Extension of hedgehog-valued functions

Javier Gutiérrez García

(joint work with T. Kubiak and M. A. de Prada Vicente)

Castellón, July 25, 2007

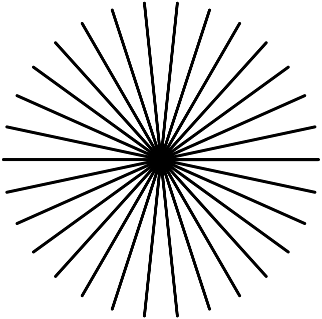
J.G.G., T. Kubiak and M.A. de Prada Vicente
Insertion of lattice-valued and hedgehog-valued functions
Topology and its Appl., 153, (2006) 1458-1475.

J.G.G., T. Kubiak and M.A. de Prada Vicente
Generating functions with values in a bounded complete domain and insertion theorems
To appear in: *Houston J. Math.*, (2007).

J.G.G., T. Kubiak and M.A. de Prada Vicente
Controlling disjointness with a hedgehog
To appear in: *Houston J. Math.*, (2008).

The hedgehog

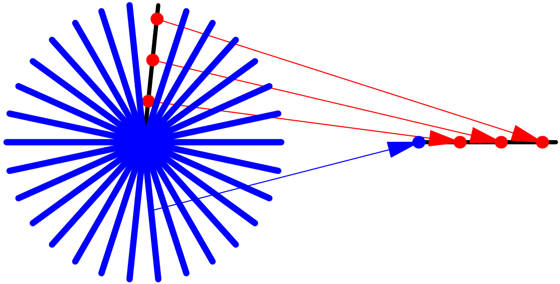
Let I be a set with the cardinality $|I| = \kappa$. The hedgehog $J(\kappa)$ is the disjoint union of κ copies (called *spines*) of the real unit interval $[0, 1]$ identified at the origin.



The hedgehog: projections

The usual projection $\pi_\kappa : J(\kappa) \longrightarrow [0,1]$, defined by $\pi_\kappa[(t,j)] = t$ for all $j \in I$, can be decomposed into new useful projections π_i so that $\pi_\kappa = \bigvee_{i \in I} \pi_i$. Define $\pi_i : J(\kappa) \to [0,1]$ by

$$\pi_i[(t,j)] = \begin{cases} t & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$



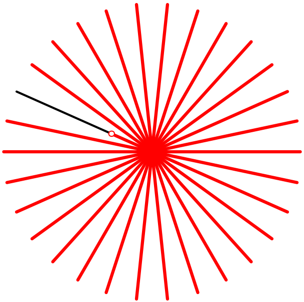
The compact hedgehog

Consider on $J(\kappa) = \{[(t,i)] : t \in [0,1],\ i \in I\}$ the partial order given by

$$[(s,j)] \leq [(t,i)] \quad \Leftrightarrow \quad s = 0 \quad \text{or} \quad j = i \text{ and } s \leq t.$$

The subbasic open sets of the *lower topology* $\omega(J(\kappa))$ are the sets of the form:

$$J(\kappa) \setminus \uparrow[(t,i)] \qquad \text{where} \qquad \uparrow[(t,i)] = \{[(s,i)] : s \geq t\}.$$



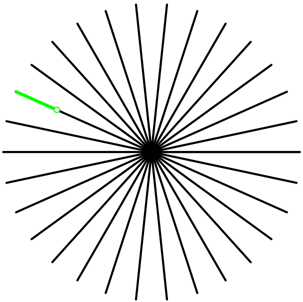
The compact hedgehog

The way-below relation $\ll$ on the poset $(J(\kappa), \leq)$ becomes the following:

$$[(s,j)] \ll [(t,i)] \quad \Leftrightarrow \quad s = 0 \quad \text{or} \quad j = i \text{ and } s < t.$$

The subbasic open sets of the *Scott* topology $\sigma(J(\kappa))$ are the sets of the form:

$$\uparrow[(t,i)] = \{[(s,i)] : s > t\}.$$

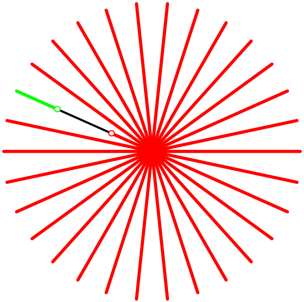


The compact hedgehog

Consequently, the typical subbasic open sets of the *Lawson topology*

$$\lambda(J(\kappa)) = \sigma(J(\kappa)) \vee \omega(J(\kappa))$$

are the sets of the form:



The compact hedgehog

It may be remarked that $\Lambda J(\kappa) = (J(\kappa), \lambda(J(\kappa)))$ is always compact. Hence we shall call the space $\Lambda J(\kappa)$ the compact hedgehog.

$\Lambda J(\kappa)$ is homeomorphic to the axes of the Tychonoff cube via the embedding $e : \Lambda J(\kappa) \hookrightarrow [0,1]^\kappa$ defined by

$$e(t,i)(j) = \begin{cases} t & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

That is, we have

$$\Lambda J(\kappa) \approxeq \bigcup_{i \in I} \{\varphi \in [0,1]^\kappa : \varphi(j) = 0 \text{ for all } j \neq i\}.$$

Frantz's original problem

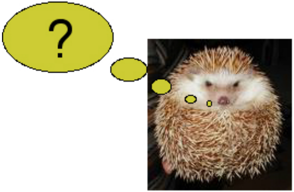
Extension of pairwise disjoint families of real-valued functions.

Problem

Let A be a closed subset of a normal space X and let $\{f_i : A \to \mathbb{R}\}_{i \in I}$ be a family of real-valued continuous and pairwise disjoint functions (i.e. $f_i \cdot f_j = 0$ for each $i \neq j$).

Do there exist pairwise disjoint continuous extensions $\left\{\widehat{f_i} : X \to \mathbb{R}\right\}_{i \in I}$ of the respective f_i over all of X?

M. Frantz.
Controlling Tietze-Urysohn extensions.
Pacific J. Math., 169 (1995), 53-73.



Frantz's first answer.

In the paper mentioned above Frantz gives the following partial answer:

- The answer is affirmative for a finite family.
- The answer is again affirmative in the case of a countable collection.
  (But he doesn't include the proof because it is too technical)
- For the case of an arbitrary infinite collection he doesn't know the answer.
- Arbitrary collections can be extended in the case when X is a metric space.

Barov and Dijkstra's contribution.

Later on S. Barov and J. Dijstra continued working on the same problem and obtained the following results:

- Every countable collection of pairwise disjoint continuous functions on a closed subset of a normal space has a pairwise disjoint extension.
- However, this is not the case for an uncountable collection of functions.

S. Barov and J. Dijkstra.
On boundary avoiding selections and some extension theorems.
Pacific J. Math., 203 (2002), no. 1, 79–87.

The problem that we studied

In view of the previous results it is natural to state the following:

Problem

Characterize the class of spaces satisfying the pairwise disjoint extension property?

Of course, a further problem would be to obtain a Tietze type theorem characterizing that class of spaces.

In order to do it, we'll first see how this problem can be reformulated in terms of hedgehog-valued functions.

But we first need to introduce some machinery.

Hedgehog-valued functions

Let $\mathcal{F} = \{f_i : X \to [0,1]\}_{i \in I}$ be a pairwise disjoint family. The map $\mathbb{F}_{\mathcal{F}} : X \to J(\kappa)$, uniquely determined by $\pi_i \circ \mathbb{F}_{\mathcal{F}} = f_i$ for all $i \in I$, will be called the hedgehog-function generated by $\mathcal{F}$:

$$\mathbb{F}_{\mathcal{F}}(x) = \begin{cases} (f_i(x), i) & \text{if } f_i(x) > 0, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

On the other hand, if $g : X \to J(\kappa)$, then $\mathcal{G} = \{\pi_i \circ g\}_{i \in I}$ is a pairwise disjoint family and $\mathbb{F}_{\mathcal{G}} = g$.

Hedgehog-valued functions

This one-to-one correspondence preserves continuity in both directions. This results which will be crucial in our work:

Result (1)

Let X be a topological space and $\mathcal{F} = \{f_i : X \to [0,1]\}_{i \in I}$ a pairwise disjoint family. Then $\mathbb{F}_{\mathcal{F}} : X \to \Lambda J(\kappa)$ is continuous if and only if $f_i : X \to ([0,1], \tau_u)$ is continuous for each $i \in I$.

Note in passing that from the above universal property it immediately follows that the Lawson topology $\lambda(J(\kappa))$ is the initial topology on $J(\kappa)$ with respect to the family $\{\pi_i\}_{i \in I}$ of functions from $J(\kappa)$ to $[0,1]$ endowed with the natural topology on $[0,1]$.

Hedgehog-valued functions

The following relates extension of a disjoint family of functions with extension of the hedgehog-function generated by the family.

Result (2)

Let A be closed in X and $\mathcal{F} = \{f_i : A \to [0,1]\}_{i \in I}$ a pairwise disjoint family of continuous functions. Then there exists a pairwise disjoint continuous extension of $\mathcal{F}$ if and only if there is a continuous $\overline{f} : X \to J(\kappa)$ such that $\overline{f}|A = \mathbb{F}_{\mathcal{F}}$.

Recall that a space Y is called an absolute extensor of X iff, given a closed $A \subset X$ and a continuous $f : A \to Y$, there is a continuous $\overline{f} : X \to Y$ such that $\overline{f}|A = f$.

Restatement of the problem

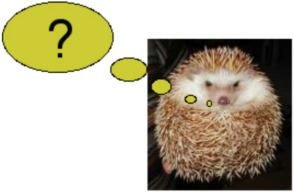
In view of the previous results we can restate the announced problem in the following terms:

Problem

Characterize the class of spaces having the compact hedgehog $\Lambda J(\kappa)$ as an absolute extensor.

Recall that the metric hedgehog is well known to be an absolute extensor for κ-collectionwise normal spaces)

Before providing an answer to this problem, we start with some partial results:



Collectionwise Normality

Before proceed we should recall a number of forms of collectionwise normality.

Definition

A space X is κ-collectionwise normal if for every discrete family $\{K_i\}_{i \in I}$ of closed subsets of X with $|I| \leq \kappa$, there is a disjoint family $\{U_i\}_{i \in I}$ of open subsets such that $K_i \subset U_i$ for every $i \in I$.

(Recall that ω-collectionwise normality = normality.)

Definition

The space X is hereditarily κ-collectionwise normal if every subset of X is κ-collectionwise normal.

Total κ-collectionwise Normality

We need more terminology. Following Aull, we will say that:

Definition

A subspace A of a space X is κ-totally z-embedded in X if every disjoint family of cozero-sets of A of power at most κ may be extended disjointly to a family of cozero-sets in X.

A space X is totally κ-collectionwise normal if every closed subset of X is κ-totally z-embedded in X.

C. E. Aull.
Extendability and expandability.
Boll. U.M.I. (6) 5-A (1986), 129–135.

Main Result

The following is our characterization of spaces having $\Lambda J(\kappa)$ as an absolute extensor.

Result (3)

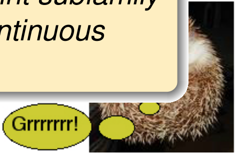
For X a space, the following are equivalent:

(1) $\Lambda J(\kappa)$ *is an absolute extensor for* X;

(2) X *is totally* κ*-collectionwise normal*;

(3) *for every closed subset* $A \subset X$ *and every continuous* $f : A \to \Lambda J(\kappa)$ *there is a continuous extension to all of* X;

(4) *for every closed subset* $A \subset X$ *and every disjoint subfamily of* $C(A, [0, 1])$ *of power at most* κ *there is a continuous disjoint extension to all of* X.



Relationship with collectionwise normality

Result (4)

The following holds: hereditary κ-collectionwise normality $\Rightarrow$ total κ-collectionwise normality $\Rightarrow$ κ-collectionwise normality.

The first implication follows from Theorem 5 in Aull's paper and has been also addressed in:

K. Yamazaki.
Controlling extensions of functions and *C*-embedding
Topology Proc. 26 (2001), 323–341.

Relationship with collectionwise normality

None of those implications is reversible in general:

Example

(1) For $\kappa = \omega$, total ω-collectionwise normality coincides with normality and the latter is weaker than hereditary ω-collectionwise normality (= hereditary normality).

(2) For $\kappa = c = |\mathbb{R}|$, the compact hedgehog $\Lambda J(c)$ is c-collectionwise normal but fails to be total c-collectionwise normal.

More information in:
http://www.ehu.es/javiergutierrezgarcia