Localic real functions: a general setting

Making the ring of continuous localic real functions into a subring of all localic real functions

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– joint work with Tomasz Kubiak (Poznan) and Jorge Picado (Coimbra)

"The set $\mathrm{C}(X)$ *of all continuous, real-valued functions on a topological space* X *will be provided with an algebraic structure and an order structure. Since their definitions do not involve continuity, we begin by imposing these structures on the collection* $\mathbb{R}^X$ *of all functions from* X *into the set* $\mathbb{R}$ *of real numbers. [...]*

In fact, it is clear that $\mathbb{R}^X$ *is a commutative ring with unity element (provided that* X *is non empty). [...]*

Therefore $\mathrm{C}(X)$ *is a commutative ring, a subring of* $\mathbb{R}^X$*."*

L. Gillman and M. Jerison,
Rings of Continuous Functions

The category of frames (locales)

pointfree topology

$$(X, \mathcal{O}X) \rightsquigarrow (\mathcal{O}X, \subseteq)$$

$$A \wedge \bigvee_i B_i = \bigvee_i (A \wedge B_i)$$

f^{-1} preserves $\bigvee$ and $\wedge$

$(Y, \mathcal{O}Y)$ $(\mathcal{O}Y, \subseteq)$

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$\mathrm{Top}(X, \Sigma L) \simeq \mathrm{Frm}(L, \mathcal{O}X)$

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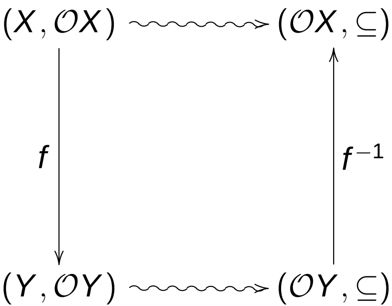

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usc	$f : X \to (\mathbb{R}, \mathcal{T}_l)$	$h : \mathfrak{L}_l(\mathbb{R}) \to L$ satisfying$(\ldots)$
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	$C(X) = USC(X) \cap LSC(X)$	???

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B. Banaschewski,
The real numbers in pointfree topology
Textos de Matemática, Série B, 12, Univ. de Coimbra, 1997.

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J. Gutiérrez García and J. Picado
On the algebraic representation of semicontinuity
Journal of Pure and Applied Algebra, 210 (2007) 299–306.

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(Q1) How to remedy this?

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- sublocale maps (i.e. onto frame homomorphisms),
- congruences,
- nuclei
- sublocale sets.

A *congruence* on a frame L, is an equivalence relation θ on L which is a subframe of $L \times L$ in the obvious sense.

The lattice of frame congruences on L under set inclusion is a frame, denoted by $\mathfrak{C}L$.

Open and *closed* congruences:

$$\Delta_a = \{(a,b) \in L \times L \mid a \wedge x = b \wedge x\}$$
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Complemented:

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$\nabla L := \{\nabla_x \mid x \in L\}$ is a subframe of $\mathfrak{C}L$.

The correspondence $x \mapsto \nabla_x$ defines an isomorphism $L \to \nabla L$.

$$\nabla : L \xrightarrow{\simeq} \nabla L \subset \mathfrak{C}L$$

Closure and interior of a congruence:

$$\overline{\theta} = \bigvee\{\nabla_a : \nabla_a \leq \theta\} \qquad \mathring{\theta} = \bigwedge\{\Delta_a : \theta \leq \Delta_a\}.$$

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Recall the isomorphisms

$$\text{Top}\,(X, (\mathbb{R}, \mathcal{T}_e)) \simeq \text{Frm}\,(\mathfrak{L}\,(\mathbb{R}), \mathcal{O}X)$$

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Slogan of pointfree topology:

Congruences in L ≡ generalized subspaces

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Definition

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- $F : \mathfrak{L}(\mathbb{R}) \to \mathfrak{C}L$ s. t. $F(\mathfrak{L}_l(\mathbb{R})) \subseteq \nabla L$ — usc — $\mathrm{USC}(L)$
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Let $\theta \in \mathfrak{C}L$ be complemented.

The characteristic function $\chi_\theta \in \mathrm{F}(L)$:

$$\chi_\theta(-,q) = \begin{cases} 0 & \text{if } q \le 0 \\ \theta & \text{if } 0 < q \le 1, \\ 1 & \text{if } q > 1 \end{cases} \qquad \chi_\theta(p,-) = \begin{cases} 1 & \text{if } p < 0 \\ \neg\theta & \text{if } 0 \le p < 1 \\ 0 & \text{if } p \ge 1. \end{cases}$$

- $\chi_\theta \in \mathrm{USC}(L)$ if and only if θ is closed.
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For $F \in \mathrm{F}(L)$ we define the *lower regularization* F°:

$$F^\circ(-,q) = \bigvee_{s<q} \neg \overline{F(s,-)} \quad \text{and} \quad F^\circ(p,-) = \bigvee_{r>p} \overline{F(r,-)}.$$

$F^\circ \leq F$

$F^{\circ\circ} = F^\circ$

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Localic real-valued functions

Achievements

- One can see semicontinuous functions as a particular kind of real-valued functions on the frame of congruences, with the same domain, namely $\mathfrak{L}(\mathbb{R})$.
- Being all upper and lower semicontinuous functions particular kinds of real-valued functions on the frame of congruences, we can compare them.
- By considering the algebraic operations of the ring $\mathrm{Frm}(\mathfrak{L}(\mathbb{R}), \mathfrak{C}L)$, we obtain, in particular, a way of defining the sum of upper and lower semicontinuous functions.
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Insertion theorems

Theorem (Katětov-Tong)

The following conditions on a frame L are equivalent:

(1) *L is normal.*

(2) *For every $F \in \mathrm{USC}(L)$ and every $G \in \mathrm{LSC}(L)$ with $F \leq G$, there exists $H \in \mathrm{C}(L)$ such that $F \leq H \leq G$.*

Theorem (Stone)

The following conditions on a frame L are equivalent:

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(3) $\mathrm{C}(L) = \{G^\circ : G \in \mathrm{USC}(L)\}$.

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(2) $\mathrm{C}(L) = \{F^- : F \in \mathrm{LSC}(L)\}$.

(3) $\mathrm{C}(L) = \{G^\circ : G \in \mathrm{USC}(L)\}$.

(4) *For every $F \in \mathrm{USC}(L)$ and every $G \in \mathrm{LSC}(L)$ with $G \leq F$, there exists $H \in \mathrm{C}(L)$ such that $G \leq H \leq F$.*

Insertion theorems

Let $\mathrm{UL}(L) = \{(F, G) \in \mathrm{USC}(L) \times \mathrm{LSC}(L) : F \leq G\}$ with the order $(F_1, G_1) \leq (F_2, G_2) \iff F_2 \leq F_1$ and $G_1 \leq G_2$.

Theorem (Monotone Katětov-Tong)

For a frame L, the following are equivalent:

(1) *L is monotonically normal.*

(2) *There exists a monotone function* $\Lambda : \mathrm{UL}(L) \to \mathrm{C}(L)$ *such that* $F \leq \Lambda(F, G) \leq G$ *for all* $(F, G) \in \mathrm{UL}(L)$.

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The following conditions on a frame L are equivalent:

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(3) *Each open sublocale of L is normal.*

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For each frame L the following are equivalent:

Strict insertion

Michael insertion theorem for perfectly normal frames. . .

Dowker insertion theorem for normal and countably paracompact frames. . .

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Extension theorems

Each $\theta \in \mathfrak{C}L$ determines a unique sublocale $S_\theta \subseteq L$ and a unique frame quotient $c_\theta \in \mathrm{Frm}(L, S_\theta)$.

$\widetilde{H} \in \mathrm{C}(L)$ is said to be a *continuous extension* of $H \in \mathrm{C}(S_\theta)$ if and only if the following diagram commutes

$$\begin{array}{ccccc} & & \nabla L & \overset{\nabla}{\longleftrightarrow} & L \\ & \overset{\widetilde{H}}{\nearrow} & & & \downarrow c_\theta \\ \mathfrak{L}(\mathbb{R}) & \overset{H}{\longrightarrow} & \nabla S_\theta & \overset{\nabla}{\longleftrightarrow} & S_\theta \end{array}$$

i.e. $c_\theta \circ \nabla \circ \widetilde{H} = \nabla \circ H$.

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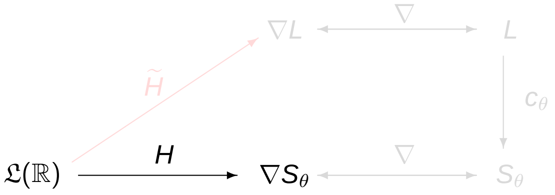

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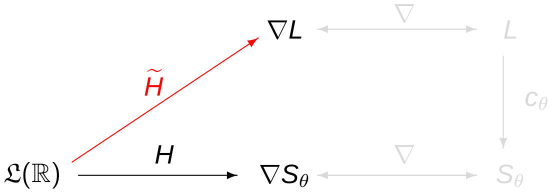

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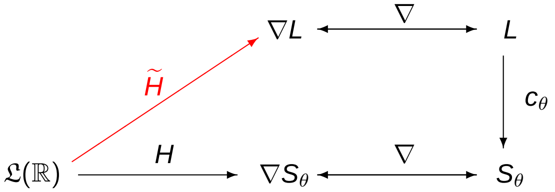

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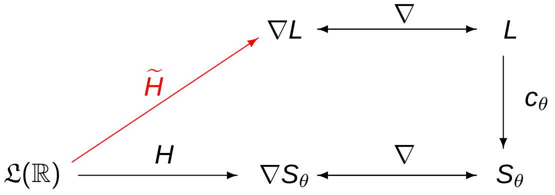

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Theorem (Tietze)

The following conditions on a frame L are equivalent:

(1) *L is normal.*

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Also versions for monotone normality, perfect normality, ...

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For a frame L, the following are equivalent:

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