets $C \subset L$ the relation $\bigwedge C \leq a$ implies the existence of $c \in C$ with $c \leq b$.

The elements $a \in L$ satisfying $a \lhd a$ (resp. $a \blacktriangleleft a$) are called completely join-irreducible (resp. completely meet-irreducible); they are called coprimes (resp. primes) if C is assumed to be finite in (2) (resp. (3)).

As mentioned above, we have the following characterization:

Lemma 2. *A lattice L is completely distributive if and only if $a = \bigvee\{b \in L : b \lhd a\}$ (equivalently, $a = \bigwedge\{b \in L : a \blacktriangleleft b\}$) for each $a \in L$. A lattice L is continuous if and only if $a = \bigvee\{b \in L : b \ll a\}$ for every $a \in L$.*

The next step is to determine which kind of elements should be selected to play the role of the countable set of rationals in the reals. It turns out that the right kind of sets are countable join-dense subsets free of completely join-irreducible elements, where a subset $D \subset L$ is called *join-dense* (or a base) if $a = \bigvee\{d \in D : d \leq a\}$ for each $a \in L$.

Completely distributive lattices which have a countable join-dense subset free of completely join-irreducible elements will be called $\lhd$-*separable* lattices. This kind of lattices will play the role of the range space for the functions in next subsection 2.2.

An example of completely distributive $\lhd$-*separable* lattice is, of course, $L = [0, 1]$. More examples come from the fact that the class of $\lhd$-separable completely distributive lattices is closed under countable products with componentwise ordering. In particular, the Hilbert cube $L(\omega) = [0,1]^\omega$ with the componentwise order is a $\lhd$-separable completely distributive lattice. Its coordinate axes form a poset denoted by $J(\omega)$ (a hedgehog with ω spines) which is a join-dense subset.

Even if $J(\omega)$ is no longer a lattice, the insertion theorem of subsection 2.2 yields an insertion theorem for $J(\omega)$-valued functions as a corollary [7] (which is independent of that of Blair and Swardson [2]). Throughout this section L denotes a (complete) completely distributive lattice. This assumption may occasionally be repeated.

2.1. **Generating lattice-valued functions.** We will follow Stone's ideas and methods. They were later on modified and simplified giving rise to the notion of "scale"(see [12] and [3]). Our next objective is to look at the essence of that notion with "lattice-eyes" and develop a method which allows to substitute lattice-valued functions by L-indexed (or yet D-indexed) families of subsets of the domain (here L denotes the whole lattice and D some dense-like subset in L).

For X a set and L a complete lattice, L^X denotes the complete lattice of all maps from X into L under pointwise ordering. Given $f \in L^X$ and $a \in L$, we write: $\twoheaduparrow a = \{b \in L : a \ll b\}$, $\downarrow a = \{b \in L : b \leq a\}$ and $[f \prec a] = \{x \in X : a \prec f(x)\}$, where $\prec \in \{\rhd, \blacktriangleleft, \leq, \geq, \ll\}$.

The following lemma recovers for lattice-valued functions what is essential in the notion of the spectral family of a real valued function.

Lemma 3. *For a subset $D \subset L$ and a family $\mathcal{E} = \{E_d \subset X : d \in D\}$ the following are equivalent:*

(1) $E_{d_1} \supset E_{d_2}$ *whenever* $d_1 \lhd d_2$. [We shall say that $\mathcal{E}$ is $\lhd$-antitone.]
(2) *There exists* $f : X \to L$ *such that* $[f \rhd d] \subset E_d \subset [f \geq d]$, *for every* $d \in D$.

We are now in a position to define the crucial concept in this research.

Definition 4. Let $D \subset L$. A $\lhd$-antitone family $\mathcal{E} = \{E_d \subset X : d \in D\}$ is called a *scale* in X. The function $f \in L^X$ defined by $f(x) = \bigvee\{d \in D : x \in E_d\}$ is said to be generated by the scale $\mathcal{E}$.

Given $f \in L^X$, both $\{[f \geq a] : a \in L\}$ and $\{[f \rhd a] : a \in L\}$ are scales that generate the function f.

A parallel study can be done starting with a $\blacktriangleleft$-isotone family $\mathcal{E} = \{E_d \subset X : d \in D\}$. In this case, the function $f \in L^X$ will be defined by $f(x) = \bigwedge\{d \in D : x \in E_d\}$ and said to be generated by the scale $\mathcal{E}$. Since complete distributivity is a self-dual property, all the results obtained here hold also for this kind of scales.

Moreover, we have lattice-valued analogous of Stone's real-valued results.

Lemma 5. *Let $D \subset L$ be join-dense. Let $f, g \in L^X$ be generated by the scales $\{F_d\}_{d \in D}$ and $\{G_d\}_{d \in D}$, respectively. Then $f \leq g$ if and only if $F_{d_1} \subset G_{d_2}$ whenever $d_2 \lhd d_1$.*

Lemma 6. *Let $D \subset L$ be join-dense and $\{F_d\}_{d \in D}$ a scale in X. Then there exists a unique $f \in L^X$ such that $[f \rhd d] \subset F_d \subset [f \geq d]$, for any $d \in D$. Moreover, for every $a \in L$ one has $[f \geq a] = \bigcap_{d \lhd a} F_d$ and $[f \rhd a] = \bigcup_{a \lhd d} F_d$.*

2.2. **Katětov-Tong-type insertion theorem.** Now we will adapt the techniques introduced by Katětov in [9]. These techniques allow to give conditions for the insertion of real-valued functions.

Let X be a topological space and L be a lattice. Let $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{F}$ be families of maps from X into L consisting of continuous functions with respect to appropriately chosen topologies on L. Assume $g \in \mathcal{G}$, $h \in \mathcal{H}$, and $g \leq h$. An insertion-type statement reads as follows: *there exists an* $f \in \mathcal{F}$ *such that* $g \leq f \leq h$.

To extend Katětov-Tong theorem to lattice-valued functions, we first need to choose appropriate definitions of semicontinuity for this kind of functions.

Among the different definitions of semicontinuities for real-valued functions, let us select the ones defined in terms of the order. It is well known that, given a topological space X, a function $f : X \to \overline{\mathbb{R}}$ is lower [upper]semicontinuous if and only if $f = f_*$ [$f = f^*$] where f_* is the lower limit function of f and is defined by $f_*(x) = \bigvee\{\bigwedge f(U) : U$ is an open nbhd of $x\}$ and f^* is the upper limit function of f, which is defined dually. Notice that they are a "logic" generalization of the interior and the closure of a set in a topological space. Indeed $(1_A)_* = 1_{\mathrm{Int}\, A}$ and $(1_A)^* = 1_{\overline{A}}$, where 1_A is the characteristic function of $A \subset X$.

We observe that no topology in the codomain is used. Therefore the definitions of lower and upper limit functions go unchanged to the case of lattice-valued functions.

Definition 7. Let X be a topological space. A function $f : X \to L$ is said to be *lower* (resp. *upper*) *semicontinuous* if and only if $f(x) = \bigvee\{\bigwedge f(U) : U$ is an open nbhd of $x\}$ (resp. $f(x) = \bigwedge\{\bigvee f(U) : U$ is an open nbhd of $x\}$) for any $x \in X$.

We denote by $LSC(X,L)$ and $USC(X,L)$ the collections of all lower and upper semicontinuous functions of L^X. Members of $C(X,L) = LSC(X,L) \cap USC(X,L)$ are called *continuous*.

Let us consider the following topologies on a lattice L:

(1) The *Scott topology* $\sigma(L)$ which has $\{\Uparrow a : a \in L\}$ as a base,

(2) the *lower topology* $\omega(L)$ which is generated from the subbase $\{L \setminus \uparrow a : a \in L\}$,

(3) the *upper topology* $\nu(L)$ which is generated from the subbase $\{L \setminus \downarrow a : a \in L\}$,

(4) the *Lawson topology* $\lambda(L)$ being the supremum of $\sigma(L)$ and $\omega(L)$.

We write $\Sigma L = (L, \sigma(L))$, $\Omega L = (L, \omega(L))$ and $\Lambda L = (L, \lambda(L))$. Also, given a topological space X, let $C(X, \Sigma L)$ denote the collection of all continuous functions from X to ΣL, and similarly for the remaining topologies on L. In particular, $C(X, \Lambda L) = C(X, \Sigma L) \cap C(X, \Omega L)$.

Next result shows that lower and upper semicontinuity are nothing but continuity when the range space is endowed with certain well-know topologies.

Proposition 8. *Let* X *be a topological space and* $f \in L^X$. *Then*

(1) $f \in USC(X,L)$ *iff* $[f \geq a]$ *is closed for each* $a \in L$ *iff* $[f \blacktriangleleft a]$ *is open for each* $a \in L$.

(2) $f \in LSC(X,L)$ *iff* $[f \leq a]$ *is closed for each* $a \in L$ *iff* $[f \triangleright a]$ *is open for each* $a \in L$.

Besides, we have a description of these notions in terms of scales.

Proposition 9. *Let* X *be a topological space,* D *a join-dense subset of* L *and* $f \in L^X$ *a map generated by the scale* $\{E_d : d \in D\}$. *The following hold:*

(1) f *is upper semicontinuous if and only if* $\overline{E_{d_1}} \subset E_{d_2}$ *whenever* $d_2 \lhd d_1$;

(2) f *is lower semicontinuous if and only if* $E_{d_1} \subset \mathrm{Int}\, E_{d_2}$ *whenever* $d_2 \lhd d_1$;

(3) f *is continuous if and only if* $\overline{E_{d_1}} \subset \mathrm{Int}\, E_{d_2}$ *whenever* $d_2 \lhd d_1$.

All the previous definitions and results highlight the necessity of establishing conditions under which there exist distinguished collections of subsets in the domain which are comparable in a precise way (cf. Proposition 9).

Definition 10. Let L be a complete lattice. A binary relation ϱ on L is a *Katětov relation* if and only if for all $a, b, c, d \in L$ the following hold:

- ϱ is an idempotent relation (transitive with interpolation property),
- $a \,\varrho\, b \Rightarrow a \le b$,
- $\{a \in L : a \,\varrho\, b\}$ is an ideal,
- $\{b \in L : a \,\varrho\, b\}$ is a filter.

Lemma 11. *Let ϱ be a Katětov relation on a complete lattice L. Let $A, B \subset L$ be two countable subsets such that*

$$\left(\bigvee A\right) \varrho\, b \quad \text{and} \quad a\, \varrho \left(\bigwedge B\right)$$

for all $a \in A$ and $b \in B$, then there is a $c \in L$ such that $a \;\varrho\; c \;\varrho\; b$ for all $a \in A$ and $b \in B$.

We generalize the original Katětov Lemma as follows:

Lemma 12. *Let ϱ be a Katětov relation on a complete lattice L. Let D be an arbitrary countable set and let $\prec$ be a transitive and irreflexive relation on D. Let $\{a_d\}_{d\in D}$ and $\{b_d\}_{d\in D}$ be two countable subsets of L such that*

$$d \prec d' \quad \text{implies} \quad a_{d'} \le a_d,\ \ b_{d'} \le b_d \ \ \text{and} \ \ a_{d'} \,\varrho\, b_d.$$

Then there is a countable subset $\{c_d\}_{d\in D}$ of L such that

$$d \prec d' \quad \text{implies} \quad (a_{d'} \vee c_{d'})\, \varrho\, (c_d \wedge b_d).$$

Given a topological space X, a Katětov relation $\Subset$ in $\mathbf{2}^X$ is said to be *strong*, if $A \Subset B$ implies $\overline{A} \subset B$ and $A \subset \operatorname{Int} B$. In particular, the relation $A \Subset B$, defined by $\overline{A} \subset \operatorname{Int} B$, is strong if and only if X is normal.

We give now a sufficient condition for insertion which is an analogue of the classical insertion theorem of Lane [10].

Theorem 13. *Let X be a topological space. Let L be a $\lhd$-separable completely distributive lattice (with $D \subset L$ being a countable join-dense subset without supercompact elements). Let $\{F_d\}_{d\in D}$ and $\{G_d\}_{d\in D}$ be scales generating $f, g : X \to L$, respectively. If there exists a strong Katětov relation $\Subset$ such that $F_{d_2} \Subset G_{d_1}$ whenever $d_1 \lhd d_2$, then there exists a continuous function $h : X \to L$ such that $f \le h \le g$.*

The following provides a fairly general extension of the classical insertion theorem of Katětov [9] and Tong [13] to functions with values in a $\lhd$-separable completely distributive lattice (see also [8]).

Theorem 14. *For X a topological space and L a $\lhd$-separable completely distributive lattice, the following are equivalent:*

(1) *X is normal*;
(2) [Katětov-Tong theorem] *If $f : X \to L$ is upper semicontinuous, $g : X \to L$ is lower semicontinuous, and $f \le g$, then there exists a continuous function $h : X \to L$ such that $f \le h \le g$*;
(3) *If $f : X \to L$ is upper semicontinuous, $g : X \to L$ is lower semicontinuous, and $f \le g$, then there exists a lower semicontinuous function $h : X \to L$ such that $f \le h \le h^* \le g$*;
(4) [Urysohn's lemma] *If $K \subset X$ is closed, $U \subset X$ is open, and $K \subset U$, then there exists a continuous function $h : X \to L$ such that $h(K) = \{1\}$ and $h(X\backslash U) = \{0\}$*;
(5) [Tietze's theorem] *Let Y be a closed subspace of X. Then each continuous $h : Y \to L$ has a continuous extension to the whole X.*

Notice that there are L-analogue results concerning characterizations of some other normality-like axioms.

Also, an interesting specialization is the one with $L(\omega) = [0,1]^\omega$ being the Hilbert cube , having the hedgehog $J(\omega)$ with the Lawson topology as join dense subset (the compact hedgehog $J(\omega)$).

2.3. **Insertion of a pair of semicontinuous functions.** The possibility of inserting a pair of semicontinuous L-valued functions gives new characterizations of some classes of topological spaces. The following is an iff criterion for a double insertion theorem.

Proposition 15. *Let X be a topological space. Let L be a completely distributive lattice and let $f, g : X \to L$ be two arbitrary functions. If $f \leq g$ and there exist two families $\{l_n\}_{n\in\mathbb{N}} \subset LSC(X,L)$ and $\{u_n\}_{n\in\mathbb{N}} \subset USC(X,L)$ such that*

$$f \leq \bigvee_{n\in\mathbb{N}} l_n \leq \bigvee_{n\in\mathbb{N}} (l_n)^* \leq g \quad \text{and} \quad f \leq \bigwedge_{n\in\mathbb{N}} (u_n)_* \leq \bigwedge_{n\in\mathbb{N}} u_n \leq g,$$

then there exist a function $h \in LSC(X,L)$ such that $f \leq h \leq h^* \leq g$.

Recall that a space X is *hereditarily normal* if and only if, whenever $\overline{A} \subset B$ and $A \subset \operatorname{Int} B$ in X, then $A \subset U \subset \overline{U} \subset B$ for some open $U \subset X$. A space is *extremally disconnected* iff every two disjoint open sets have disjoint closures.

Proposition 16. *Let L be a lattice such that both L and L^{op} have countable join-dense subsets. For X a topological space, the following are equivalent:*

(1) *X is hereditarily normal*;
(2) *If $f, g : X \to L$ with $f^* \leq g$ and $f \leq g_*$, then there exists an $h \in LSC(L)$ such that $f \leq h \leq h^* \leq g$.*

Proposition 17. *Let X be a topological space and L a lattice which has a countable join-dense subset. Then the following are equivalent:*

(1) *X is extremally disconnected*;
(2) *If $f \in LSC(X,L)$, $g \in USC(X,L)$, and $f \leq g$, then there exists an $h \in C(X,L)$ such that $f \leq h \leq g$*;
(3) *If $f \in LSC(X,L)$, $g \in USC(X,L)$, and $f \leq g$, then there exists an $h \in USC(X,L)$ such that $f \leq h \leq h_* \leq g$.*

Proposition 18. *Let L be a lattice such that both L and L^{op} have countable join-dense subsets. For X a topological space, the following are equivalent:*

(1) *X is extremally disconnected and hereditarily normal*;
(2) *If $f, g : X \to L$ with $f^* \leq g$ and $f \leq g_*$, then there exists an $h \in C(X,L)$ such that $f \leq h \leq g$.*

As well as extension of continuous real-valued functions depends on the complete separation of certain subsets (cf. [5]), insertion depends on the complete separation of the Lebesgue sets of the comparable functions (see [1]).

We have an analogous of the classical insertion theorem of Blair [1] for L-valued functions, where L is a $\lhd$-separable completely distributive lattice.

Theorem 19. *Let $D \subset L$ be countable join-dense in L, and let $C \subset L$ be countable join-dense in L^{op}. For X a topological space and $f, g : X \to L$ such that $f \leq g$, the following are equivalent:*

(1) *$f \leq h \leq h^* \leq g$ for some $h \in LSC(X,L)$;*

(2) *For every* $d \in D$ *the sets* $[f \rhd d]$ *and* $[g \not\geq d]$ *have disjoint open neighborhoods, and for every* $c \in C$ *the sets* $[g \blacktriangleleft c]$ *and* $[f \not\leq c]$ *have disjoint open neighborhoods.*

3. Bounded complete domains

Up to now, we have been working with lattices in which all the sups (equivalently all the infs) existed. The concepts of scale and lower an upper limits depend on such existence (remember we have started with a characterization of semicontinuity of a function with values in the extended real line).

What happens if such existence is not guaranteed? This is the case of the real line, which is only Dedekind complete or conditionally complete.

The class of lattices we are going to consider in this section share this property with the real line. We focus on functions having values in bounded complete domains.

Before embarking on the task of developing a theory which serves for this class of lattices, we need a few definitions.

Definition 20. A poset L is called a *bounded complete domain* if it has the following properties:

(1) each directed subset of L has a sup,
(2) L satisfies the *axiom of approximation* with respect to $\ll$, i.e.,

$$a = \bigvee\{b \in L : b \ll a\} \quad \text{for all } a \in L$$

(3) each subset of L that is bounded above has a sup (i.e., L is *conditionally complete*; in particular, each bounded complete domain has a bottom element 0).

We introduce the concept of a $\ll$-basis for L, which is weaker than that of [4]. The reason of changing the original definition of a basis in the sense of [4] is that our range spaces for inserting functions will, among others, include several hedgehog-like structures which have $\ll$-bases which fail to be bases in the sense of [4].

Definition 21. Let L be a bounded complete domain. A subset $D \subset L$ is called a $\ll$*-basis* of L if and only if for all $a \in L$ the following hold:

(1) $a = \bigvee(D \cap \downarrow a)$,
(2) If $a \in L$ and $d_1 \ll a$ with $d_1 \in D$, then there exists a $d_2 \in D$ such that $d_1 \ll d_2 \ll a$.

3.1. **Generating domain-valued maps.** In this subsection we shall discuss the procedure of generating L-valued functions by monotone families of subsets in the context of a bounded complete domain L. We first develop a theory of generating such functions from certain scales or prescales of subsets. The situation is much different from that in a completely distributive lattice, due to the fact that a bounded complete domain needs not have the top element.

Definition 22. Let X be a nonempty set and let D be an arbitrary nonempty subset of a bounded complete domain L. A family $\mathcal{F} = \{F_d \subset X : d \in D\}$ is called a *prescale* if

(1) $\{d \in D : x \in F_d\}$ is bounded in L for all $x \in X$.

If one additionally assumes that:

(2) for any nonempty subset $C \subset D$ for which $\bigvee C$ does exist, one has $\bigcap_{c \in C} F_c \subset F_d$ whenever $d \ll \bigvee C$,

then $\mathcal{F}$ is called a *scale.*

Without loss of generality the set C can be assumed to be finite. It follows immediately (with C being a singleton) that each scale is $\ll$-*antitone*, i.e., $F_{d_1} \supset F_{d_2}$ whenever $d_1 \ll d_2$.

The following shows how to generate an L-valued function from a (pre)scale.

Lemma 23. *Let X be a set, L a bounded complete domain, D a nonempty subset of L, and let $\mathcal{F} = \{F_d \subset X : d \in D\}$.*

(1) *If $\mathcal{F}$ is a prescale, then $f : X \to L$, with $f(x) = \bigvee\{d \in D : x \in F_d\}$ for all $x \in X$, is a well-defined function.*

(2) *$\mathcal{F}$ is a scale if and only if there exists a function $f : X \to L$ such that for every $d \in D$:*

$$[f \gg d] \subset F_d \subset [f \geq d].$$

(3) *If D is a $\ll$-basis consisting of coprime elements and $\mathcal{F}$ is a prescale, then $\mathcal{F}$ is a scale if and only if it is $\ll$-antitone.*

Definition 24. Let L be a bounded complete domain, let $D \subset L$ and let $\mathcal{F} = \{F_d \subset X : d \in D\}$ be a (pre)scale. The function $f \in L^X$ defined by $f(x) = \bigvee\{d \in D : x \in F_d\}$ is said to be *generated* by the (pre)scale $\mathcal{F}$.

We notice that for every $f \in L^X$, both $\{[f \geq a] : a \in L$ and $\{[f \gg a] : a \in L\}$ are scales that generate f.

Now, we can recover for bounded complete domain Lemmas 5 and 6 in section 2.

Lemma 25. *Let L be a bounded complete domain and let $D \subset L$ be a $\ll$-basis in L. Let $f, g \in L^X$ be generated by the scales $\{F_d\}_{d\in D}$ and $\{G_d\}_{d\in D}$. Then the following hold:*

$$f \leq g \quad \text{iff} \quad F_{d_1} \subset G_{d_2} \text{ whenever } d_2 \ll d_1.$$

Lemma 26. *Let L be a bounded complete domain and let $D \subset L$ be a $\ll$-basis in L. Let $\{F_d\}_{d\in D}$ be a scale. Then there exists a unique $f \in L^X$ such that $[f \gg d] \subset F_d \subset [f \geq d]$, for any $d \in D$. Moreover, one has $[f \geq a] = \bigcap_{d \ll a} F_d$ whenever $a \in L$, and $[f \gg d_1] = \bigcup_{d_1 \ll d} F_d$ whenever $d_1 \in D$.*

3.2. **Katětov-Tong-type insertion theorem.** Now we introduce the lower and upper limits of bounded complete domain valued functions. Again, things become more complex than in the case of lattice-valued maps. This is particularly noticeable in the case of an upper limit function (lower limit functions can be defined as previously), which generally may fail to exist. We characterize the limit functions in terms of the (pre)scales generating the original ones.

Since the most flexible versions of semicontinuity for lattice-valued functions on a topological space seem to be the continuities with respect to the upper topology $\nu(L)$ and the lower topology $\omega(L)$, the following characterization of members of $C(X, \Omega L)$ suggests a possible definition.

Proposition 27. *Let X be a topological space and let L be a bounded complete domain. Then $f \in C(X, \Omega L)$ if and only if $f(x) = \bigvee\{\bigwedge_{U\in\mathcal{N}_x} f(\varphi(U)) : \varphi \in \prod_{U\in\mathcal{N}_x} U\}$ for all $x \in X$.*

Definition 28. Let L be a bounded complete domain. We define for each $f \in L^X$ and $x \in X$ (assuming the sup exists):

$$f^*(x) = \bigvee \left\{ \bigwedge_{U\in\mathcal{N}_x} f(\varphi(U)) : \varphi \in \prod_{U\in\mathcal{N}_x} U \right\}.$$

We shall refer to the element $f^*(x)$ as to the upper limit of f at x. If it exists for all x in X, we have a new function $f^* \in L^X$ called the upper limit function of f.

We have the following results:

Proposition 29. *Let X be a topological space and let L be a bounded complete domain with a $\ll$-basis D. Let $f \in L^X$ be generated by the scale $\mathcal{F} = \{F_d \subset X : d \in D\}$. Then:*

(1) $\operatorname{Int}\mathcal{F} := \{\operatorname{Int} F : F \in \mathcal{F}\}$ *is a scale.*
(2) $\operatorname{Int}\mathcal{F}$ *generates* f_*.
(3) f *has the upper limit at* $x \in X$ *iff* $\{d \in D : x \in \overline{F_d}\}$ *is bounded.*
(4) *If* f *has the upper limit function* f^*, *then* f^* *is generated by the prescale* $\{\overline{F} : F \in \mathcal{F}\}$.

Corollary 30. *Let X be a topological space, L a bounded complete domain with a $\ll$-basis D. If $f \in L^X$ is generated by the scale $\mathcal{F} = \{F_d : d \in D\}$, then the following hold:*

(1) $f \in C(X, \Sigma L)$ *iff* $F_{d_1} \subset \operatorname{Int} F_{d_2}$ *whenever* $d_2 \ll d_1$;
(2) $f \in C(X, \Omega L)$ *iff* $\overline{F_{d_1}} \subset F_{d_2}$ *whenever* $d_2 \ll d_1$;
(3) $f \in C(X, \Lambda L)$ *iff* $\overline{F_{d_1}} \subset \operatorname{Int} F_{d_2}$ *whenever* $d_2 \ll d_1$.

Definition 31. A bounded complete domain L will be called *$\ll$-separable* if it has a countable $\ll$-basis $D \subset L$ consisting of noncompact coprimes (an element $a \in L$ is called *compact* iff $a \ll a$).

An example of a bounded complete domain with a countable base is the hedgehog $J(\omega)$.

Next, we give a sufficient condition for inserting a Lawson continuous function between two comparable L-valued functions.

Theorem 32. *Let X be a topological space. Let L be a $\ll$-separable bounded complete domain with a $\ll$-basis D. Let $f, g : X \to L$ be such that $f \leq g$. Let $\mathcal{F} = \{F_d\}_{d\in D}$ and $\mathcal{G} = \{G_d\}_{d\in D}$ be scales generating f and g. If there exists a strong Katětov relation $\Subset$ on $\mathcal{P}(X)$ such that $F_{d_2} \Subset G_{d_1}$ whenever $d_1 \ll d_2$, then there exists a continuous function $h : X \to \Lambda L$ such that $g \leq h \leq f$.*

As in section 2, we have a general extension of the classical insertion theorem of Katětov [9] and Tong [13].

Theorem 33. *For X a topological space and L a $\ll$-separable bounded complete domain, the following are equivalent:*

(1) X *is normal*;
(2) [Katětov-Tong theorem] *If* $f \in C(X, \Omega L)$, $g \in C(X, \Sigma L)$ *and* $f \leq g$, *then there exists an* $h \in C(X, \Lambda L)$ *such that* $f \leq h \leq g$;
(3) [Urysohn's lemma] *If* $K \subset X$ *is closed,* $U \subset X$ *is open and* $K \subset U$, *then, for any* $a \in L \setminus \{0\}$, *there exists* $h_a \in C(X, \Lambda L)$ *such that* $h_a(K) = \{a\}$ *and* $h_a(X \setminus U) = \{0\}$.

It is a heuristic principle, for real-valued functions, that insertion theorems usually have Tietze-type extension theorems as corollaries too. We got such a Tietze-type extension theorem in section 2, for $\lhd$-separable completely distributive lattices.

However, a procedure similar to the usual one to get extension from insertion is no longer valid for an arbitrary $\ll$-separable bounded complete domain, as the following example shows:

Example 34. Let $X = \mathbb{R}$ be endowed with the usual topology, $Y = [-1, 1]$ and $J(2)$ the hedgehog with 2 spines. Let $F : Y \to J(2)$ be defined as $F(x) = (x, i_1)$ if $x \in [0, 1]$ and $F(x) = (-x, i_2)$ if $x \in [-1, 0]$. Then $F \in C(X, \Lambda L)$. Mimicking the usual procedure of extending a continuous $[0,1]$-valued function to the whole space, led to consider two maps $g, h : X \to J(2)$, the first one being upper semicontinuous and the second one lower semicontinuous in such a way that the continuous map in-between extends the given F to the whole X. The first map is usually defined as $g = F$ on Y and $g = \mathbf{0}$ on $X \setminus Y$ and the second one as $h = F$ on Y and the top element on $X \setminus Y$. In the case of bounded complete domains, since there is no top element but there are maximal elements, one could think on defining h equal to any of these maximal elements on $X \setminus Y$. However, the map $h : \mathbb{R} \to J(2)$ defined by $h = F$ on Y and $h = (1, i_1)$ on $X \setminus Y$ is not lower semicontinuous. Analogously if we consider $h = F$ on Y and $h = (1, i_2)$ on $X \setminus Y$.

The techniques developed up to now do not allow to prove the Tietze theorem nor to disprove it for an arbitrary $\ll$-separable bounded complete domain. We have a partial result in this direction:

Theorem 35. *A topological space X is normal iff for every closed A in X and each continuous $f : A \to J(\omega)$ there exists a continuous extension to the whole X.*

Nevertheless the proof of this statement requires techniques which go beyond the scope of this talk. They have been published in [8].

References

[1] R. L. Blair, *Extension of Lebesgue sets and of real-valued functions*, Czech Math. J. **31** (1981), no. 106, 63–74.

[2] R. L. Blair, M. A. Swardson, *Insertion and extension of hedgehog-valued functions*, Indian J. Math. **29** (1987), 229–250.

[3] F. Brooks, *Indefinite Cut Sets for Real Functions*, Am. Math. Mon. **78** (1971), 1007–1010.

[4] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove and D. S. Scott, Continuous Lattices and Domains, Cambridge University Press, Cambridge, 2003.

[5] L. Gillman, M. Jerison, Rings of Continuous Functions, (Van Nostrand Reinhold Company, 1960).

[6] J. Gutiérrez García, T. Kubiak, M. A. de Prada Vicente, *Insertion of lattice-valued and hedgehog-valued functions*, Topol. Appl. **153** (2006), no. 9, 1458–1475.

[7] J. Gutiérrez García, T. Kubiak, M. A. de Prada Vicente, *Generating and inserting continuous functions with values in bounded complete domains and hedgehog-like structures*, Houston J. Math. **34** (2008), no. 1, 123–144.

[8] J. Gutiérrez García, T. Kubiak, M. A. de Prada Vicente, *Controlling disjointness with a hedgehog*, Houston J. Math. **35** (2009), no. 2, 469–484.

[9] M. Katětov, *On Real-Valued Functions in Topological Spaces*, Fund Math, **38** (1951), 85–91; *Correction to "On Real-Valued Functions in Topological Spaces"*, Fund Math, **40** (1953), 203–205.

[10] E. P. Lane, *A sufficient condition for the insertion of a continuous function*, P. Am. Math. Soc. **49** (1975), 90–94.

[11] Y.-M. Liu, M.-K. Luo, *Lattice-valued Hahn-Dieudonné-Tong insertion theorem and stratification structure*, Topol. Appl. **45** (1992), 173–188.

[12] M. H. Stone, *Boundedness Properties in Function-Lattices*, Can. J. Math. **1** (1949), 176–186.

[13] H. Tong, *Some Characterizations of Normal and Perfectly Normal Spaces*, Duke Math. J. **19** (1952), 289–292.