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Unified representability of total preorders, semiorders and interval orders through scales and a single map

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Abstract

We introduce a new approach that deals, jointly and in a unified manner, with the topics of the (continuous) numerical representability of total preorders, semiorders and interval orders. This setting is based on the consideration of increasing scales and the systematic use of a particular kind of codomain based on a subset of the real plane. The key fact is that this canonical codomain has a theoretical structure of a completely distributive lattice that allows us to use a single function (taking values in that codomain) in order to represent the three kinds of binary relations.

1. Introduction

The present paper can be considered as a natural continuation of the analysis, initiated in [5, 2, 6, 3] about the possibility of finding representations of several kinds of binary relations, and in particular of interval orders, using only *a single map* that takes values on a set (different, if necessary, from the real line $\mathbb{R}$).

The techniques used here are essentially the same as those used in [3], where we were able to provide a unified treatment in the case of total preorders and interval orders. The key idea in [3] was to observe that when representing total preorders and interval orders through a single map, despite of being different *the two typical codomains* (namely, the real line for the case of total preorders, and a particular subset of the real plane for the case of interval orders) *shared the same lattice theoretical structure*. Indeed both codomains were a particular type of completely distributive lattices. After noticing it, it was quite natural to try to use the techniques of *scales* coming from [8] in order to provide the desired uniform treatment. However, in [3] the possibility of extending that kind of results to capture also the representability of semiorders (in some manner that should be equivalent to the classical representability of semiorders in the sense of Scott and Suppes [11] through a real valued map and a nonnegative threshold) was left as an *open problem*.

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The main originality in the present paper is that we can actually also deal with the representation of typical (or intransitive) semiorders. The concept of a *semiorder* was introduced in [10] to deal with inaccuracies in measurements where a nonnegative *threshold* of discrimination is involved. The original idea was that of presenting a mathematical model of preferences enable to capture situations of "*intransitive indifference with a threshold of discrimination*". Semiorders are an intermediate type of binary relations that lie between total preorders and interval orders, in the sense that semiorders are a particular case of interval orders, whereas total preorders are precisely the transitive semiorders.

The structure of the paper goes as follows: In section 2 we include the necessary preliminaries to deal with total preorders, semiorders and interval orders along the manuscript. We also present the canonical interval order and typical semiorder. Both of them have the algebraical structure of completely distributive lattice that will play a crucial role in the rest of the paper. Section 3 consists of previous results on completely distributive lattices that are needed in the subsequent sections to benefit from the algebraical properties of the canonical lattices introduced in section 2. The proof of all the results can be found in [3]. Finally, Section 4 deals with the continuous representability of total preorders, semiorders and interval orders through scales.

2. Total preorders, semiorders and interval orders

In what follows X denotes a nonempty set, and $\mathcal{R}$ a binary relation on X. The *asymmetric part* $\mathcal{P}$ of $\mathcal{R}$ is defined for each $x, y \in X$ as $x\mathcal{P}y$ if and only if $x\mathcal{R}y$ and $\neg(y\mathcal{R}x$.

Definition 1. Let $\mathcal{R}$ a binary relation on X and $x, y, z, t \in X$. $\mathcal{R}$ is said to be:

(i) An *interval order* if it is reflexive and whenever $x\mathcal{P}y$ and $z\mathcal{P}t$ either $x\mathcal{P}t$ or $z\mathcal{P}y$. Notice that $\mathcal{R}$ is, in particular, complete.
(ii) A *semiorder* if it is an interval order and whenever $x\mathcal{P}y$ and $y\mathcal{P}z$ either $x\mathcal{P}t$ or $t\mathcal{P}z$.
(iii) A *preorder* if it is reflexive and transitive. If in addition it is complete, $\mathcal{R}$ is said to be a *total preorder*.

Clearly enough we have the following relation between the three notions:

$$\text{total preorder} \quad \Longrightarrow \quad \text{semiorder} \quad \Longrightarrow \quad \text{interval order}.$$

Concerning the first implication, it is well-known that a semiorder is a total preorder if and only if it is transitive. In this sense a semiorder is said to be *typical* if $\mathcal{P}$ is *not* a total preorder, i.e. if it is not transitive. However the converse of the implications above are not necessarily true as the following examples show.

Examples 2. (1) (Canonical interval order) Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ denote the extended real line and $\mathcal{Y} = \{(a_1, a_2) \in \overline{\mathbb{R}}^2 : a_1 \leq a_2\}$. Endow $\mathcal{Y}$ with the relation $\mathcal{R}_{i.o.}$ given by

$$(a_1, a_2)\mathcal{R}_{i.o.}(b_1, b_2) \iff a_1 \leq b_2 \qquad (a_1, a_2), (b_1, b_2) \in \mathcal{Y}.$$

The corresponding asymmetric part $\mathcal{P}_{i.o.}$ of $\mathcal{R}_{i.o.}$ is given by

$$(a_1, a_2)\mathcal{P}_{i.o.}(b_1, b_2) \iff a_2 < b_1 \qquad (a_1, a_2), (b_1, b_2) \in \mathcal{Y}.$$

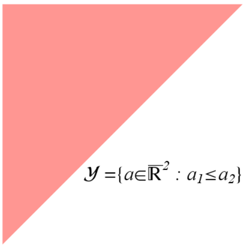

Figure 1(1)

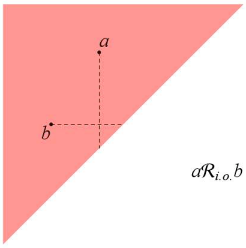

Figure 1(2)

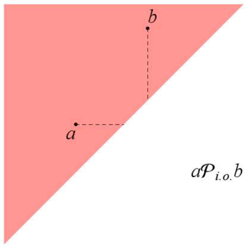

Figure 1(3)

It is easy to check that $\mathcal{R}_{i.o}$ is an interval-order which fails to be a semiorder.

(2) (Canonical typical semiorder) Endow $\overline{\mathbb{R}}$ with the binary relation $\mathcal{R}_{s.o.}$ given by

$$a\mathcal{R}_{s.o.}b \iff a \leq b+1 \qquad a, b \in \overline{\mathbb{R}}.^3$$

The corresponding assymetric part $\mathcal{P}_{s.o.}$ of $\mathcal{R}_{s.o.}$ is given by

$$a\mathcal{P}_{s.o.}b \iff a < b-1 \qquad a, b \in \overline{\mathbb{R}}.^4$$

It is easy to check that $\mathcal{R}_{s.o.}$ is a typical (non transitive) semiorder.

Remarks 3. (1) Note that the restriction of the interval order $\mathcal{R}_{i.o.}$ on $\mathcal{Y}$ to the diagonal $\Delta = \{a \in \mathcal{Y} : a_1 = a_2\}$ becomes a total preorder on Δ. Clearly enough, the pair $(\Delta, \mathcal{R}_{i.o.|\Delta})$ is isomorphic to $(\overline{\mathbb{R}}, \leq)$.

(2) Analogously, the restriction of $\mathcal{R}_{i.o.}$ to $\Delta_1 = \{a \in \mathcal{Y} : a_2 = a_1 + 1\}$ becomes a typical semiorder and the pair $(\Delta_1, \mathcal{R}_{i.o.|\Delta_1})$ is trivially isomorphic to $(\overline{\mathbb{R}}, \mathcal{R}_{s.o.})$.

These examples are in the very origin of the notions of interval order and semiorder: When a set of closed intervals of the reals is partially ordered by decreeing that $A < B$ when A lies strictly to the left of B, the resulting structure is an interval order (the name of interval order comes precisely from this example). Semiorders may be viewed as interval orders that arise from closed intervals having a fixed length. Finally, when the fixed length is equal to 0 the semiorder becomes a total preorder.

2.1. **Representability of total preorders, semiorders and interval orders.**

Definition 4. (1) A total preorder $\mathcal{R}$ on (X, τ) is said to be *(continuously) representable* if there exists a (continuous) $u : (X, \tau) \to (\mathbb{R}, \tau_u)$ (also called *"utility function"*) such that

$$x\mathcal{R}y \iff u(x) \leq u(y) \quad (x, y \in X).$$

(2) A semiorder $\mathcal{R}$ on (X, τ) is said to be *(continuously) representable in the sense of Scott and Suppes* (see [11]) if there exist a (continuous) $u : (X, \tau) \to (\mathbb{R}, \tau_u)$ and a nonnegative constant or "discrimination threshold" $K \geq 0$ such that

$$x\mathcal{R}y \iff u(x) \leq u(y) + K \quad (x, y \in X).$$

We shall say that $\mathcal{R}$ *(continuously) representable in* $\overline{\mathbb{R}}$ if there exist a (continuous) $u : (X, \tau) \to (\overline{\mathbb{R}}, \tau_u)$ and a discrimination threshold $K \geq 0$ such that

$$x\mathcal{R}y \iff u(x) \leq u(y) + K \quad (x, y \in X).$$

[3]Here we understand $+\infty + 1 = +\infty = +\infty - 1$, i.e. $\mathcal{R}_{s.o.}$ could be equivalently defined as $a\mathcal{R}_{s.o.}b \iff a = -\infty$ or $(a, b \in \mathbb{R}$ and $a \leq b+1)$ or $b = +\infty \qquad a, b \in \overline{\mathbb{R}}$.

[4]Equivalently,
$a\mathcal{P}_{s.o.}b \iff (a = -\infty$ and $b \in \mathbb{R})$ or $(a, b \in \mathbb{R}$ and $a < b-1)$ or $(a \in \mathbb{R}$ and $b = +\infty) \qquad a, b \in \overline{\mathbb{R}}$.

(3) An interval order $\mathcal{R}$ on X is said to be *(continuously) representable* if there exists a pair of (continuous) $u, v : (X, \tau) \to (\mathbb{R}, \tau_u)$ such that

$$x\mathcal{R}y \iff u(x) \leq v(y) \quad (x, y \in X).$$

The following result constitutes a first step towards the unified continuous representability of total preorders, semiorders and interval orders:

Proposition 5. *Let (X, τ) a topological space. Then*

(1) *A total preorder $\mathcal{R}$ on (X, τ) is continuously representable if there exists a continuous $f : (X, \mathcal{R}, \tau) \to (\overline{\mathbb{R}}, \leq, \tau_u)$ such that $x\mathcal{R}y \iff f(x) \leq f(y) \quad (x, y \in X)$.*

(2) *A semiorder $\mathcal{R}$ on (X, τ) is continuously representable in $\overline{\mathbb{R}}$ if there exists a continuous $f : (X, \mathcal{R}, \tau) \to (\overline{\mathbb{R}}, \mathcal{R}_{s.o.}, \tau_u)$ such that $x\mathcal{R}y \iff f(x)\mathcal{R}_{s.o.}f(y) \quad (x, y \in X)$.*

(3) *An interval order $\mathcal{R}$ on (X, τ) is continuously representable if there exists a continuous $f : (X, \mathcal{R}, \tau) \to (\mathcal{Y}, \mathcal{R}_{i.o.}, \tau_u)$ such that $x\mathcal{R}y \iff f(x)\mathcal{R}_{i.o.}f(y) \quad (x, y \in X)$.*

The analogy comes from the fact that both $\overline{\mathbb{R}}$, and $\mathcal{Y}$ are completely distributive lattices. Even more, they are $\blacktriangleleft$-separable in the sense of Definition 9 below. In the case of $\overline{\mathbb{R}}$ this is nothing but the separability in the usual sense. We'll see what it means in the second case in Example 10 (2).

Remarks 6. There are still another two interesting things to be commented here in connection with Remarks 3.

(1) It is easy to check that an interval order $\mathcal{R}$ is a (continuously) representable total preorder if and only if it is (continuously) representable as an interval order through a map *that takes values inside* Δ (see [3, Remark 1, Proposition 3] for details).

(2) Analogously, in the case of semiorders we can prove that an interval order $\mathcal{R}$ is a semiorder (continuously) representable in $\overline{\mathbb{R}}$ if and only if it is (continuously) representable as an interval order through a map *that takes values inside* Δ_1.

In view of the previous comments we have now another unified look at the (continuous) representability of all these kinds of ordered structures. In this case we have that the (continuous) representability of semiorders in $\overline{\mathbb{R}}$ and of total preorders can be interpreted as a particular case of the (continuous) representability when considered as interval orders.

3. Some known results on completely distributive lattices

In what follows L always denotes a complete lattice. The top and bottom elements are denoted by $\top = \bigwedge \varnothing$ and $\bot = \bigvee \varnothing$, respectively. Our main reference for general concepts regarding lattices and complete distributivity is [7].

Given a lattice L and $a, b \in L$, we write

$$a \blacktriangleleft b \iff \text{ for each } A \subseteq L \text{ with } \bigwedge A \leq a, \text{ there is } c \in A \text{ with } c \leq b.$$

This relation (or its dual version) has several names in the literature: *well-above relation*, *long-way-above relation* or simply *Raney relation*.

For each $a \in L$ we write

$$\begin{array}{lll} U_{\leq}(a) = \{b \in L : a \leq b\}, & U_{<}(a) = \{b \in L : a < b\}, & U_{\blacktriangleleft}(a) = \{b \in L : a \blacktriangleleft b\}, \\ L_{\leq}(a) = \{b \in L : b \leq a\}, & L_{<}(a) = \{b \in L : b < a\}, & L_{\blacktriangleleft}(a) = \{b \in L : b \blacktriangleleft a\}, \end{array}$$

etc.

It is well-known that a lattice L is completely distributive if and only if

$$a = \bigwedge U_{\blacktriangleleft}(a) = \bigwedge\{b \in L : a \blacktriangleleft b\} \quad \text{ for each } a \in L.$$

Examples 7. (1) Let $(L, \leq)$ be a complete chain (a totally ordered set). An element $a \in L$ is said to be *isolated from above* if $a < \bigwedge U_<(a)$. We now have for each $a, b \in L$:

$$a \blacktriangleleft b \iff a < b \quad \text{or} \quad a = b \text{ is isolated from above.}$$

It follows that $U_\blacktriangleleft(a) = U_\leq(a)$ if a is isolated from above and $U_\blacktriangleleft(a) = U_<(a)$ otherwise. We conclude the well-known fact that any chain is completely distributive. In the particular case $L = \overline{\mathbb{R}}$, one just has $a \blacktriangleleft b \iff a < b$ for each $a, b \in \overline{\mathbb{R}}$. Hence for each $a \in \overline{\mathbb{R}}$

$$U_\blacktriangleleft(a) = U_<(a) = (a, +\infty] \quad \text{and} \quad L_\blacktriangleleft(a) = L_<(a) = [\infty, a).$$

(2) Let $(L, \leq_L)$ be a completely distributive lattice and

$$\mathcal{L} = \{a \equiv (a_1, a_2) \in L^2 : a_1 \leq_L a_2\}$$

endowed with the componentwise order given by

$$a \leq_\mathcal{L} b \iff a_1 \leq_L b_1 \text{ and } a_2 \leq_L b_2 \qquad a \equiv (a_1, a_2), b \equiv (b_1, b_2) \in L.$$

In this case $\bot_\mathcal{L} = (\bot_L, \bot_L)$ and $\top_\mathcal{L} = (\top_L, \top_L)$. Given $a, b \in \mathcal{L}$, we have

$$a \blacktriangleleft_\mathcal{L} b \iff \big(a_1 \blacktriangleleft_L b_1 \text{ and } b_2 = \top_L\big) \quad \text{or} \quad a_2 \blacktriangleleft_L b_1.$$

Consequently, for each $a \in \mathcal{L}$ we have

$$U_{\blacktriangleleft_\mathcal{L}}(a) = \big(U_{\blacktriangleleft_L}(a_1) \times \{\top_L\}\big) \cup \big((U_{\blacktriangleleft_L}(a_2) \times L) \cap \mathcal{L}\big)$$

and so $\bigwedge U_{\blacktriangleleft_\mathcal{L}}(a) = \bigwedge\big(U_{\blacktriangleleft_L}(a_1) \times \{\top_L\}\big) \wedge \bigwedge\big((U_{\blacktriangleleft_L}(a_2) \times L) \cap \mathcal{L}\big) = (a_1, \top_L) \wedge (a_2, a_2) = a$ for each $a \in \mathcal{L}$. We conclude that $(\mathcal{L}, \leq_\mathcal{L})$ is a completely distributive lattice. In the particular case $L = \overline{\mathbb{R}}$, we have $\mathcal{L} = \mathcal{Y}$ and one just has for $a, b \in \mathcal{Y}$:

$$a \blacktriangleleft_\mathcal{Y} b \iff \big(a_1 < b_1 \text{ and } b_2 = +\infty\big) \quad \text{or} \quad a_2 < b_1.$$

Hence $U_{\blacktriangleleft_\mathcal{Y}}(a) = \big((a_1, +\infty] \times \{+\infty\}\big) \cup \big((a_2, +\infty] \times \overline{\mathbb{R}}) \cap \mathcal{Y}\big)$ and

$$L_{\blacktriangleleft_\mathcal{Y}}(a) = \begin{cases} (\overline{\mathbb{R}} \times [-\infty, a_1)) \cap \mathcal{Y}, & \text{if } b_2 < +\infty \\ ([-\infty, a_1) \times \overline{\mathbb{R}}) \cap \mathcal{Y}, & \text{if } b_2 = +\infty \end{cases}$$

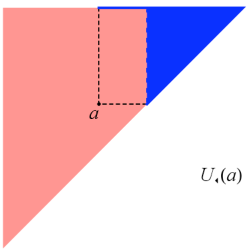

Figure 2(1)

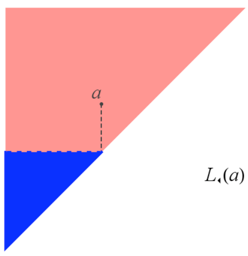

Figure 2(2)

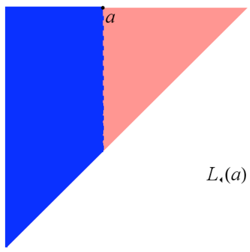

Figure 2(3)

Definition 8. A subset $D \subseteq L$ is called *meet-dense* if each element $a \in L$ there exists some $D_a \subseteq D$ such that $a = \bigwedge D_a$.

Definition 9. A completely distributive lattice is said to be $\blacktriangleleft$*-separable* if it has a countable meet-dense subset.

Examples 10. (1) $\overline{\mathbb{R}}$ is $\blacktriangleleft$-separable with $D = \mathbb{Q}$.

(2) $\mathcal{Y} = \{a \in \overline{\mathbb{R}}^2 : a_1 \leq a_2\}$ endowed with the componentwise order is $\blacktriangleleft$-separable with (see Figure 3(1)).

$$D = D_1 \cup D_2 = \big(\{a \in \mathcal{Y} : a_1 = a_2 \in \mathbb{Q}\}\big) \cup \big(\{a \in \mathcal{Y} : a_1 \in \mathbb{Q} \text{ and } a_2 = +\infty\}\big).$$

Indeed, for each $a \in \mathcal{Y}$ we have $a = \bigwedge D_a$ where (see Figure 3(2))

$D_a = \{b \in \mathcal{Y} : b_1 = b_2 \in \mathbb{Q} \cap (a_2, +\infty)\} \cup \{b \in \mathcal{Y} : b_1 \in \mathbb{Q} \cap (a_1, +\infty)$ and $b_2 = +\infty\}$.

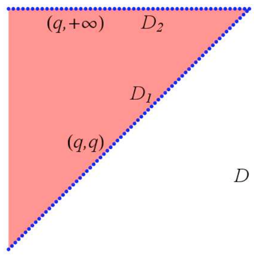

Figure 3(1)

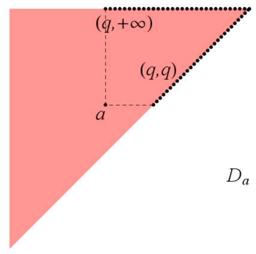

Figure 3(2)

3.1. **Generating lattice-valued functions by scales.** Our standing assumption on L is – as has already been mentioned – the complete distributivity. For X a set, a map f from X into L and $a \in L$, we standardly write:

$$[f \leq a] = \{x \in X : f(x) \leq a\} \quad \text{and} \quad [f \blacktriangleleft a] = \{x \in X : f(x) \blacktriangleleft a\}.$$

We recall here in the context of lattice-valued functions what is known about generating real-valued functions by monotone families of subsets (Stone-Urysohn's procedure).

Definition 11. Let X be a set, L be a completely distributive lattice, $D \subseteq L$ meet-dense and $\mathcal{F} = \{F_d \subseteq X : d \in D\}$. $\mathcal{F}$ is said to be a $\blacktriangleleft$-*scale* if $\mathcal{F}$ is $\blacktriangleleft$-*increasing*, i.e.

$$F_{d_1} \subseteq F_{d_2} \text{ whenever } d_1 \blacktriangleleft d_2.$$

We can prove now the following key result:

Proposition 12. *Let X be a set and L be a completely distributive lattice. For a meet-dense $D \subseteq L$ and a family $\mathcal{F} = \{F_d \subseteq X : d \in D\}$. Then the following are equivalent:*

(1) *$\mathcal{F}$ is a $\blacktriangleleft$-scale.*

(2) *There exists a function $f : X \to L$ such that for every $d \in D$:*

$$[f \blacktriangleleft d] \subseteq F_d \subseteq [f \leq d].$$

Remarks 13. (1) Given a $\blacktriangleleft$-scale $\mathcal{F}$, the function $f : X \to L$ defined by $f(x) = \bigwedge\{d \in D : x \in F_d\}$ is said to be *generated by* $\mathcal{F}$.

(2) Given an $f : X \to L$, both $\{[f \blacktriangleleft d] : d \in D\}$ and $\{[f \leq d] : d \in D\}$ are $\blacktriangleleft$-scales that generate the function f. We mainly use the first one, $\{[f \blacktriangleleft d] : d \in D\}$, but it is important to note here that this is not the only possible choice.

(3) Proposition 12 means that $\blacktriangleleft$-scales on X and L-valued functions on X are equivalent notions; given a $\blacktriangleleft$-scale $\mathcal{F}$ we have the function $f : X \to L$ generated by $\mathcal{F}$ and given a function $f : X \to L$ we have the $\blacktriangleleft$-scale $\{[f \blacktriangleleft d] : d \in D\}$.

(4) If L is $\blacktriangleleft$-separable then we can choose D to be separable and so we conclude that we can identify L-valued functions on X with countable $\blacktriangleleft$-scales on X.

3.2. **Lattice-valued semicontinuous functions.** Any poset $(L, \leq)$ carries three well-known topologies:

- the *upper topology* $\nu(L)$ having $\{L \setminus L_{\leq}(a) : a \in L\}$ as a subbase.
- the *lower topology* $\omega(L)$ having $\{L \setminus U_{\leq}(a) : a \in L\}$ as a subbase.
- the *interval topology* $\nu(L) \vee \omega(L)$.

It is easy to check the following result:

Proposition 14. *Let L be a $\blacktriangleleft$-separable completely distributive lattice and $D \subseteq L$ a meet-dense subset. Then:*

(1) $\{L \setminus L_{\leq}(d) : d \in D\}$ *is a subbase of* $\nu(L)$.

(2) $\{L_{\blacktriangleleft}(d) : d \in D\}$ *is a subbase of* $\omega(L)$.

Once again, we are particularly interested in the following:

Examples 15. (1) In the case of the extended real line $\overline{\mathbb{R}}$ with $D = \mathbb{Q}$ the previous proposition just says that $\{(q, +\infty] : q \in \mathbb{Q}\}$ (resp. $\{[-\infty, q) : q \in \mathbb{Q}\}$) is a subbase of the upper (resp. lower) topology. In fact, in this particular case, both families are not only subbases but also bases. It is important to emphasize here that this is not true in general, as shown by the following example.

(2) Let $\mathcal{Y}$ and D be as in Example 10 (2). By Proposition 14 (1) we have the following subbase of the upper topology $\nu(\mathcal{Y})$ on $\mathcal{Y}$:

$$\{(\mathcal{Y} \setminus L_{\leq}((q,q)) : q \in \mathbb{Q}\} \cup \{(\mathcal{Y} \setminus L_{\leq}((q,+\infty)) : q \in \mathbb{Q}\}.$$

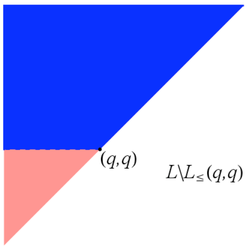

Figure 4(1)

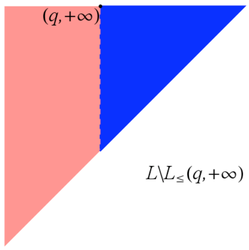

Figure 4(2)

Similarly, by Proposition 14 (2) we have the following subbase of the lower topology $\omega(\mathcal{Y})$ on $\mathcal{Y}$:

$$\{L_{\blacktriangleleft}(q,q) : q \in \mathbb{Q}\} \cup \{L_{\blacktriangleleft}(q,1) : q \in \mathbb{Q}\}.$$

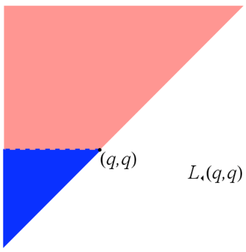

Figure 5(1)

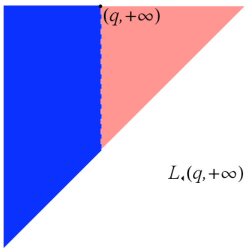

Figure 5(2)

Definition 16. Given a topological space (X, τ) and $f : X \to L$ we say that:

(1) f is *lower semicontinuous* iff it is continuous with respect to the upper topology $\nu(L)$;

(2) f is *upper semicontinuous* iff it is continuous with respect to the lower topology $\omega(L)$;

(3) f is *continuous* iff it is continuous with respect to the interval topology.

We have now the following immediate corollary of Proposition 14.

Corollary 17. *Let L be a $\blacktriangleleft$-separable completely distributive lattice and $D \subseteq L$ a meet-dense subset. Given a topological space (X, τ) and $f : X \to L$ we say that:*

(1) *f is lower semicontinuous iff $[f \leq d]$ is closed for all $d \in D$;*

(2) *f is upper semicontinuous iff $[f \blacktriangleleft d]$ is open for all $d \in D$.*

After the equivalence stated in Proposition 12 between $\blacktriangleleft$-scales on X and L-valued functions on X it is natural to ask wether given an L-valued function $f : X \to L$ generated by a $\blacktriangleleft$-scale $\{F_d \subseteq X : d \in D\}$ it is possible to characterize the (semi)continuity of f in terms of the elements of the $\blacktriangleleft$-scale.

In this sense, we have now the following result. It is essentially proved in [8].

Theorem 18. *For X a topological space, D a meet-dense subset of L, and $f : X \to L$ being generated by the $\blacktriangleleft$-scale $\{F_d \subseteq X : d \in D\}$, the following hold:*

(1) *f is lower semicontinuous iff $\overline{F_{d_1}} \subseteq F_{d_2}$ whenever $d_1 \blacktriangleleft d_2$;*
(2) *f is upper semicontinuous iff $F_{d_1} \subset \operatorname{Int} F_{d_2}$ whenever $d_1 \blacktriangleleft d_2$;*
(3) *f is continuous iff $\overline{F_{d_1}} \subset \operatorname{Int} F_{d_2}$ whenever $d_1 \blacktriangleleft d_2$.*

4. Continuous representation of interval orders, semiorders and total preorders by means of scales

Finally, we can use the results in previous sections in order to deal with the (continuous) representability of all these kinds of binary relations by means of countable increasing scales.

For the sake of completeness, we shall also include here the statements in the case of total preorders and interval orders, whose proofs can be found in [3].

Let us start by particularizing the results in the previous section to simplest situation: the case of the extended real line. This will serve us to obtain the characterizations of (continuous) representability of both total preorders and semiorders.

Definition 19. Let X be a set. We say that a family $\mathcal{F} = \{F_q \subseteq X\}_{q\in\mathbb{Q}}$ is a *scale* if it is a $\blacktriangleleft$-scale in the sense of Definition 11, i.e. if $\mathcal{F}$ is *$<$-increasing*, i.e.

$$F_{q_1} \subseteq F_{q_2} \quad \text{whenever } q_1 < q_2.$$

The function $f : X \to \overline{\mathbb{R}}$ defined by

$$f(x) = \bigwedge\{q \in \mathbb{Q}\, x \in F_q\}$$

is said to be generated by the scale $\mathcal{F}$.

4.1. Total preorders.

We start with the following theorem in the case of total preorders

Theorem 20 ([3], see also . . .)**.** *Let (X, τ) be a topological space and $\mathcal{R}$ an total preorder on X. Then the following are equivalent:*

(1) *$\mathcal{R}$ is continuously representable*;
(2) *There exists a scale $\{F_q\}_{q\in\mathbb{Q}}$ satisfying for each $x, y \in X$*
    (a) *$x\mathcal{P}y \iff \exists q_1 < q_2 \in \mathbb{Q}$ such that $x \in F_{q_1}$ and $y \notin F_{q_2}$.* *(representability)*
    (b) *$\overline{F_{q_1}} \subseteq \operatorname{Int} F_{q_2}$ whenever $q_1 < q_2 \in \mathbb{Q}$.* *(continuity)*

4.2. Typical semiorders.

Since the representability of semiorders is the main originality of the present paper comparing with [3], we shall include here a more detailed analysis of the whole situation in this particular case. Note that the corresponding results could be also stated in the case of total preorders and interval orders.

Theorem 21. *Let $\mathcal{R}$ an typical semiorder on X. Then the following are equivalent:*

(1) *$\mathcal{R}$ is representable in $\overline{\mathbb{R}}$*;
(2) *There exists a scale $\{F_q\}_{q\in\mathbb{Q}}$ satisfying for each $x, y \in X$*
    (a) *$x\mathcal{P}y \iff \exists q \in \mathbb{Q}$ such that $x \in F_q$ and $y \notin F_{q+1}$.* *(representability)*

Proof. (1) $\Longrightarrow$ (2): Let $f : (X, \mathcal{R}) \to (\overline{\mathbb{R}}, \mathcal{R}_{s.o.})$ be such that $x\mathcal{R}y \iff f(x)\mathcal{R}_{s.o.}f(y)$ for each $x, y \in X$. It follows from Remarks 13 (2) that $\{F_q = [f < q]\}_{q\in\mathbb{Q}}$ is a scale.

Given $x, y \in X$ we have

$$\begin{aligned} x\mathcal{P}y &\iff f(x)\mathcal{P}_{s.o.}f(y)\\ &\iff f(x) < f(y) - 1\\ &\iff \text{there exists } q \in \mathbb{Q} \text{ with } f(x) < q \text{ and } q + 1 \leq f(y)\\ &\iff \text{there exists } q \in \mathbb{Q} \text{ with } x \in F_q \text{ and } y \notin F_{q+1}. \end{aligned}$$

(2) $\Longrightarrow$ (1): Let $\{F_q\}_{q\in\mathbb{Q}}$ be a scale satisfying condition (a). Let $f : X \to \overline{\mathbb{R}}$ be the function generated by the scale, that is, $f(x) = \bigwedge\{q \in \mathbb{Q} : x \in F_q\}$ for each $x \in X$. Let $x, y \in X$. If $x\mathcal{P}y$, then there exists $q \in \mathbb{Q}$ such that $x \in F_q$ and $y \notin F_{q+1}$, then $f(x) \leq q$ and $y \notin F_{q'}$ for all $q' \leq q + 1$. Hence $f(x) \leq q < q + 1 \leq f(y)$. We conclude that $f(x)\mathcal{P}_{s.o.}f(y)$. Conversely, if $f(x)\mathcal{P}_{s.o.}f(y)$, one can always find an $r \in \mathbb{R}$ such that $f(x) < r$ and $r + 1 \leq f(y)$. Consequently there exists $q \in \mathbb{Q}$ such that $q < r$ and $x \in F_q$ and $y \notin F_{q+1}$ $\square$

Following Definition 4 (2), we say that a typical semiorder $\mathcal{R}$ on X is *lower* (resp. *upper*) *$\mathcal{R}$ semicontinuously representable in* $\overline{\mathbb{R}}$ if it is representable through an lower (resp. upper) semicontinuous function $u : (X, \tau) \to (\overline{\mathbb{R}}, \tau_u)$. Note that in this case the notion of lower (upper) semicontinuity is the usual notion in the case of real-valued functions.

Theorem 22. *Let (X, τ) be a topological space and $\mathcal{R}$ a typical semiorder on X. Then the following are equivalent:*

(1) *$\mathcal{R}$ is lower semicontinuously representable in $\overline{\mathbb{R}}$;*
(2) *There exists a scale $\{F_q\}_{q\in\mathbb{Q}}$ satisfying for each $x, y \in X$*
  (a) *$x\mathcal{P}y \iff \exists q \in \mathbb{Q}$ such that $x \in F_q$ and $y \notin F_{q+1}$.* *(representability)*
  (b$_1$) *$\overline{F_{q_1}} \subseteq F_{q_2}$ whenever $q_1 < q_2 \in \mathbb{Q}$.* *(lower semicontinuity)*

Proof. It follows immediately from Theorem 21 and Theorem 18 (1). $\square$

Clearly enough we also have the dual result:

Theorem 23. *Let (X, τ) be a topological space and $\mathcal{R}$ a typical semiorder on X. Then the following are equivalent:*

(1) *$\mathcal{R}$ is upper semicontinuously representable in $\overline{\mathbb{R}}$;*
(2) *There exists a scale $\{F_q\}_{q\in\mathbb{Q}}$ satisfying for each $x, y \in X$*
  (a) *$x\mathcal{P}y \iff \exists q \in \mathbb{Q}$ such that $x \in F_q$ and $y \notin F_{q+1}$.* *(representability)*
  (b$_2$) *$F_{q_1} \subseteq \operatorname{Int} F_{q_2}$ whenever $q_1 < q_2 \in \mathbb{Q}$.* *(upper semicontinuity)*

Finally, combining Theorems 22 and 23 we obtain:

Theorem 24. *Let (X, τ) be a topological space and $\mathcal{R}$ an typical semiorder on X. Then the following are equivalent:*

(1) *$\mathcal{R}$ is continuously representable in $\overline{\mathbb{R}}$;*
(2) *There exists a scale $\{F_q\}_{q\in\mathbb{Q}}$ satisfying for each $x, y \in X$*
  (a) *$x\mathcal{P}y \iff \exists q_1 < q_2 \in \mathbb{Q}$ such that $x \in F_q$ and $y \notin F_{q+1}$.* *(representability)*
  (b) *$\overline{F_{q_1}} \subseteq \operatorname{Int} F_{q_2}$ whenever $q_1 < q_2 \in \mathbb{Q}$.* *(continuity)*

4.3. **Interval orders.** In this case we can proceed as in [3] and introduce the following definition. (See [3] for a more exhaustive presentation of these results).

Definition 25. Let X be a set. A family $\mathcal{F} = \{F_{(q,q)}\}_{q\in\mathbb{Q}} \cup \{F_{(q,1)}\}_{q\in\mathbb{Q}}$ of subsets of X is said to be an *i.o. scale* if it is a $\blacktriangleleft$-scale in the sense of Definition 11 i.e. if $\mathcal{F}$ is $\blacktriangleleft$-*increasing*:

$$F_{(q_1,q_1)} \subseteq F_{(q_2,q_2)},\ F_{(q_1,q_1)} \subseteq F_{(q_2,1)} \text{ and } F_{(q_1,1)} \subseteq F_{(q_2,1)} \quad \text{whenever } q_1 < q_2.$$

The function $f : X \to L$ defined by $f(x) = \bigwedge\{d \in D : x \in F_d\}$ is said to be generated by the i.o. scale $\mathcal{F}$.

Proposition 26. *Let $\{F_{(q,q)}\}_{q\in\mathbb{Q}} \cup \{F_{(q,1)}\}_{q\in\mathbb{Q}}$ be an i.o. scale on X. Then for each $x \in X$ we have $f(x) = (u(x), v(x))$ where $u(x) = \bigwedge\{q : x \in F_{(q,1)}\}$ and $v(x) = \bigwedge\{q : x \in F_{(q,q)}\}$.*

Theorem 27. *Let (X, τ) be a topological space and $\mathcal{R}$ an interval order on X. The following are equivalent:*

(1) *$\mathcal{R}$ is continuously representable through a pair of continuous real-valued functions u and v with values in $[0,1]$, where u is a representation for the total preorder $\mathcal{R}^{**}$ and v is a representation for the total preorder $\mathcal{R}^{*}$;*
(2) *There exists an i.o. scale $\{F_{(q,q)}\}_{q\in\mathbb{Q}} \cup \{F_{(q,1)}\}\}_{q\in\mathbb{Q}}$ satisfying for each $x, y \in X$*
   (a) *$x\mathcal{P}y \iff \exists q_1 < q_2 \in \mathbb{Q}$ with $x \in F_{(q_1,q_1)}$ and $y \notin F_{(q_2,1)}$.* *(representability)*
   (b) *$F_{(q,q)}$ is $\mathcal{R}^*$-decreasing and $F_{(q,1)}$ is $\mathcal{R}^{**}$-decreasing for every $q \in \mathbb{Q}$.* *(representability for $\mathcal{R}^*$ and $\mathcal{R}^{**}$)*
   (c) *$\overline{F_{(q_1,q_1)}} \subseteq \operatorname{Int} F_{(q_2,q_2)}$ and $\overline{F_{(q_1,1)}} \subseteq \operatorname{Int} F_{(q_2,1)}$ whenever $q_1 < q_2 \in \mathbb{Q}$.* *(continuity)*

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