The frame of the metric hedgehog and a cardinal extension of normality

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[1]Joint work with I. Mozo Carollo, J. Picado, and J. Walters-Wayland.

The metric hedgehog

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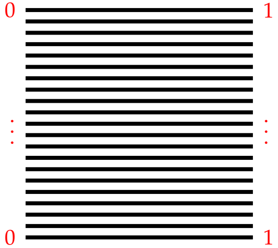

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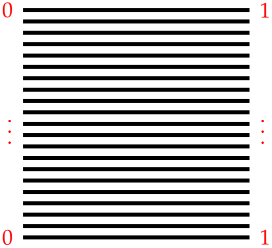

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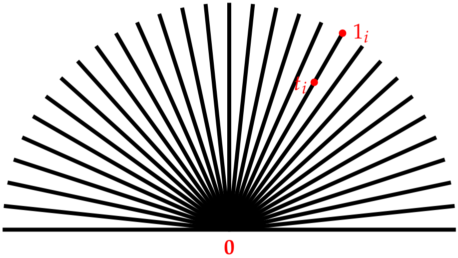

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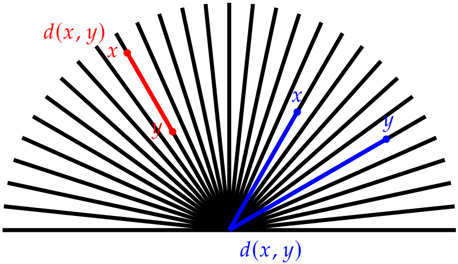

Now we identify all the copies (the spines) of the real unit interval at the origin and obtain the hedgehog $J(\kappa)$. The metric on $J(\kappa)$ is

$$d(x,y)=\begin{cases}|t-s|, & \text{if } x=t_i \text{ and } y=s_i,\\ t+s, & \text{if } x=t_i \text{ and } y=s_j \text{ with } j\neq i.\end{cases}$$

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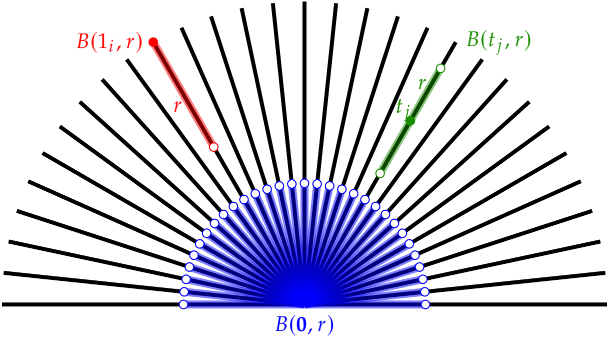

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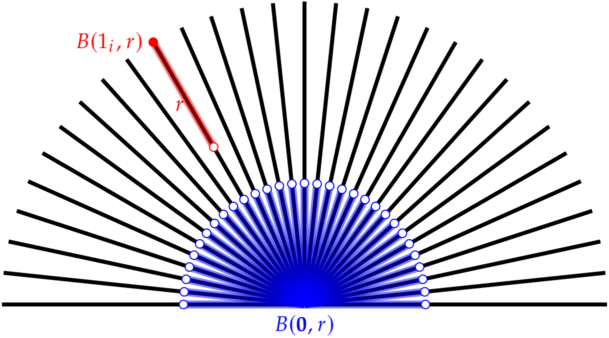
form a subbase for the metric topology.



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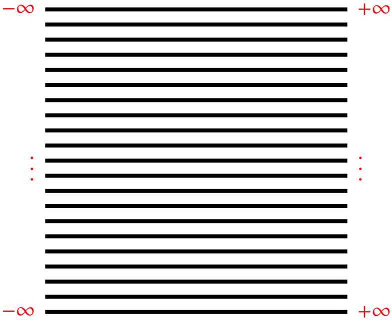

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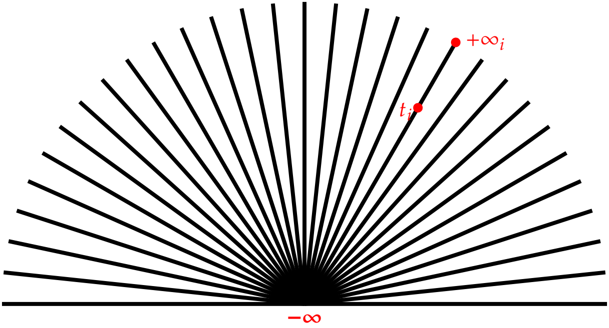

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$$\{B(-\infty, r) \mid r \in \mathbb{Q}\} \cup \{B(+\infty_i, r) \mid r \in \mathbb{Q} \text{ and } i \in I\}$$

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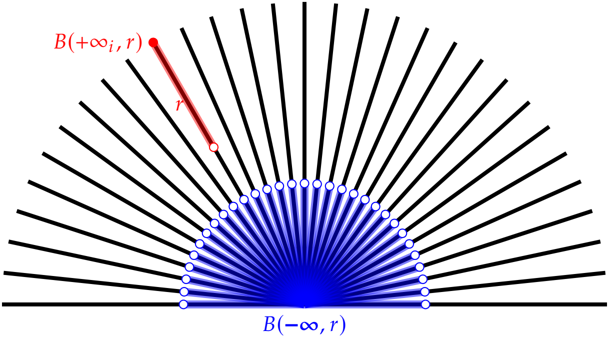

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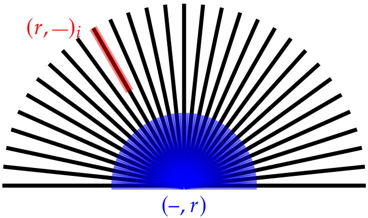
The frame of the metric hedgehog with κ spines is the frame $\mathfrak{L}(J(\kappa))$ presented by generators $(r, -)_i$ and $(-, r)$ for $r \in \mathbb{Q}$ and $i \in I$, subject to the defining relations:



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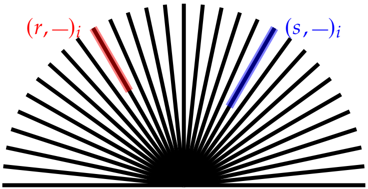

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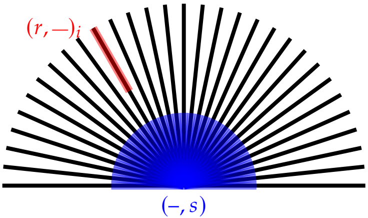

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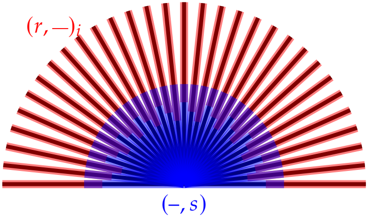

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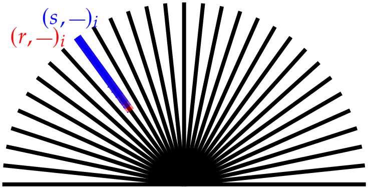

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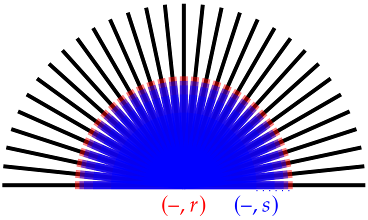

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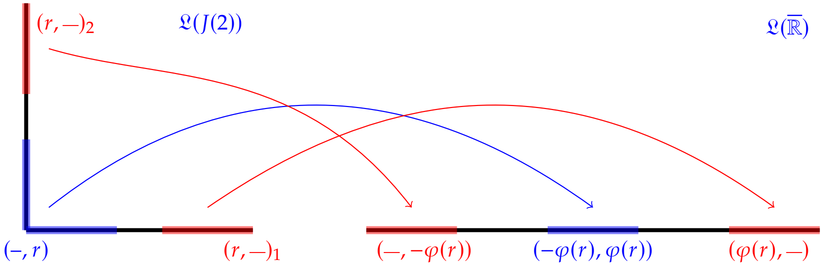
- B. Banaschewski, J.G.G. and J. Picado, Extended real functions in pointfree topology, *J. Pure Appl. Algebra* 216 (2012) 905–922.

– $\mathfrak{L}(J(1)) = \mathfrak{L}(\overline{\mathbb{R}}) \simeq \mathfrak{L}(J(2)$.

The isomorphism is induced by the following correspondence (where φ denotes any increasing bijection between $\mathbb{Q}$ and $\mathbb{Q}^+$):

$$(r,\text{—})_1 \longmapsto (\varphi(r),\text{—}),\ \ (r,\text{—})_2 \longmapsto (\text{—},-\varphi(r)),$$
$$(\text{—},r) \longmapsto (-\varphi(r),\text{—}) \wedge (\text{—},\varphi(r)).$$



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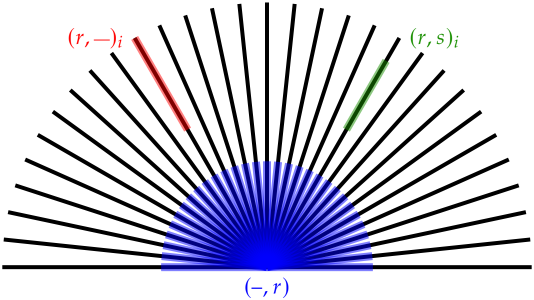

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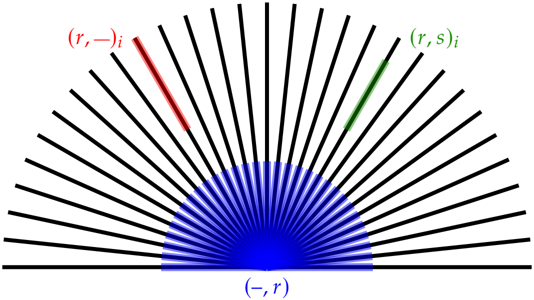
- The weight of $\mathfrak{L}(J(\kappa))$ is $\kappa \cdot \aleph_0$.



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Proof: For each $h \in \Sigma\mathfrak{L}(J(\kappa))$ define

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The homeomorphism $\pi \colon \Sigma\mathfrak{L}(J(\kappa)) \longrightarrow J(\kappa)$ is given by:

$$h \longmapsto \pi(h) = \begin{cases} (\varphi(\alpha_h), i_h), & \text{if } \alpha(h) \neq -\infty, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

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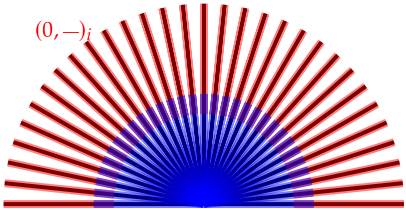
$$C = \{(\text{--}, 1)\} \cup \{(0, \text{--})_i \mid i \in I\}$$

is an infinite cover of $\mathfrak{L}(J(\kappa))$ with no proper subcover. ■



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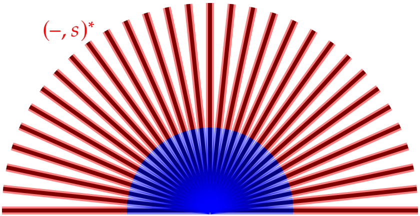

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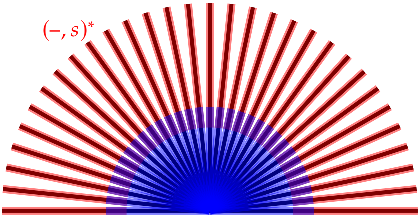

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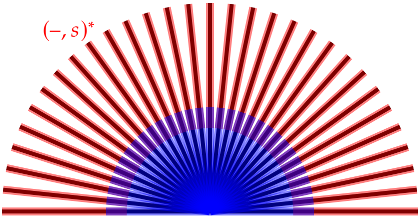

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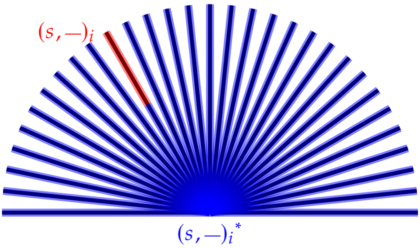

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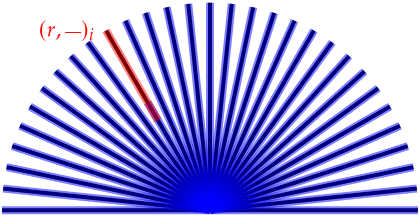

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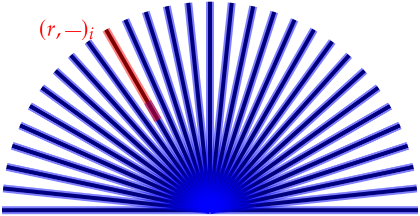

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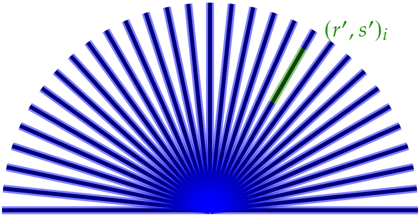
(3) $(r',s')_i^* = \bigvee_{\substack{j\neq i \\ t\in\mathbb{Q}}} (t,-)_j \vee (-,r') \vee (s',-)_i.$



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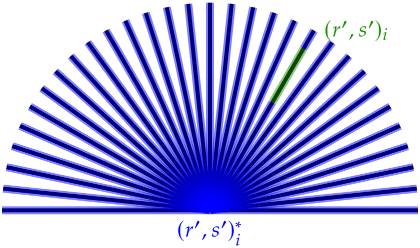
Proof: Since $B_\kappa = \{(-,r)\}_{r\in\mathbb{Q}} \cup \{(r,-)_i\}_{r\in\mathbb{Q},\, i\in I} \cup \{(r,s)_i\}_{r<s \text{ in } \mathbb{Q},\, i\in I}$ is a base of $\mathfrak{L}(J(\kappa))$, it is enough to prove that $b = \bigvee_{a\prec b} a$ for all $b \in B_\kappa$.

(3) $(r',s')_i^* = \bigvee_{\substack{j\neq i \\ t\in\mathbb{Q}}} (t,-)_j \vee (-,r') \vee (s',-)_i$. Hence $(r',s')_i \prec (r,s)_i$ whenever $r < r' < s' < s$ and $(r,s)_i = \bigvee_{r<r'<s'<s} (r',s')_i$. ∎



The frame of the metric hedgehog

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For each cardinal κ, the frame of the metric hedgehog $\mathfrak{L}(J(\kappa))$ is a metric frame of weight $\kappa \cdot \aleph_0$.

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(2) $\mathfrak{L}(J(\kappa))$ is a regular frame.

(3) For each $n \in \mathbb{N}$, let $C_n = C_n^1 \cup C_n^2 \cup C_n^3 \subseteq B_\kappa$ with

$$C_n^1 = \{(-,r) \mid r < -n\}, \quad C_n^2 = \{(r,-)_i \mid r > n,\, i \in I\} \quad \text{and}$$
$$C_n^3 = \left\{(r,s)_i \mid 0 < s - r < \tfrac{1}{n},\, i \in I\right\}.$$

These C_n determine an admissible countable system of covers of $\mathfrak{L}(J(\kappa))$. ■

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- J. R. Isbell, Atomless parts of spaces, *Math. Scand.* 31 (1972) 5–32.

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The right adjoint h_* is also a frame homomorphism, hence h is an isomorphism. ■

Continuous hedgehog-valued functions

By the familiar (dual) adjunction between the contravariant functors $\mathfrak{O}\colon \mathsf{Top} \to \mathsf{Frm}$ and $\Sigma\colon \mathsf{Frm} \to \mathsf{Top}$ there is a natural isomorphism $\mathsf{Top}(X, \Sigma L) \simeq \mathsf{Frm}(L, \mathfrak{O}X)$.

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An extended continuous real-valued function on a frame L is a frame homomorphism $\mathfrak{L}(\overline{\mathbb{R}}) \longrightarrow L$.

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Hence we define:

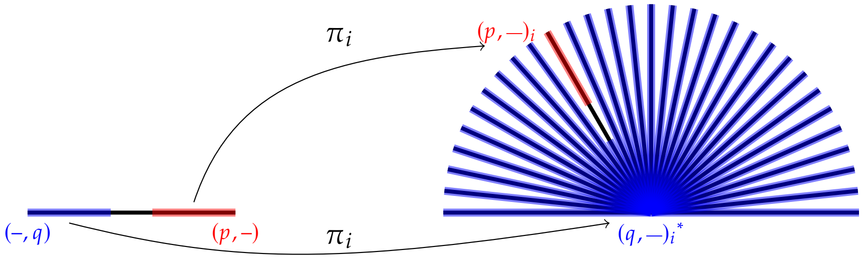
An extended continuous real-valued function on a frame L is a frame homomorphism $\mathfrak{L}(\overline{\mathbb{R}}) \longrightarrow L$.

A continuous (metric) hedgehog-valued function on a frame L is a frame homomorphism $\mathfrak{L}(J(\kappa)) \longrightarrow L$.

Continuous hedgehog-valued functions

For each $i \in I$ let $\pi_i \colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{L}(J(\kappa))$ be given by

$$\pi_i(p, -) = (p, -)_i \quad \text{and} \quad \pi_i(-, q) = (q, -)_i^*$$



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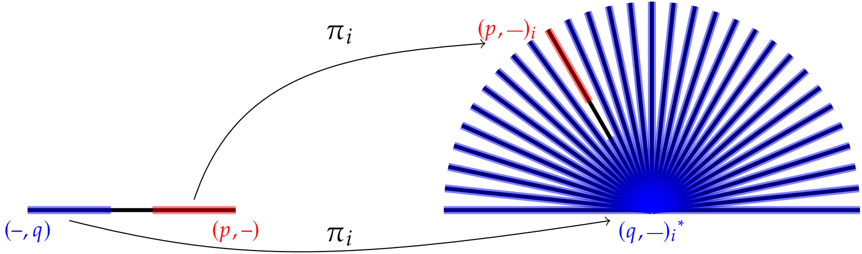

π_i turns the defining relations in $\mathfrak{L}(\overline{\mathbb{R}})$ into identities in $\mathfrak{L}(J(\kappa))$:

(r1) $\pi_i(p,-) \wedge \pi_i(-,q) = 0$ if $q \le p$,

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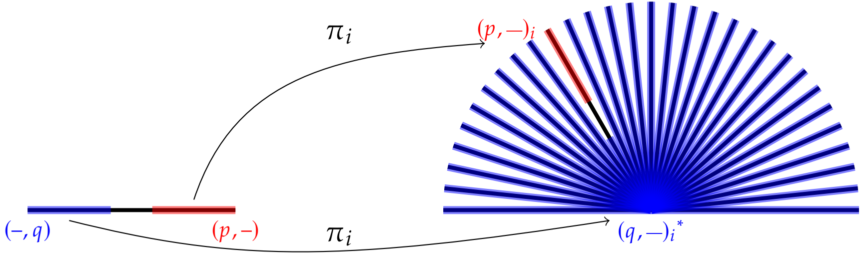

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Hence π_i is a frame homomorphism, i.e. an extended continuous real-valued function on $\mathfrak{L}(J(\kappa))$, called the *i-th projection*.

Continuous hedgehog-valued functions

Furthermore, let $\pi_\kappa \colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{L}(J(\kappa))$ be given by

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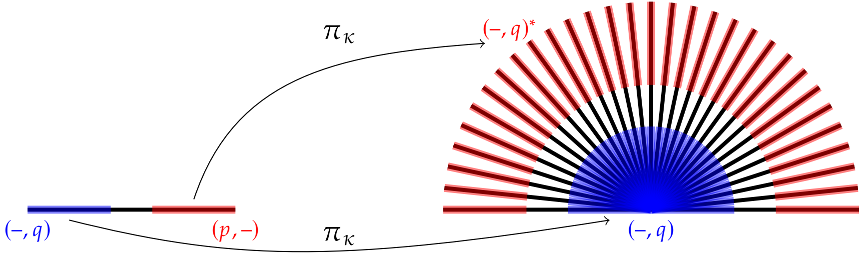

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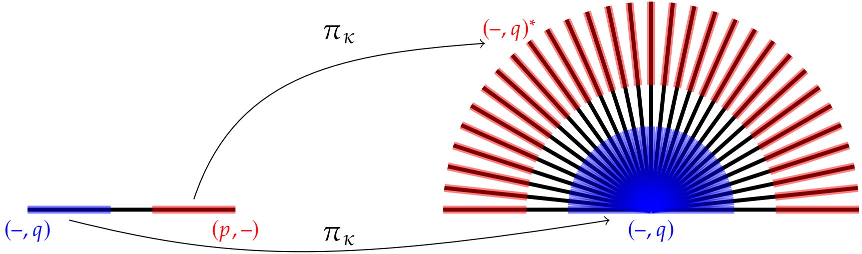

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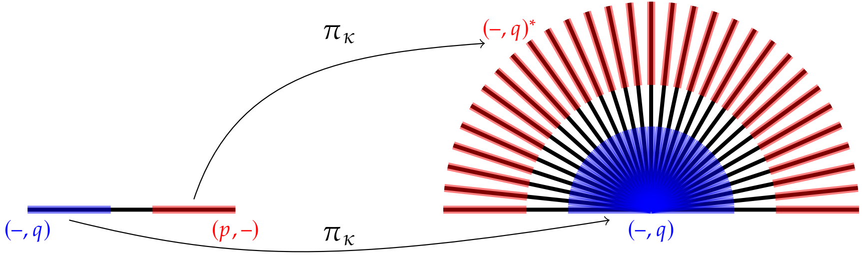

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and

$$h_\kappa(p,\_) = h((\_,p)^{*}) \quad \text{and} \quad h_\kappa(\_,q) = h(\_,q)$$

are extended continuous real-valued functions.

Note also that

$$h_\kappa = \bigvee_{i\in I} h_i$$

Join cozero κ-families

Recall that a cozero element of a frame L is an element of the form

$$\operatorname{coz} h = h((-,0) \vee (0,-)) = \bigvee\{h(p,0) \vee h(0,q) \mid p < 0 < q \text{ in } \mathbb{Q}\}$$

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Proposition

Let L be a frame and $a \in L$. TFAE:

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The equivalence "(2) $\iff$ (3)" can be easily checked by considering an increasing bijection φ between $\mathbb{Q} \cap (0,1)$ and $\mathbb{Q}$.

Join cozero κ-families

Let $h\colon \mathfrak{L}(J(\kappa)) \to L$ be a continuous hedgehog-valued function and

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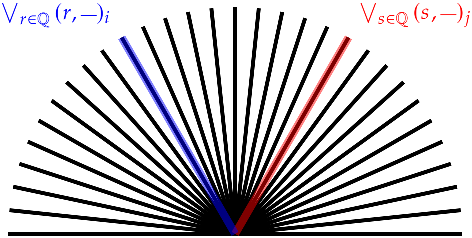
Hence $\{a_i\}_{i\in I}$ is a disjoint family.



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(3) $h_\kappa = h \circ \pi_\kappa \colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ is an extended continuous real-valued function and hence $\bigvee_{r\in\mathbb{Q}} h_\kappa(r,\_) = \bigvee_{r\in\mathbb{Q}} \bigvee_{i\in I} h((r,\_)_i) = \bigvee_{i\in I} a_i$ is again a cozero element.

Join cozero κ-families

Conversely, let $\{a_i\}_{i\in I} \subseteq L$, $|I| = \kappa$, be a disjoint family of cozero elements such that $\bigvee_{i\in I} a_i$ is again a cozero element.

Then:

Join cozero κ-families

Conversely, let $\{a_i\}_{i\in I} \subseteq L$, $|I| = \kappa$, be a disjoint family of cozero elements such that $\bigvee_{i\in I} a_i$ is again a cozero element.

Then:

(1) Since a_i is a cozero element for each $i \in I$, there exists $h_i\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ such that $\bigvee_{r\in\mathbb{Q}} h_i(r,\text{--}) = a_i$.

Conversely, let $\{a_i\}_{i\in I} \subseteq L$, $|I| = \kappa$, be a disjoint family of cozero elements such that $\bigvee_{i\in I} a_i$ is again a cozero element.

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(2) Since also $\bigvee_{i\in I} a_i$ is a cozero element, there exists $h_0 \colon \mathfrak{L}(\overline{\mathbb{R}}) \longrightarrow L$ such that $\bigvee_{r\in\mathbb{Q}} h_0(r, \text{–}) = \bigvee_{i\in I} a_i$.

Join cozero κ-families

Conversely, let $\{a_i\}_{i\in I} \subseteq L$, $|I| = \kappa$, be a disjoint family of cozero elements such that $\bigvee_{i\in I} a_i$ is again a cozero element.

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(3) The formulas

$$h((r,\text{–})_i) = h_0(r,\text{–}) \wedge h_i(r,\text{–}) \quad \text{and}$$
$$h(\text{–},r) = h_0(\text{–},r) \vee \Big(\bigvee_{i\in I} h_i(\text{–},r)\Big)$$

determine a continuous hedgehog-valued function $h \colon \mathfrak{L}(J(\kappa)) \longrightarrow L$ such that $a_i = \bigvee_{r\in\mathbb{Q}} h((r,\text{–})_i)$ for each $i \in I$.

Join cozero κ-families

Proposition

Let L be a frame and $\{a_i\}_{i\in I} \subseteq L$, $|I| = \kappa$. TFAE:

(1) $\{a_i\}_{i\in I}$ is a disjoint family of cozero elements such that $\bigvee_{i\in I} a_i$ is again a cozero element.

(2) There exists a continuous hedgehog-valued function $h\colon \mathfrak{L}(J(\kappa)) \to L$ such that $a_i = \bigvee_{r\in\mathbb{Q}} h((r,\_)_i)$ for each $i \in I$.

Join cozero κ-families

Let κ be a cardinal. We say that a disjoint collection $\{a_i\}_{i\in I}$, $|I| = \kappa$, of cozero elements of a frame L is a join cozero κ-family if $\bigvee_{i\in I} a_i$ is again a cozero element.

Proposition

Let L be a frame and $\{a_i\}_{i\in I} \subseteq L$, $|I| = \kappa$. TFAE:

(1) $\{a_i\}_{i\in I}$ is a join cozero κ-family.

(2) There exists a continuous hedgehog-valued function $h\colon \mathfrak{L}(J(\kappa)) \to L$ such that $a_i = \bigvee_{r\in\mathbb{Q}} h((r,\text{–})_i)$ for each $i \in I$.

Let κ be a cardinal. We say that a disjoint collection $\{a_i\}_{i\in I}$, $|I| = \kappa$, of cozero elements of a frame L is a join cozero κ-family if $\bigvee_{i\in I} a_i$ is again a cozero element.

Proposition

Let L be a frame and $a \in L$. TFAE:

(1) a is a cozero element.

(2) There exists an extended continuous real-valued function $h\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ such that $a = \bigvee_{r\in\mathbb{Q}} h(r, -)$.

(1) If $\kappa = 1$, a join cozero κ-family is precisely a cozero element. Since $\mathfrak{L}(J(1)) = \mathfrak{L}(\overline{\mathbb{R}})$ it follows that this result generalizes the previous one for arbitrary cardinals.

Let κ be a cardinal. We say that a disjoint collection $\{a_i\}_{i\in I}$, $|I| = \kappa$, of cozero elements of a frame L is a join cozero κ-family if $\bigvee_{i\in I} a_i$ is again a cozero element.

Proposition

Let L be a frame and $\{a_i\}_{i\in I} \subseteq L$, $|I| = \kappa \leq \aleph_0$. TFAE:

(1) $\{a_i\}_{i\in I}$ is a a disjoint collection of cozero elements.

(2) There exists a continuous hedgehog-valued function $h\colon \mathfrak{L}(J(\kappa)) \to L$ such that $a_i = \bigvee_{r\in\mathbb{Q}} h((r,-)_i)$ for each $i \in I$.

(2) Since any finite or countable suprema of cozero elements is a cozero element, it follows that in the case $\kappa \leq \aleph_0$, a join cozero κ-family is precisely a disjoint collection of cozero elements.

Let κ be a cardinal. We say that a disjoint collection $\{a_i\}_{i\in I}$, $|I| = \kappa$, of cozero elements of a frame L is a join cozero κ-family if $\bigvee_{i\in I} a_i$ is again a cozero element.

Proposition

Let L be a frame and $\{a_i\}_{i\in I} \subseteq L$, $|I| = \kappa$. TFAE:

(1) $\{a_i\}_{i\in I}$ is a join cozero κ-family.

(2) There exists a continuous hedgehog-valued function $h\colon \mathfrak{L}(J(\kappa)) \to L$ such that $a_i = \bigvee_{r\in\mathbb{Q}} h((r,\text{--})_i)$ for each $i \in I$.

(3) Perfectly normal frames are precisely those frames in which every element is cozero.

Let κ be a cardinal. We say that a disjoint collection $\{a_i\}_{i\in I}$, $|I| = \kappa$, of cozero elements of a frame L is a join cozero κ-family if $\bigvee_{i\in I} a_i$ is again a cozero element.

Proposition

Let L be a perfectly normal frame and $\{a_i\}_{i\in I} \subseteq L$, $|I| = \kappa$. TFAE:

(1) $\{a_i\}_{i\in I}$ is a disjoint family.

(2) There exists a continuous hedgehog-valued function $h\colon \mathfrak{L}(J(\kappa)) \to L$ such that $a_i = \bigvee_{r\in\mathbb{Q}} h((r,\text{--})_i)$ for each $i \in I$.

(3) Perfectly normal frames are precisely those frames in which every element is cozero.
Therefore, in any perfectly normal frame a join cozero κ-family is precisely a disjoint collection of elements.

Universality: Kowalsky's Hedgehog Theorem

A family of frame homomorphisms $\{h_i \colon M_i \to L\}_{i\in I}$ is said to be separating in case

$$a \leq \bigvee_{i\in I} h_i((h_i)_*(a))$$

for every $a \in L$.

- L. Español, J.G.G. and T. Kubiak, Separating families of locale maps and localic embeddings, *Algebra Univ.* 67 (2012) 105–112.

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A family of standard continuous functions $\{f_i \colon X \to Y_i\}_{i\in I}$ separates points from closed sets if for every closed set $K \subseteq X$ and every $x \in X \setminus K$, there is an i such that $f_i(x) \notin f_i[K]$.

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Proposition

The family $\{f_i \colon X \longrightarrow Y_i\}_{i\in I}$ separates points from closed sets if and only if the corresponding family of frame homomorphisms $\{\mathfrak{O} f_i \colon \mathfrak{O} Y_i \longrightarrow \mathfrak{O} X\}_{i\in I}$ is separating.

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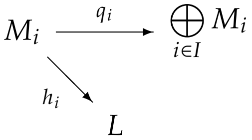
Let $\{h_i \colon M_i \to L\}_{i\in I}$ be a family of frame homomorphisms and let $q_i \colon M_i \to \bigoplus_{i\in I} M_i$ be the i^{th} injection map.



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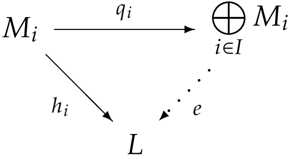
Then there is a frame homomorphism $e\colon \bigoplus_{i\in I} M_i \to L$ such that, for each i, the diagram commutes



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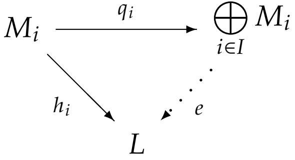

The map e need not be a quotient map, but one has the following:

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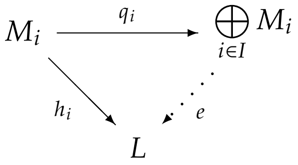

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Theorem

If $\{h_i \colon M_i \to L\}_{i\in I}$ is separating then e is a quotient map.

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Universality: Kowalsky's Hedgehog Theorem

For a class $\mathbb{L}$ of frames, a frame T in $\mathbb{L}$ is said to be universal in $\mathbb{L}$ if for every $L \in \mathbb{L}$ there exists a quotient map from T onto L.

- T. Dube, S. Iliadis, J. van Mill, I. Naidoo, Universal frames, *Topol. Appl.* 160 (2013) 2454–2464.

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For each cardinal κ, the coproduct $\bigoplus_{n\in\mathbb{N}} \mathfrak{L}(J(\kappa))$ is universal in the class of metric frames of weight $\kappa \cdot \aleph_0$.

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For each cardinal κ, the coproduct $\bigoplus_{n\in\mathbb{N}} \mathfrak{L}(J(\kappa))$ is universal in the class of metric frames of weight $\kappa \cdot \aleph_0$.

Proof: (1) $\bigoplus_{n\in\mathbb{N}} \mathfrak{L}(J(\kappa))$ is a metric frame of weight $\kappa \cdot \aleph_0$.

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For each cardinal κ, the coproduct $\bigoplus_{n\in\mathbb{N}} \mathfrak{L}(J(\kappa))$ is universal in the class of metric frames of weight $\kappa \cdot \aleph_0$.

Proof: (2) Let L be a metric frame of weight κ.
Then L has a σ-discrete base, i.e. there exists a base $B \subseteq L$ such that $B = \bigcup_{n\in\mathbb{N}} B_n$, where $B_n = \{a_n^i\}_{i\in I_n}$ is a discrete family.
We can assume with no loss of generality that the cardinality of $\bigcup_{n\in\mathbb{N}} I_n$ is precisely κ.

- J. Picado, A. Pultr, *Frames and Locales* Springer Basel AG, 2012.

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For each cardinal κ, the coproduct $\bigoplus_{n\in\mathbb{N}} \mathfrak{L}(J(\kappa))$ is universal in the class of metric frames of weight $\kappa \cdot \aleph_0$.

Proof: (3) Any metric frame is perfectly normal.
Hence, for each $n \in \mathbb{N}$ there exists a continuous hedgehog-valued function $h_n : \mathfrak{L}(J(\kappa)) \to L$ such that

$$a_n^i = \bigvee_{r\in\mathbb{Q}} h_n((r, \_)_i)$$

for every $i \in I$.

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Theorem

For each cardinal κ, the coproduct $\bigoplus_{n\in\mathbb{N}} \mathfrak{L}(J(\kappa))$ is universal in the class of metric frames of weight $\kappa \cdot \aleph_0$.

Proof: (4) The family $\{h_n : \mathfrak{L}(J(\kappa)) \to L\}_{n\in\mathbb{N}}$ is separating.

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Theorem

For each cardinal κ, the coproduct $\bigoplus_{n\in\mathbb{N}} \mathfrak{L}(J(\kappa))$ is universal in the class of metric frames of weight $\kappa \cdot \aleph_0$.

Proof: (5) The frame homomorphism $e\colon \bigoplus_{n\in\mathbb{N}} \mathfrak{L}(J(\kappa)) \to L$ such that, for each $n \in \mathbb{N}$, the diagram

$$\begin{array}{ccc} \mathfrak{L}(J(\kappa)) & \xrightarrow{\;q_n\;} & \bigoplus_{n\in\mathbb{N}} \mathfrak{L}(J(\kappa)) \\ & {\scriptstyle h_n}\searrow \quad \swarrow{\scriptstyle e} & \\ & L & \end{array}$$

commutes, is a quotient map. ■

Collectionwise normality: a cardinal extension of normality

- A space is normal if for any pair of disjoint closed subsets F_1, F_2 there exist disjoint open subsets V_1, V_2 such that $F_1 \subseteq U_1$ and $F_2 \subseteq U_2$.

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- Question: What about countable families of pairwise disjoint closed subsets?
  It is not true. Just consider the family $\big\{\{q\}\big\}_{q\in\mathbb{Q}}$ of all rational atoms in $\mathbb{R}$.

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(A family $\{A_i\}_{i\in I}$ of subsets of X is discrete if for all $x \in X$ there exists a neighborhood U_x such that $U_x \cap A_i = \emptyset$ for all i with possibly one exception, or, equivalently,
if there exists an open cover $\mathscr{C}$ of X such that for each $U \in \mathscr{C}$, $U \cap A_i = \emptyset$ for all i, with possibly one exception.)

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- Question: What about arbitrary discrete families closed subsets? It fails again. The Bing space is an example of a normal space in which there exist discrete families of closed subsets which cannot be separated by disjoint open subsets.

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- For $\kappa \geq 2$, a space is κ-collectionwise normal if for any discrete family of closed subsets $\{F_i\}_{i\in I}$ with $|I| \leq \kappa$ there exists a discrete family of open subsets $\{U_i\}_{i\in I}$ such that $F_i \subseteq U_i$ for all i.

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▸ A. Pultr, Remarks on metrizable locales, *Proc. of the 12th Winter School on Abstract Analysis* (1984) 247–258.

Localic collectionwise normality: a cardinal extension of normality

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  - disjoint if $x_i \wedge x_j = 0$ for every $i \neq j$.

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- A frame is κ-collectionwise normal if for any co-discrete family $\{x_i\}_{i\in I}$, $|I| \leq \kappa$, there is a discrete family $\{u_i\}_{i\in I}$ such that $x_i \vee u_i = 1$ for all i.

▸ A. Pultr, Remarks on metrizable locales, *Proc. of the 12th Winter School on Abstract Analysis* (1984) 247–258.

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Localic collectionwise normality: a cardinal extension of normality

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- Each metric frame is collectionwise normal.
- Each regular and paracompact frame is collectionwise normal.

▸ S.-H. Sun, On paracompact locales and metric locales, *Comment. Math. Univ. Carolinae* 30 (1989) 101–107.

Localic collectionwise normality

Lemma

A frame is κ-collectionwise normal if and only if for any co-discrete family $\{x_i\}_{i\in I}$, $|I| \leq \kappa$, there is a disjoint family $\{u_i\}_{i\in I}$ such that $x_i \vee u_i = 1$ for all i.

Lemma

A frame is κ-collectionwise normal if and only if for any co-discrete family $\{x_i\}_{i\in I}$, $|I| \leq \kappa$, there is a disjoint family $\{u_i\}_{i\in I}$ such that $x_i \vee u_i = 1$ for all i.

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Proof: The implication '⇒' is obvious since any discrete family is disjoint.

Conversely, let $\{x_i\}_{i\in I}$ be a co-discrete family and $\{u_i\}_{i\in I}$ disjoint such that $x_i \vee u_i = 1$ for every i. Now let

$$D = \{x \in L \mid x \wedge u_i \neq 0 \text{ for at most one } i\}$$

and $\overline{d} = \bigvee D$.

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and $\overline{d} = \bigvee D$. Then, $\overline{d} \vee \bigwedge_I x_i = 1$ and since L is normal, there are $u, v \in L$ such that $u \vee \bigwedge_{i\in I} x_i = 1 = v \vee \overline{d}$ and $u \wedge v = 0$.

Lemma

A frame is κ-collectionwise normal if and only if for any co-discrete family $\{x_i\}_{i\in I}$, $|I| \leq \kappa$, there is a disjoint family $\{u_i\}_{i\in I}$ such that $x_i \vee u_i = 1$ for all i.

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Conversely, let $\{x_i\}_{i\in I}$ be a co-discrete family and $\{u_i\}_{i\in I}$ disjoint such that $x_i \vee u_i = 1$ for every i. Now let

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and $\overline{d} = \bigvee D$. Then, $\overline{d} \vee \bigwedge_I x_i = 1$ and since L is normal, there are $u, v \in L$ such that $u \vee \bigwedge_{i\in I} x_i = 1 = v \vee \overline{d}$ and $u \wedge v = 0$. The system

$$\{y_i := u_i \wedge u\}_{i\in I}$$

is a discrete system such that $x_i \vee y_i = 1$ for all i. ■

An $S \subseteq L$ is a sublocale of L if S is closed under arbitrary infima and moreover $x \to s \in S$ for every $x \in L$ and $s \in S$.

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The set $\mathcal{S}(L)$ of all sublocales of L forms a coframe (i.e., the dual of a frame) under inclusion, in which arbitrary infima coincide with intersections, $\{1\}$ is the bottom element and L is the top element.

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There are two special classes of sublocales: the closed and the open ones, defined respectively as

$$\mathfrak{c}(a) = \uparrow a \quad \text{and} \quad \mathfrak{o}(a) = \{a \to b \mid b \in L\}, \qquad a \in L.$$

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The F_σ-sublocales are the countable joins of closed subocales in $\mathcal{S}(L)$.

Any sublocale S of a frame L is a frame itself with meets (and hence the partial order) as in L, but joins may differ.

Localic collectionwise normality

Proposition

Any closed sublocale of a normal frame is normal.

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Any F_σ-sublocale of a κ-collectionwise normal frame is κ-collectionwise normal.

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Any F_σ-sublocale of a κ-collectionwise normal frame is κ-collectionwise normal.

This is the pointfree counterpart of the classical result of Šedivă, that κ-collectionwise normality is hereditary with respect to F_σ-sets. (It may be worth emphasizing that the localic proof is much simpler.)

- V. Šedivă, On collectionwise normal and hypocompact spaces, *Czechoslovak Math. J.* 9 (84) (1959) 50–62 (in Russian).

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This is the pointfree counterpart of the classical result of Šedivă, that κ-collectionwise normality is hereditary with respect to F_σ-sets. (It may be worth emphasizing that the localic proof is much simpler.)

In particular, it follows that any closed sublocale of a collectionwise normal locale is collectionwise normal.

- V. Šedivă, On collectionwise normal and hypocompact spaces, *Czechoslovak Math. J.* 9 (84) (1959) 50–62 (in Russian).

Proposition

Any F_σ-sublocale of a κ-collectionwise normal frame is κ-collectionwise normal.

Recall that a frame homomorphism $h\colon M \longrightarrow L$ is closed if $h_*(x \vee h(y)) = h_*(x) \vee y$ for every $x \in L$ and $y \in M$, where $h_*\colon L \longrightarrow M$ is the right adjoint of h.

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Proposition

Let $h\colon M \longrightarrow L$ be a one-to-one closed frame homomorphism and κ a cardinal. If L is κ-collectionwise normal, then so is M.

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Let $h\colon M \longrightarrow L$ be a one-to-one closed frame homomorphism and κ a cardinal. If L is κ-collectionwise normal, then so is M.

Formulated in terms of locales, this result states that the image of a collectionwise normal locale under any closed localic map is collectionwise normal.

Theorem (Urysohn's Lemma)

Let X be a topological space. TFAE:

(1) X is normal.

(2) For every disjoint closed sets F_1 and F_2, there exists a continuous $f\colon X \to \overline{\mathbb{R}}$ such that $F_1 \subseteq f^{-1}((-\infty, 0])$ and $F_2 \subseteq f^{-1}([1, +\infty))$.

Theorem (Localic Urysohn's Lemma)

Let L be a frame. TFAE:

(1) L is normal.

(2) For each pair $x_1, x_2 \in L$ such that $x_1 \vee x_2 = 1$, there exists a a frame homomorphism $h\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ such that $h((-,0)^*) \le x_1$ and $h((1,-)^*) \le x_2$.

- C.H. Dowker, D. Papert. On Urysohn?s lemma. *Proc. Second Prague Topological Sympos.*, 1966.
- B. Banaschewski, *The real numbers in Pointfree Topology*, Textos de Matemática, Vol. 12, University of Coimbra, 1997.
- R. N. Ball, J. Walters-Wayland, *C*-and *C**-quotients in pointfree topology, *Diss. Math.* 412 (2002) 1–62.

Theorem (Urysohn-type theorem)

Let L be a frame. TFAE:

(1) L is κ-collectionwise normal.

(2) For each co-discrete system $\{x_i\}_{i\in I}$, $|I| \leq \kappa$, there exists a a frame homomorphism $h\colon \mathfrak{L}(J(\kappa)) \to L$ such that $h((0,\_)_i^*) \leq x_i$ for each $i \in I$.

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Proof: (1) $\implies$ (2): (i) Let $\{x_i\}_{i\in I} \subseteq L$ be a co-discrete system.
By hypothesis there is a disjoint $\{u_i\}_{i\in I}$ such that $u_i \vee x_i = 1$ for every $i \in I$.

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Proof: (1) $\Longrightarrow$ (2): (i) Let $\{x_i\}_{i\in I} \subseteq L$ be a co-discrete system.
By hypothesis there is a disjoint $\{u_i\}_{i\in I}$ such that $u_i \vee x_i = 1$ for every $i \in I$.
By the localic Urysohn's lemma, there is, for each $i \in I$, a frame homomorphism $h_i\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ such that

$$\bigvee_{r\in\mathbb{Q}} h_i(\text{--}, r) \leq x_i \quad \text{and} \quad \bigvee_{r\in\mathbb{Q}} h_i(r, \text{--}) \leq u_i.$$

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Proof: (1) $\Longrightarrow$ (2): (ii) The required frame homomorphism $h\colon \mathfrak{L}(J(\kappa)) \to L$ is determined on generators by

$$h(\text{–}, r) = \bigvee_{t<r} \bigwedge_{i\in I} h_i(\text{–}, t) \quad \text{and} \quad h((r,\text{–})_i) = h_i(r,\text{–}), \qquad r \in \mathbb{Q},\ i \in I.$$

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Proof: (2) $\implies$ (1): Let $\{x_i\}_{i\in I} \subseteq L$ be a co-discrete system.
By hypothesis, there exists a frame homomorphism $h\colon \mathfrak{L}(J(\kappa)) \to L$ such that $h((0,\_)_i^*) \leq x_i$ for all $i \in I$.
Let $u_i = h((-1,\_)_i)$ for each $i \in I$.

Theorem (Urysohn-type theorem)

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By hypothesis, there exists a frame homomorphism $h\colon \mathfrak{L}(J(\kappa)) \to L$ such that $h((0,\_)_i^*) \leq x_i$ for all $i \in I$.
Let $u_i = h((-1,\_)_i)$ for each $i \in I$. The family $\{u_i\}_{i\in I}$ is disjoint and

$$u_i \vee x_i \geq h((-1,\_)_i) \vee h((0,\_)_i^*) \geq h\Big((-1,\_)_i \vee \bigvee_{j\neq i} (-1,\_)_j \vee (\_,0)\Big) = 1$$

for every $i \in I$. Hence L is κ-collectionwise normal. ■

Finally we prove a Tietze-type extension theorem for continuous hedgehog-valued functions.

To prove it we need first to introduce some terminology and a glueing result for localic maps defined on closed sublocales (that we reformulate here in terms of frame homomorphisms).

Collectionwise normality and the metric hedgehog

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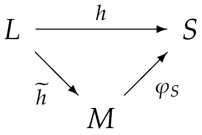
For each sublocale S of a frame M, we say that a frame homomorphism $h\colon L \to S$ has an extension to M if there exists a further frame homomorphism $\widetilde{h}\colon L \to M$ such that the diagram



commutes (where φ_S is the left adjoint of the embedding $S \hookrightarrow M$). In that case we say that $\widetilde{h}\colon L \to M$ extends h.

Collectionwise normality and the metric hedgehog

Finally we prove a Tietze-type extension theorem for continuous hedgehog-valued functions.

To prove it we need first to introduce some terminology and a glueing result for localic maps defined on closed sublocales (that we reformulate here in terms of frame homomorphisms).

Proposition

Let L and M be frames, $a_1, a_2 \in M$, and let $h_i\colon L \longrightarrow \mathfrak{c}(a_i)$ $(i = 1, 2)$ be frame homomorphisms such that $h_1(x) \vee a_2 = h_2(x) \vee a_1$ for all $x \in L$.

Then the map $h\colon L \longrightarrow \mathfrak{c}(a_1) \vee \mathfrak{c}(a_2)$ given by $h(x) = h_1(x) \wedge h_2(x)$ is a frame homomorphism that extends both h_1 and h_2.

- J. Picado, A. Pultr, Localic maps constructed from open and closed parts, *Categ. Gen. Algebr. Struct. Appl.* 6 (2017) 21–35.

Theorem (Tietze)

Let X be a topological space. TFAE:

(1) X is normal.

(2) For each closed subset F of X, each continuous $f\colon F \longrightarrow \overline{\mathbb{R}}$ has an extension to X.

Collectionwise normality and the metric hedgehog

Theorem (Localic Tietze)

Let L be a frame. TFAE:

(1) L is normal.

(2) For each closed sublocale $\mathfrak{c}(a)$ of L, each frame homomorphism $h\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{c}(a)$ has an extension to L.

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Collectionwise normality and the metric hedgehog

Theorem (Tietze-type theorem)

Let L be a frame. TFAE:

(1) L is κ-collectionwise normal.

(2) For each closed sublocale $\mathfrak{c}(a)$ of L, each frame homomorphism $h\colon \mathfrak{L}(J(\kappa)) \to \mathfrak{c}(a)$ has an extension to L.

Theorem (Tietze-type theorem)

Let L be a frame. TFAE:

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(2) For each closed sublocale $\mathfrak{c}(a)$ of L, each frame homomorphism $h\colon \mathfrak{L}(J(\kappa)) \to \mathfrak{c}(a)$ has an extension to L.

Proof: (1) $\Longrightarrow$ (2): (i) Let $a \in L$ and $h\colon \mathfrak{L}(J(\kappa)) \to \mathfrak{c}(a)$. By composing with $\pi_\kappa\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{L}(J(\kappa))$ we have a continuous extended real-valued function $h_\kappa = h \circ \pi_\kappa\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{c}(a)$ given by

$$h_\kappa(r, \text{–}) = h((\text{–}, r)^*) \quad \text{and} \quad h_\kappa(\text{–}, r) = h(\text{–}, r).$$

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$$h_\kappa(r,\text{–}) = h((\text{–},r)^*) \quad \text{and} \quad h_\kappa(\text{–},r) = h(\text{–},r).$$

By the localic Tietze's lemma, h_κ has a continuous extension $\widetilde{h_\kappa}\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$.

Theorem (Tietze-type theorem)

Let L be a frame. TFAE:

(1) L is κ-collectionwise normal.

(2) For each closed sublocale $\mathfrak{c}(a)$ of L, each frame homomorphism $h\colon \mathfrak{L}(J(\kappa)) \to \mathfrak{c}(a)$ has an extension to L.

Proof: (1) $\Longrightarrow$ (2): (ii) Let

$$F = \bigvee_{r\in\mathbb{Q}} \mathfrak{c}\big(\widetilde{h_\kappa}(-,r)\big) = \bigvee_{r\in\mathbb{Q}} \mathfrak{o}\big(\widetilde{h_\kappa}(r,-)\big) = \mathfrak{o}\Big(\bigvee_{r\in\mathbb{Q}} \widetilde{h_\kappa}(r,-)\Big).$$

This is an open F_σ-sublocale of L, hence κ-collectionwise normal.

Theorem (Tietze-type theorem)

Let L be a frame. TFAE:

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(2) For each closed sublocale $\mathfrak{c}(a)$ of L, each frame homomorphism $h\colon \mathfrak{L}(J(\kappa)) \to \mathfrak{c}(a)$ has an extension to L.

Proof: (1) $\Longrightarrow$ (2): (iii) For each $i \in I$, let

$$x_i = \bigwedge_{r\in\mathbb{Q}} h((r,\text{--})_i^*).$$

The system $\{x_i\}_{i\in I}$ is co-discrete in F.

Theorem (Tietze-type theorem)

Let L be a frame. TFAE:

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$$x_i = \bigwedge_{r\in\mathbb{Q}} h((r,\_)_i^*).$$

The system $\{x_i\}_{i\in I}$ is co-discrete in F.

Then there is a disjoint $\{u_i\}_{i\in I} \subseteq F$ such that $u_i \overset{F}{\vee} x_i = 1$ for every $i \in I$.

Theorem (Tietze-type theorem)

Let L be a frame. TFAE:

(1) L is κ-collectionwise normal.

(2) For each closed sublocale $\mathfrak{c}(a)$ of L, each frame homomorphism $h\colon \mathfrak{L}(J(\kappa)) \to \mathfrak{c}(a)$ has an extension to L.

Proof: (1) $\Longrightarrow$ (2): (iv) Let $g\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{c}\big(a \wedge \bigvee_{i\in I} u_i\big)$ be the frame homomorphism given by

$$g(r,\text{–}) = h((\text{–},r)^*) \wedge \bigvee_{i\in I} u_i \quad \text{and} \quad g(\text{–},r) = h(\text{–},r).$$

Then, by the pointfree Tietze's extension theorem again, g has a continuous extension to L, say $\widetilde{g}\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$.

Theorem (Tietze-type theorem)

Let L be a frame. TFAE:

(1) L is κ-collectionwise normal.

(2) For each closed sublocale $\mathfrak{c}(a)$ of L, each frame homomorphism $h\colon \mathfrak{L}(J(\kappa)) \to \mathfrak{c}(a)$ has an extension to L.

Proof: (1) $\Longrightarrow$ (2): (v) The required extension $\widetilde{h}\colon \mathfrak{L}(J(\kappa)) \to L$ is determined on generators by

$$\widetilde{h}((r,\_)_i) = \widetilde{g}(r,\_) \wedge u_i \quad \text{and} \quad \widetilde{h}(\_,r) = \widetilde{g}(\_,r).$$

Theorem (Tietze-type theorem)

Let L be a frame. TFAE:

(1) L is κ-collectionwise normal.

(2) For each closed sublocale $\mathfrak{c}(a)$ of L, each frame homomorphism $h\colon \mathfrak{L}(J(\kappa)) \longrightarrow \mathfrak{c}(a)$ has an extension to L.

Proof: (2) $\Longrightarrow$ (1): (i) Let $\{x_i\}_{i\in I} \subseteq L$ be a co-discrete system. Further, let $a = \bigwedge_{i\in I} x_i$, $a_i = \bigwedge_{j\neq i} x_j$ for each $i \in I$ and let $h\colon \mathfrak{L}(J(\kappa)) \longrightarrow \mathfrak{c}(a)$ be the frame homomorphism determined on generators by

$$h(\_, r) = a \quad \text{and} \quad h((r, \_)_i) = a_i$$

Theorem (Tietze-type theorem)

Let L be a frame. TFAE:

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Proof: (2) $\implies$ (1): (ii) By hypothesis, there exists an extension $\widetilde{h}\colon \mathfrak{L}(J(\kappa)) \to L$ such that $\varphi_{\mathfrak{c}(a)} \circ \widetilde{h} = h$. In particular,

$$\widetilde{h}((0,\_)_i^*) \leq \left(\varphi_{\mathfrak{c}(a)} \circ \widetilde{h}\right)((0,\_)_i^*) = h((0,\_)_i^*) \leq x_i$$

for each $i \in I$.

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The conclusion that L is κ-collectionwise normal follows now from the previous Theorem. ∎

Thank you!