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On monotone normality and T_1 axiom

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Abstract

Monotone normality is usually defined in the class of T_1 spaces. In this work we extend well-known characterizations of these kind of spaces to the T_1-free context and besides, we generalize such results considering lattice-valued maps instead of real-valued maps. Among the new T_1-free characterizations of monotonically normal spaces we provide are a Katetov-Tong-type insertion theorem and Tietze-type extension theorem for lattice-valued functions.

1. Introduction

There has been an extensive literature devoted to monotonically normal spaces (see the surveys [3, 5] and the references on them) since the notion was introduced in [1, 8, 17]. With the exception of [9], monotone normality has always been studied in the restricted class of T_1 spaces.

The influence of computer science not only has given relevance to those spaces not satisfying T_1 axiom, but also has focussed attention in functions with values in ordered sets rather than in the reals. Continuous lattices or domains with their Scott topology are an important class among the spaces which do not satisfy the T_1 axiom.

In concordance with those ideas, the present work explores monotone normality in a T_1 free context. Also lattice-valued functions rather than real-valued functions are considered throughout. The techniques established in [6] will allow us to give lattice-valued counterparts of some known characterizations of monotonically normal spaces given in terms of real-valued functions, and all of them will be free of the T_1 axiom.

After some lattice theoretic preliminaries, the notion of monotone normality, free of T_1 axiom, is studied. Several characterizations of monotone normality in this context are provided and some deviation from T_1-monotonically normal spaces is exhibited. It is well known that in the class of normal spaces (either T_1 or not), complete normality and hereditary normality are equivalent concepts as well as the fact that open subsets inherit the property [16]. As to the class of T_1-monotonically normal spaces is concerned, it has been proved [2, 8, 13] that monotone normality is equivalent to any one of the following notions: complete monotone normality, hereditary monotone normality, open subsets inherit the property. The proof of these equivalences depends strongly on the

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axiom T_1. It relies upon a new property, also equivalent to monotone normality, which can be properly called monotone regularity and implies the Hausdorff axiom. The question as to whether the above equivalences hold in spaces not satisfying the axiom T_1 is answered in negative. The answer is based on a construction of a non T_1 monotonically normal compactification associated to any topological space. It is important to notice that, when characterizing monotone normality, the role of points will now be played by the closure of singletons (the minimal closed sets in a non T_1-space). This idea is as simple as effective. It is also used to provide an extension property of lattice-valued functions for monotonically normal spaces. This extension property is obtained as a consequence of a monotone and lattice-valued version of the Katětov-Tong's insertion theorem and Urysohn's lemma that we shall also obtain.

This presentation is a summary of the two papers [12, 7] already published by the authors.

2. Preliminaries

2.1. **Lattices.** In the sequel L denotes a completely distributive lattice (with bounds 0 and 1). For general concepts regarding lattices and complete distributivity we refer the reader to [4]. We shall use the Raney's characterization of complete distributivity in terms of an extra order $\lhd$ with the approximation property:

Given a complete lattice L and $a, b \in L$, take the following binary relation: $a \lhd b$ if and only if, whenever $C \subseteq L$ and $b \leq \bigvee C$, there exists some $c \in C$ with $a \leq c$. The lattice L is said to be *completely distributive* if and only if $a = \bigvee\{b \in L : b \lhd a\}$ for each $a \in L$. The previous relation has the following properties: (1) $a \lhd b$ implies $a \leq b$; (2) $c \leq a \lhd b \leq d$ implies $c \lhd d$; (3) $a \lhd b$ implies $a \lhd c \lhd b$ for some $c \in L$ (Interpolation Property).

A subset $D \subset L$ is called *join-dense* (or a *base*) if $a = \bigvee\{d \in D : d \leq a\}$ for each $a \in L$. An element $a \in L$ is called *supercompact* if $a \lhd a$ holds. As in [6], any completely distributive lattice which has a countable join-dense subset free of supercompact elements will be called $\lhd$-*separable*.

2.2. **Semicontinuous lattice-valued functions.** Given a set X, L^X denotes the collection of all functions from X into L ordered pointwisely, i.e., $f \leq g$ in L^X iff $f(x) \leq g(x)$ in L for each $x \in X$. Given $f \in L^X$ and $a \in L$, we write $[f \geq a] = \{x \in X : a \leq f(x)\}$ and similarly for $[f \rhd a]$.

Let X be a topological space. A function $f \in L^X$ is said to be *upper* (resp. *lower*) *semicontinuous* if $[f \geq a]$ is closed (resp. $[f \rhd a]$ is open) for each $a \in L$ (cf. [6, 11]).

The collections of all upper [lower] semicontinuous functions of L^X will be denoted by $\mathrm{USC}(X, L)$ [$\mathrm{LSC}(X, L)$]. Elements of $\mathrm{C}(X, L) = \mathrm{USC}(X, L) \cap \mathrm{LSC}(X, L)$ are called *continuous*.

3. Monotone normality in a T_1 free context

Let X be a topological space with topology $o(X)$ and let us denote by $\kappa(X)$ the family of closed subsets of X. We shall need the following sets (the notation comes from [2] and [9]):

$$\mathcal{D}_X = \{(K, U) \in \kappa(X) \times o(X) : K \subset U\},$$

$$\mathcal{S}_X = \{(A, B) \in 2^X \times 2^X : \overline{A} \subset B \text{ and } A \subset \operatorname{Int} B\},$$

$$\widehat{\mathcal{S}_X} = \Big\{(A, B) \in 2^X \times 2^X : \overline{A} \subset \bigcap_{y \in X \setminus B} \operatorname{Int}(X \setminus \{y\}) \text{ and } \bigcup_{x \in A} \overline{\{x\}} \subset \operatorname{Int} B\Big\}.$$

All these sets are partially ordered considering the componentwise order.
Note that $\mathcal{D}_X \subset \widehat{\mathcal{S}_X} \subset \mathcal{S}_X$ and besides, $\mathcal{S}_X = \widehat{\mathcal{S}_X}$ if X is T_1.

Definition 1. [8]. A topological space X is called *monotonically normal* if there exists and order-preserving function $\Delta : \mathcal{D}_X \to o(X)$ such that

$$K \subset \Delta(K,U) \subset \overline{\Delta(K,U)} \subset U$$

for any $(K,U) \in \mathcal{D}_X$. The function Δ is called a *monotone normality operator.*

Remark 2. The spaces described in the previous definition were originally assumed to be T_1. However, as in [9], we will not consider the axiom T_1 as part of the definition of monotone normality. A trivial example of a monotonically normal space, not satisfying T_1 axiom, is provided by the reals endowed with the right-order topology (Kolmogorov's line). We will show some more relevant examples after Proposition 4.

Let us recall the following characterizations of monotone normality (under condition T_1). The first one was originally called complete monotone normality [17]. The second one could be properly called monotone regularity. Many of the known results on monotonically normal spaces rely on these characterizations.

Proposition 3. [2, 8]. *Let X be T_1. The following statements are equivalent:*

(1) X is monotonically normal;
(2) There exists an order-preserving function $\Sigma : \mathcal{S}_X \to o(X)$ such that $A \subset \Sigma(A,B) \subset \overline{\Sigma(A,B)} \subset B$ for any $(A,B) \in \mathcal{S}_X$.
(3) There exists a function H which assigns to each ordered pair (x,U) (with $x \in U$ and $U \in o(X)$) an open set $H(x,U)$ such that:
  (a) $x \in H(x,U) \subset U$,
  (b) If $x \in U \subset V$, then $H(x,U) \subset H(x,V)$,
  (c) If $x \neq y$ are points of X, then $H(x, X \setminus \{y\}) \cap H(y, X \setminus \{x\}) = \emptyset$.

The proposition below gives the counterpart of Proposition 3 when T_1 axiom is not assumed. We would like to point out that it is the key to extend many known results to the T_1-free context.

Proposition 4. *Let X be a topological space. The following are equivalent:*

(1) X is monotonically normal;
(2) There exists an order preserving function $\widehat{\Sigma} : \widehat{\mathcal{S}_X} \to o(X)$ such that

$$A \subset \widehat{\Sigma}(A,B) \subset \overline{\widehat{\Sigma}(A,B)} \subset B$$

for any $(A,B) \in \widehat{\mathcal{S}_X}$.
(3) For each point x and open set U containing $\overline{\{x\}}$ we can assign an open set $H(x,U)$ such that:
  (a) $\overline{\{x\}} \subset H(x,U) \subset U$;
  (b) if V is open and $\overline{\{x\}} \subset U \subset V$, then $H(x,U) \subset H(x,V)$;
  (c) if $\overline{\{x\}} \cap \overline{\{y\}} = \varnothing$, then $H(x, X \setminus \overline{\{y\}}) \cap H(y, X \setminus \overline{\{x\}}) = \varnothing$.

Remark 5. As it was said in the introduction, under the T_1 axiom, monotone normality and hereditary monotone normality are equivalent axioms (see Proposition 3 in [13], or Lemma 2.2 in [8] or Theorem 2.4 in [2]). The proof of this result is based on Proposition 3. However, in the absence of the T_1 axiom, the equivalence between monotone normality and hereditary monotone normality cannot be derived directly from Proposition 4. Even more, for spaces not satisfying the axiom T_1, this equivalence does not hold, as the following construction shows:

Any topological space has a monotonically normal non T_1 compactification. Indeed, for a topological space (X, τ), let Y be a set such that $X \subset Y$ and $Y \setminus X \neq \emptyset$. Define on Y the topology $\tau^\star = \tau \cup \{Y\}$. Then, X is an open, dense subspace of the monotonically normal non T_1 compact space Y.

Some other interesting examples of monotonically normal non T_1 spaces, come from the field of quasi-pseudo-metrics (where by a quasi-pseudo-metric we mean a map $d : X \times X \to [0, \infty)$ such that $d(x, y) = d(y, x) = 0$ iff $x = y$ and $d(x, z) \leq d(x, y) + d(y, z)$ for any $x, y, z \in X$).

Example 6. Let $K > 0$ and $X = (-\infty, 0] \cup [K, +\infty)$. Define the map $d : X \times X \to [0, \infty)$ as follows:

$$d(x, y) = \begin{cases} |x - y| & \text{if } x, y \leq 0 \text{ or } x, y \geq K, \\ y - x - K & \text{if } x \leq 0 \text{ and } y \geq K, \\ x - y & \text{if } y \leq 0 \text{ and } x \geq K. \end{cases}$$

The map d defined above is a quasi-pseudo-metric and the collection $\{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}$ (where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$) forms a base for a topology τ_d on X.

Clearly the space (X, τ_d) is not T_1 (notice that $\overline{\{0^*\}} = \{0, 0^*\}$). Even if monotone normality is not a property easy to manage with, condition (3) of Proposition 4 turns out to be very effective to prove that the previous space is monotonically normal.

4. Monotone normality and lattice-valued functions

Monotonically normal spaces will now be characterized in terms of insertion and extension of some kind of lattice-valued functions. Before doing so, we shall need some more notation. Let us consider the following families:

$$UL(X, L) = \{(f, g) \in USC(X, L) \times LSC(X, L) : f \leq g\},$$

$$SF(X, L) = \{(f, g) \in L^X \times L^X : f^* \leq g \text{ and } f \leq g_*\},$$

$$\widehat{SF}(X, L) = \Big\{(f, g) \in L^X \times L^X : \bigvee_{y \in \overline{\{x\}}} f^*(y) \leq g(x) \text{ and } f(x) \leq \bigwedge_{y \in \overline{\{x\}}} g_*(y) \text{ for each } x \in X\Big\},$$

which are partially ordered considering the componentwise order.

Remark 7. (a) $UL(X, L) \subset \widehat{SF}(X, L) \subset SF(X, L)$ and besides, $SF(X, L) = \widehat{SF}(X, L)$ if X is T_1.
(b) $(A, B) \in \mathcal{S}_X$ $(\widehat{\mathcal{S}_X})$ if and only if $(1_A, 1_B) \in SF(X, L)$ $(\widehat{SF}(X, L))$.

The proposition below is a characterization of monotonically normal spaces in terms of insertion of semicontinuous lattice-valued functions. For the case of real-valued functions, the equivalence (1) $\Leftrightarrow$ (2) in the T_1-free context was obtained in [9].

Proposition 8. *Let X be a topological space and L be a completely distributive lattice. The following are equivalent:*

(1) X is monotonically normal;
(2) There exists an order preserving function $\Gamma : UL(X,L) \to LSC(X,L)$ such that $f \le \Gamma(f,g) \le \Gamma(f,g)^* \le g$ for any $(f,g) \in UL(X,L)$.
(3) There exists an order preserving function $\widehat{\Theta} : \widehat{SF}(X,L) \to LSC(X,L)$ such that $f \le \widehat{\Theta}(f,g) \le \widehat{\Theta}(f,g)^* \le g$ for any $(f,g) \in \widehat{SF}(X,L)$;

Thanks to the previous proposition, the following equivalent results hold for arbitrary topological spaces. The first one is a monotone and lattice-valued version of the well known Katetov-Tong insertion theorem and generalizes a result obtained by Kubiak [9] (see also Lane and Pan [10]). The second one is a monotone lattice-valued version of Urysohn's lemma, which for the case of real-valued function was obtained by Borges [1, 2].

Theorem 9. *Let X be a topological space and L be a completely distributive $\lhd$-separable lattice. The following statements are equivalent:*

(1) X is monotonically normal;
(2) [Monotone Katětov-Tong theorem] *There exists an order-preserving function $\Lambda : UL(X,L) \to C(X,L)$ such that $f \le \Lambda(f,g) \le g$ for any $(f,g) \in UL(X,L)$;*
(3) [Monotone Urysohn's lemma] *There exists an order-preserving function $\Psi : \mathcal{D}_X \to C(X,L)$ such that $\Psi(K,U)(K) = \{1\}$ and $\Psi(K,U)(X-U) = \{0\}$ for each $(K,U) \in \mathcal{D}_X$.*

As a consequence, we have the following result, which shows that monotonically normal spaces satisfy the monotone extension property for lattice-valued functions.

Corollary 10. *Let X be a topological space and L be a completely distributive $\lhd$-separable lattice. If X is monotonically normal, then for every closed subspace $A \subset X$ there exists an order-preserving function $\Phi : C(A,L) \to C(X,L)$ such that $\Phi(f)|_A = f$ for all $f \in C(A,L)$.*

Remark 11. In [8] Heath, Lutzer and Zenor proved the previous extension property for $L = [0,1]$ (under T_1 axiom). In the same paper they raised the question of whether the converse was true. It was Van Douwen [15] who proved that, for real-valued functions, the previous property does not characterize monotone normality. Later, in 1995, Stares [14] pointed out that the problem for the converse not to hold seemed to be that the above property does not link continuous functions defined in different closed subspaces. Taking this fact into account, he gave an additional condition which solved the situation and obtained an analogue of the Tietze-Urysohn theorem for monotonically normal spaces [14, Theorem 2.3]. The proof of Stares depends on the axioms T_1. Our final result extends to the T_1-free context and generalizes to lattice-valued functions the extension theorem given by Stares. We include the proof to highlight the importance of Proposition 4.

Theorem 12. *Let X be a topological space and L be a completely distributive $\lhd$-separable lattice. The following are equivalent:*

(1) X is monotonically normal,
(2) For every closed subspace $A \subset X$ there exists an order-preserving function $\Phi_A : C(A,L) \to C(X,L)$ such that $\Phi_A(f)|_A = f$ for all $f \in C(A,L)$ and which satisfies the following two conditions:
  (a) If $A_1 \subset A_2$ are closed subspaces and $f_1 \in C(A_1,L)$, $f_2 \in C(A_2,L)$ are such that $f_2|_{A_1} \ge f_1$ and $f_2(x) = 1$ for any $x \in A_2 \setminus A_1$, then $\Phi_{A_2}(f_2) \ge \Phi_{A_1}(f_1)$.

(b) *If $A_1 \subset A_2$ are closed subspaces and $f_1 \in C(A_1, L)$, $f_2 \in C(A_2, L)$ are such that $f_2|_{A_1} \leq f_1$ and $f_2(x) = 0$ for any $x \in A_2 \setminus A_1$, then $\Phi_{A_2}(f_2) \leq \Phi_{A_1}(f_1)$.*

Proof. (1) $\Rightarrow$ (2): For any closed $A \subset X$ let $\Phi_A : C(A, L) \to C(X, L)$ be defined by $\Phi_A(f) = \Lambda(h_f, g_f)$, where $h_f, g_f : X \to L$ are such that $h_f = f = g_f$ on A, $h_f = 0$ and $g_f = 1$ on $X \setminus A$ and Λ the monotone inserter of Theorem 9 (2). If $A_1 \subset A_2$ are closed subspaces and $f_1 \in C(A_1, L)$, $f_2 \in C(A_2, L)$ are such that $f_2|_{A_1} \geq f_1$ and $f_2(x) = 1$ for any $x \in A_2 \setminus A_1$, then $h_{f_1} \leq h_{f_2}$ and $g_{f_1} \leq g_{f_2}$ so

$$\Phi_{A_1}(f_1) = \Lambda(h_{f_1}, g_{f_1}) \leq \Lambda(h_{f_2}, g_{f_2}) = \Phi_{A_2}(f_2)$$

and hence condition (a) is satisfied. Condition (b) is proved similarly.

(2) $\Rightarrow$ (1): In order to prove monotone normality we will use (3) of Proposition 4. Let U be open and $x \in X$ such that $\overline{\{x\}} \subset U$. We take the closed subspace $A_U^x = \overline{\{x\}} \cup (X \setminus U)$ and define the maps

$$f_{A_U^x}, g_{A_U^x} : \overline{\{x\}} \cup (X \setminus U) \to L$$

as $f_{A_U^x} = 1_{X \setminus U}$ and $g_{A_U^x} = 1_{\overline{\{x\}}}$. Then $f_{A_U^x}, g_{A_U^x} \in C(A_U^x, L)$ and hence the extensions $\Phi_{A_U^x}(f_{A_U^x}), \Phi_{A_U^x}(g_{A_U^x})$ belong to $C(X, L)$. Let $a \in L \setminus \{0\}$ be such that $0 \lhd a \lhd 1$ and define

$$H(x, U) = (X \setminus [\Phi_{A_U^x}(f_{A_U^x}) \geq a]) \cap [\Phi_{A_U^x}(g_{A_U^x}) \rhd a].$$

Then, clearly $H(x, U)$ is open and $\overline{\{x\}} \subset H(x, U) \subset U$. Now, if V is open and $\overline{\{x\}} \subset U \subset V$, by property (a) it easy to prove that $[\Phi_{A_V^x}(f_{A_V^x}) \geq a] \subset [\Phi_{A_U^x}(f_{A_U^x}) \geq a]$ and property (b) yields the inclusion $[\Phi_{A_U^x}(g_{A_U^x}) \rhd a] \subset [\Phi_{A_V^x}(g_{A_V^x}) \rhd a]$ so

$$H(x, U) \subset H(x, V).$$

Moreover, if $x, y \in X$ are such that $\overline{\{x\}} \cap \overline{\{y\}} = \varnothing$, one easily checks that

$$H(x, X \setminus \overline{\{y\}}) \cap H(y, X \setminus \overline{\{x\}}) = \emptyset.$$

By Proposition 4 the space is monotonically normal. $\square$

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