How to deal with the ring of (continuous) real functions in terms of scales

Javier Gutiérrez García

Departament of Mathematics, UPV-EHU

(joint work with Jorge Picado (Coimbra))

Outline

What do we understand by “scale”?

The word scale has been used in many different situations to denote completely different notions.

In our case, we are speaking of the kind of families appearing in the proof of Urysohn Lemma when constructing a real valued function.

Urysohn Lemma

A topological space $(X, \mathcal{O}X)$ is normal if and only if whenever E and F are closed and disjoint, there exists a continuous $f : X \to \mathbb{R}$ such that $f(E) = \{0\}$ and $f(F) = \{1\}$.

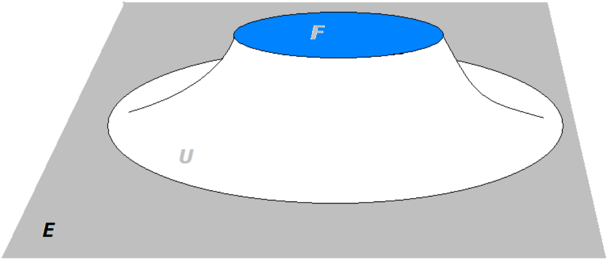
P. Urysohn
Uber die Mächtigkeit der zusammenhängenden Mengen
Mathematische Annalen **94** (1925) 262–295.

Lemma di Urysohn: (From Wikipedia)

Se X è uno spazio normale, per ogni coppia di chiusi disgiunti (E, F) di X, esiste una funzione continua

$$f : X \to [0, 1]$$

a valori nell'intervallo $I = [0, 1]$, che valga 0 su tutto E e 1 su F.



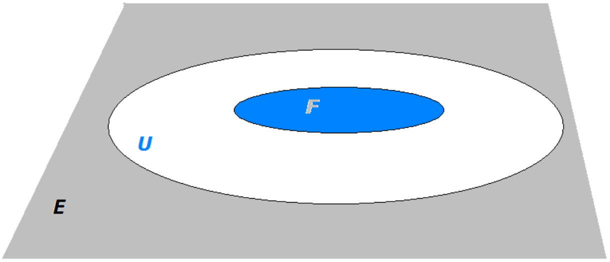
Dimostrazione: (From Wikipedia)

A discapito della profondità della tesi, la dimostrazione del teorema si rivela estremamente semplice ed intuitiva. In molti manuali, tuttavia, la semplicità viene sacrificata ad un infelice eccesso di notazione fino a renderla letteralmente oscura.

Dimostrazione: (From Wikipedia)

A discapito della profondità della tesi, la dimostrazione del teorema si rivela estremamente semplice ed intuitiva. In molti manuali, tuttavia, la semplicità viene sacrificata ad un infelice eccesso di notazione fino a renderla letteralmente oscura.

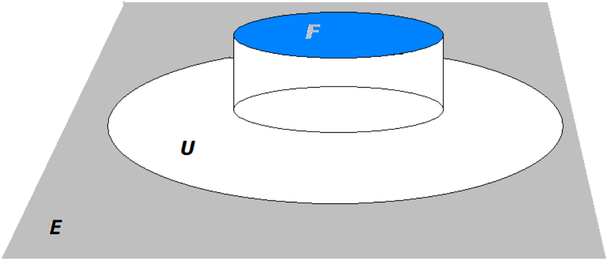
L'idea di fondo consiste nell'immaginare gli insiemi E e F su cui cofunzione come in figura:



Dimostrazione: (From Wikipedia)

Per arrivare al risultato finale si procede con delle funzioni, per così dire, a gradoni. La prima di esse sarà:

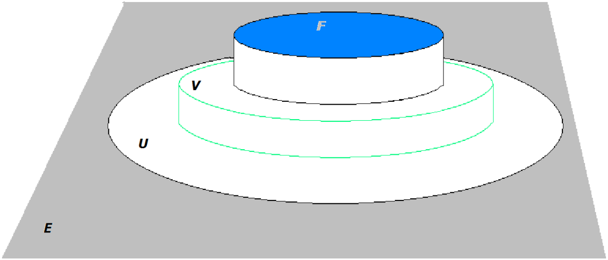
$$f_0(x) = 1 \text{ se } x \in F, \qquad f_0(x) = 0 \text{ se } x \notin F.$$



Dimostrazione: (From Wikipedia)

Si procede con un raffinamento della funzione: Si trova un aperto V tale che $F \subseteq V \subseteq \mathrm{Cl}\, V \subseteq U$. Allora si definisce:

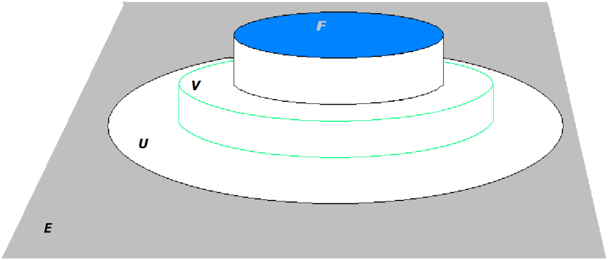
$$f_1(x) = 0 \text{ se } x \in F,\ f_1(x) = \frac{1}{2} \text{ se } x \in \mathrm{Cl}\, V \setminus F,\ f_1(x) = 0 \text{ se } x \notin \mathrm{Cl}\, V.$$



Dimostrazione: (From Wikipedia)

L'intersezione fra $\mathbb{Q}$ e l'intervallo $[0, 1]$, sia detta $D = \{d_0, \ldots, d_n \ldots\}$, è numerabile perchè, insieme dei numeri razionali, lo è.

Costruiremo una successione crescente, indicizzata da D, di aperti tra F e il complementare di E, che godrà di determinate proprietà. Posto innanzitutto $d_0 = 0$ e $d_1 = 1$, definisco per ogni numero naturale n l'insieme $D_n = \{d_0, \ldots, d_n\}$, cosicché risulta che D è l'unione di tutti gli D_n.



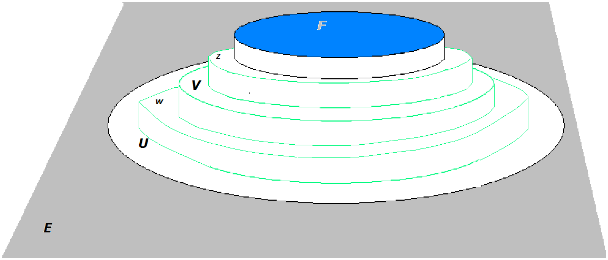
Dimostrazione: (From Wikipedia)

Siccome E e F sono due chiusi disgiunti, allora F è un chiuso contenuto in quell'aperto che è il complementare di E: dunque, per la normalità, esiste un aperto V che contiene F e la cui chiusura è contenuta nel complementare di E. Ponendo allora $V(0) := V$ e $V(1) := X \setminus E$, si ha che: $F \subseteq V(0) \subseteq \mathrm{Cl}\,(V(0)) \subseteq V(1)$. Ciò signfica che per $n = 1$, cioè per D_1, ho costruito una successione di aperti tale che:

$$\begin{cases} (i)_n \quad \mathrm{Cl}\,(V(d_i)) \subseteq V(d_k), \quad \text{allorquando } d_i < d_k \text{ per ogni } i, k < n; \\ (ii) \quad E \subseteq V(0), \quad V(1) \subseteq X \setminus F, \end{cases}$$

Dimostrazione: (From Wikipedia)

Essendo D_n finito esistono infatti in esso due razionali, siano detti d_l e d_m, che sono più vicini a d_{n+1} di qualunque altro in D_n, e tali che $d_l < d_{n+1} < d_m$. Ad essi sono associati due aperti, $V(d_l)$ e $V(d_m)$, tali che $\mathrm{Cl}\,(V(d_l)) \subseteq V(d_m)$: per normalità, esiste un aperto W tali che $\mathrm{Cl}\,(V(d_l)) \subseteq W \subseteq \mathrm{Cl}\,W \subseteq V(d_m)$. Ponendo $V(d_{n+1}) = W$, verifico facilmente che anche per D_{n+1} sono verificate le proprietà $(i)_{n+1}$ e (ii). In definitiva, per il principio di induzione, essendo D numerabile, posso concludere che esiste una successione $\{V(d)\}_{d\in D}$, che soddisfa le proprietà (i) e (ii).

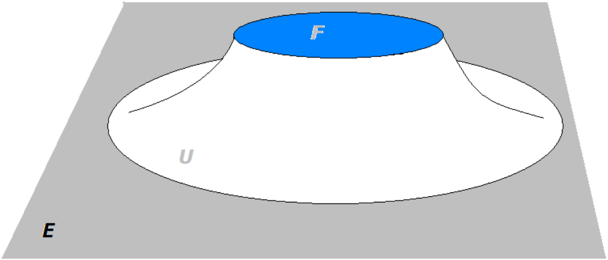


Dimostrazione: (From Wikipedia)

Posso considerare ora una funzione f così definita:

- $f(x) = 1$, se x appartiene a F;
- $f(x) = \inf\{d \in D : x \in V(d)\}$, se x appartiene a $V(1)$, ossia non appartiene a F.

Tale funzione soddisfa i requisiti: vale 1 su tutto F, vale 0 su tutto E e la funzione f è continua.



What do we understand by "scale"?

The word scale has been used in many different situations to denote completely different notions.

In our case, we are speaking of the kind of families appearing in the proof of Urysohn Lemma when constructing a real valued function

Hence, imprecisely speaking:

> *A scale is a monotone family of subsets of a given set X of the form $\mathcal{S} = \{S_d\}_{d \in D}$ with $D \subset \mathbb{Q}$ (hence countable) and dense in $\mathbb{R}$ which determine a real valued function $f_{\mathcal{S}} : X \to \mathbb{R}$, in a certain sense.*

What do we understand by "scale"?

Increasing or decreasing?

But even in this context there are different uses of this term:

What do we understand by “scale”?

Increasing or decreasing?

But even in this context there are different uses of this term:

(1) The families can be either decreasing or increasing, i.e. either

$$d_1 < d_2 \text{ implies } S_{d_1} \supseteq S_{d_2} \text{ or } d_1 < d_2 \text{ implies } S_{d_1} \subseteq S_{d_2}.$$

This is related with the way in which the real valued function $f_{\mathcal{S}}$ is generated by the scale $\mathcal{S}$:

- if $\mathcal{S}$ is decreasing then $f_{\mathcal{S}}(x) = \bigvee\{d \in D : f(x) \in S_d\}$;
- if $\mathcal{S}$ is increasing then $f_{\mathcal{S}}(x) = \bigwedge\{d \in D : f(x) \in S_d\}$;

What do we understand by "scale"?

Increasing or decreasing?

But even in this context there are different uses of this term:

(1) The families can be either decreasing or increasing, i.e. either

$$d_1 < d_2 \text{ implies } S_{d_1} \supseteq S_{d_2} \text{ or } d_1 < d_2 \text{ implies } S_{d_1} \subseteq S_{d_2}.$$

This is related with the way in which the real valued function $f_{\mathcal{S}}$ is generated by the scale $\mathcal{S}$:

- if $\mathcal{S}$ is decreasing then $f_{\mathcal{S}}(x) = \bigvee\{d \in D : f(x) \in S_d\}$;
- if $\mathcal{S}$ is increasing then $f_{\mathcal{S}}(x) = \bigwedge\{d \in D : f(x) \in S_d\}$;

Here we will always deal with decreasing families.

What do we understand by "scale"?

The index set

(2) The index set D can be $\mathbb{Q}$, $\mathbb{Q} \cap [0, 1]$, the dyadic numbers ...

The dyadic numbers have the advantage that the bijection with $\mathbb{N}$ can be easily stated. This is important when using induction.

$\mathbb{Q} \cap [0, 1]$ are used mainly when dealing with bounded functions.

What do we understand by “scale”?

The index set

(2) The index set D can be $\mathbb{Q}$, $\mathbb{Q} \cap [0,1]$, the dyadic numbers . . .

The dyadic numbers have the advantage that the bijection with $\mathbb{N}$ can be easily stated. This is important when using induction.

$\mathbb{Q} \cap [0,1]$ are used mainly when dealing with bounded functions.

But note that any decreasing family $\{S_d \subseteq X\}_{d \in D}$ indexed by a subset $D \subseteq \mathbb{Q}$ can be obviously extended to a monotone $\mathbb{Q}$-indexed family:

$$S_q = \begin{cases} \varnothing, & \text{if } q < d \text{ for all } d \in D; \\ X. & \text{if } q > d \text{ for all } d \in D; \\ \bigcup\{S_d : d \geq q\}, & \text{otherwise,} \end{cases}$$

generating the same function.

What do we understand by "scale"?

The index set

(2) The index set D can be $\mathbb{Q}$, $\mathbb{Q} \cap [0, 1]$, the dyadic numbers . . .

The dyadic numbers have the advantage that the bijection with $\mathbb{N}$ can be easily stated. This is important when using induction.

$\mathbb{Q} \cap [0, 1]$ are used mainly when dealing with bounded functions.

But note that any decreasing family $\{S_d \subseteq X\}_{d\in D}$ indexed by a subset $D \subseteq \mathbb{Q}$ can be obviously extended to a monotone $\mathbb{Q}$-indexed family:

$$S_q = \begin{cases} \varnothing, & \text{if } q < d \text{ for all } d \in D; \\ X. & \text{if } q > d \text{ for all } d \in D; \\ \bigcup\{S_d : d \geq q\}, & \text{otherwise,} \end{cases}$$

generating the same function.

Here we will always deal with $\mathbb{Q}$-indexed families.

What do we understand by “scale”?

$C(X)$ versus $F(X)$

(3) As it has already been mentioned, the origin of the notion of scale goes back to the work of P. Urysohn and it is based on his approach to the construction of a continuous function on a topological space from a given family of open sets.

P. Urysohn
Uber die Mächtigkeit der zusammenhängenden Mengen
Mathematische Annalen **94** (1925) 262–295.

What do we understand by “scale”?

$C(X)$ versus $F(X)$

(3) As it has already been mentioned, the origin of the notion of scale goes back to the work of P. Urysohn and it is based on his approach to the construction of a continuous function on a topological space from a given family of open sets.

P. Urysohn
Uber die Mächtigkeit der zusammenhängenden Mengen
Mathematische Annalen **94** (1925) 262–295.

It was probably M.H. Stone who initiated the study of an arbitrary (not necessarily continuous) real function by considering what he called the spectral family of the function.

M. H. Stone
Boundedness Properties in Function-Lattices
Can. J. Math. **1** (1949) 176–186

What do we understand by “scale”?

$C(X)$ versus $F(X)$

Hence, we have two different approaches:

- scales of open subsets generating continuous functions and
- scales of arbitrary subsets generating functions not necessarily continuous.

So, which one is the right approach?

What do we understand by “scale”?

$C(X)$ versus $F(X)$

Hence, we have two different approaches:

- scales of open subsets generating continuous functions and
- scales of arbitrary subsets generating functions not necessarily continuous.

So, which one is the right approach?

This is directly related with the following question:

Let us denote by $C(X, \mathcal{O}X)$ (or simply $C(X)$) the ring of continuous real functions on a topological space $(X, \mathcal{O}X)$ and by $F(X)$ the collection of all real functions on X.

Question

What is more general, the study of the rings $C(X, \mathcal{O}X)$ or that of the rings $F(X)$?

What do we understand by “scale”?

$C(X)$ versus $F(X)$

A first obvious answer immediately comes to our mind:

First answer

For a given topological space $(X, \mathcal{O}X)$, the family $F(X)$ is much bigger than $C(X, \mathcal{O}X)$.

Hence the study of the rings of real functions is more general than the study of the rings of continuous real functions.

What do we understand by "scale"?

$C(X)$ versus $F(X)$

A first obvious answer immediately comes to our mind:

• For a given topological space $(X, \mathcal{O}X)$, $F(X)$ is much more general than $C(X, \mathcal{O}X)$.

What do we understand by “scale”?

$C(X)$ versus $F(X)$

A first obvious answer immediately comes to our mind:

• For a given topological space $(X, \mathcal{O}X)$, $F(X)$ is much more general than $C(X, \mathcal{O}X)$.

But looking at this question from a different perspective:

Second answer

For each set X we have that $F(X) = C(X, \mathfrak{D}(X))$ (where $\mathfrak{D}(X)$ denotes the discrete topology on X), i.e. the real functions on X are precisely the continuous real functions on $(X, \mathfrak{D}(X))$.

Hence the study of all $F(X)$ is the study of all $C(X, \mathcal{O}X)$ for discrete topological spaces, a particular case of the study of all $C(X, \mathcal{O}X)$.

What do we understand by “scale”?

$C(X)$ versus $F(X)$

A first obvious answer immediately comes to our mind:

• For a given topological space $(X, \mathcal{O}X)$, $F(X)$ is much more general than $C(X, \mathcal{O}X)$.

But looking at this question from a different perspective:

• Hence the study of all $F(X)$ is just a particular case of the study of all $C(X, \mathcal{O}X)$.

What do we understand by "scale"?

$C(X)$ versus $F(X)$

A first obvious answer immediately comes to our mind:

• For a given topological space $(X, \mathcal{O}X)$, $F(X)$ is much more general than $C(X, \mathcal{O}X)$.

But looking at this question from a different perspective:

• Hence the study of all $F(X)$ is just a particular case of the study of all $C(X, \mathcal{O}X)$.

Final answer

The study of all rings of the form $C(X, \mathcal{O}X)$ is equivalent to the study of all rings of the form $F(X)$.
However, for a fixed topological space $(X, \mathcal{O}X)$, the study of $F(X)$ is clearly more general than that of $C(X, \mathcal{O}X)$.

What do we understand by "scale"?

$C(X)$ versus $F(X)$

The reason to include here the previous question is that it is directly related with the issue of dealing with real functions in terms of scales that we want to address in this talk.

What do we understand by “scale”?

$C(X)$ versus $F(X)$

The reason to include here the previous question is that it is directly related with the issue of dealing with real functions in terms of scales that we want to address in this talk.

Depending of the focus of the study, that of $C(X)$ or that of $F(X)$, different notions of scale can be found in the literature.

What do we understand by “scale”?

$C(X)$ versus $F(X)$

The reason to include here the previous question is that it is directly related with the issue of dealing with real functions in terms of scales that we want to address in this talk.

Depending of the focus of the study, that of $C(X)$ or that of $F(X)$, different notions of scale can be found in the literature.

If the focus of the study, is $C(X)$, then scales of open subsets should be considered.

What do we understand by "scale"?

$C(X)$ versus $F(X)$

The reason to include here the previous question is that it is directly related with the issue of dealing with real functions in terms of scales that we want to address in this talk.

Depending of the focus of the study, that of $C(X)$ or that of $F(X)$, different notions of scale can be found in the literature.

If the focus of the study, is $C(X)$, then scales of open subsets should be considered.

If the focus of the study, is $F(X)$, then scales of arbitrary subsets should be considered.

What do we understand by “scale”?

In our case, a scale is a family $\mathcal{S} = \{S_q\}_{q\in\mathbb{Q}}$ of subsets of a given set X such that

(1) $\mathcal{S}$ is decreasing, i.e.

$$S_q \subseteq S_p \text{ whenever } p < q.$$

(2) $\bigcup_{q\in\mathbb{Q}} S_q = X$ and $\bigcap_{q\in\mathbb{Q}} S_p = \varnothing$.

What do we understand by “scale”?

In our case, a scale is a family $\mathcal{S} = \{S_q\}_{q\in\mathbb{Q}}$ of subsets of a given set X such that

(1) $\mathcal{S}$ is decreasing, i.e.

$$S_q \subseteq S_p \text{ whenever } p < q.$$

(2) $\bigcup_{q\in\mathbb{Q}} S_q = X$ and $\bigcap_{q\in\mathbb{Q}} S_p = \varnothing$.

The function $f_{\mathcal{S}} : X \to \mathbb{R}$ defined by

$$f_{\mathcal{S}}(x) = \bigvee\{q \in \mathbb{Q} \mid x \in S_q\}$$

for each $x \in X$, is said to be the real function generated by $\mathcal{S}$.

Yet another look at Dedekind cuts

Original description

The purpose of Dedekind with the introduction of the notion of cut was to provide a logical foundation for the real number system.

Dedekind's motivation: a real number r is completely determined by the rationals strictly smaller than r and those strictly larger than r.

Yet another look at Dedekind cuts

Original description

The purpose of Dedekind with the introduction of the notion of cut was to provide a logical foundation for the real number system.

Dedekind's motivation: a real number r is completely determined by the rationals strictly smaller than r and those strictly larger than r.

A Dedekind cut is a pair (A, B) of $\mathbb{Q}$ such that

- A and B are nonempty;
- A is closed downwards and B is closed upwards;
- Every element in A is strictly below every element in B;
- A contains no greatest element.

$\mathbb{Q}$

$A = \{q \in \mathbb{Q} : q < 0 \text{ or } q^2 < 2\}$ $\qquad$ $B = \{q > 0 : q^2 > 2\}$

J.W.R. Dedekind,
Stetigkeit und irrationale zahlen, 1872.
Translation by W.W. Beman: *Continuity and irrational numbers*, in
Essays on the Theory of Numbers by Richard Dedekind. Dover 1963.

Yet another look at Dedekind cuts

One-side cuts

In fact, (assuming excluded middle) we may take the lower part A as the representative of any given cut (A, B) since the upper part of the cut B is completely determined by A. Hence one can consider the following equivalent description of the real numbers:

Definition (Dedekind's construction of the reals)

A real number is a Dedekind cut, i.e. a subset $A \subseteq \mathbb{Q}$ such that

(D1) A is a down-set, i.e. if $p < q$ in $\mathbb{Q}$ and $q \in A$, then $p \in A$;

(D2) $\varnothing \neq A \neq \mathbb{Q}$;

(D3) A contains no greatest element, i.e. if $q \in A$, then there is some $p \in A$ such that $q < p$.

Yet another look at Dedekind cuts

Total order on $\mathbb{R}$

The set of Dedekind cuts (or real numbers) is denoted by $\mathbb{R}$ and define a total ordering on the set $\mathbb{R}$ as

$$A \leq B \iff A \subseteq B.$$

We also write $A < B$ to denote the negation of $B \subseteq A$, that is

$$A < B \iff A \subseteq B \text{ and } A \neq B.$$

Yet another look at Dedekind cuts

Total order on $\mathbb{R}$

The set of Dedekind cuts (or real numbers) is denoted by $\mathbb{R}$ and define a total ordering on the set $\mathbb{R}$ as

$$A \leq B \iff A \subseteq B.$$

We also write $A < B$ to denote the negation of $B \subseteq A$, that is

$$A < B \iff A \subseteq B \text{ and } A \neq B.$$

$\mathbb{Q}$ can be embedded into $\mathbb{R}$ by identifying for each rational number q:

$$q \equiv A_q = (\leftarrow, q) = \{p \in \mathbb{Q} \mid p < q\}.$$

Clearly enough, a real number A is rational if and only if $\mathbb{Q} \setminus A$ contains a least element.

A real number A is said to be irrational if $\mathbb{Q} \setminus A$ contains no least element.

Yet another look at Dedekind cuts

$\mathbb{R}$ as a complete ordered field

Sometimes the description of the algebraic operation is really easy:

Addition. Let $A, B \in \mathbb{R}$ and define

$$A + B = \{p + q \in \mathbb{Q} : p \in A \text{ and } q \in B\}.$$

It is easy to check that $A + B$ is a Dedekind cut and the operation so defined extend that of $\mathbb{Q}$, i.e. $A_p + A_q = A_{p+q}$.

Yet another look at Dedekind cuts

$\mathbb{R}$ as a complete ordered field

Sometimes the description of the algebraic operation is really easy:

Addition. Let $A, B \in \mathbb{R}$ and define

$$A + B = \{p + q \in \mathbb{Q} : p \in A \text{ and } q \in B\}.$$

It is easy to check that $A + B$ is a Dedekind cut and the operation so defined extend that of $\mathbb{Q}$, i.e. $A_p + A_q = A_{p+q}$.

But in some other cases the situation is more complicated:

Opposite. Let $A \in \mathbb{R}$. How to define $-A$?

Yet another look at Dedekind cuts

$\mathbb{R}$ as a complete ordered field

Sometimes the description of the algebraic operation is really easy:

Addition. Let $A, B \in \mathbb{R}$ and define

$$A + B = \{p + q \in \mathbb{Q} : p \in A \text{ and } q \in B\}.$$

It is easy to check that $A + B$ is a Dedekind cut and the operation so defined extend that of $\mathbb{Q}$, i.e. $A_p + A_q = A_{p+q}$.

But in some other cases the situation is more complicated:

Opposite. Let $A \in \mathbb{R}$. How to define $-A$?

$$-A = \{-p \in \mathbb{Q} : p \notin A\}?$$

Yet another look at Dedekind cuts

$\mathbb{R}$ as a complete ordered field

Sometimes the description of the algebraic operation is really easy:

Addition. Let $A, B \in \mathbb{R}$ and define

$$A + B = \{p + q \in \mathbb{Q} : p \in A \text{ and } q \in B\}.$$

It is easy to check that $A + B$ is a Dedekind cut and the operation so defined extend that of $\mathbb{Q}$, i.e. $A_p + A_q = A_{p+q}$.

But in some other cases the situation is more complicated:

Opposite. Let $A \in \mathbb{R}$. How to define $-A$?

$$-A = \{-p \in \mathbb{Q} : p \notin A\}?$$

It is fine if A is irrational.

Yet another look at Dedekind cuts

$\mathbb{R}$ as a complete ordered field

Sometimes the description of the algebraic operation is really easy:

Addition. Let $A, B \in \mathbb{R}$ and define

$$A + B = \{p + q \in \mathbb{Q} : p \in A \text{ and } q \in B\}.$$

It is easy to check that $A + B$ is a Dedekind cut and the operation so defined extend that of $\mathbb{Q}$, i.e. $A_p + A_q = A_{p+q}$.

But in some other cases the situation is more complicated:

Opposite. Let $A \in \mathbb{R}$. How to define $-A$?

$$-A = \{-p \in \mathbb{Q} : p \notin A\}?$$

But not if it is rational!! (for example $-(\rightarrow, 0) = (\rightarrow, 0]$

Yet another look at Dedekind cuts

$\mathbb{R}$ as a complete ordered field

Sometimes the description of the algebraic operation is really easy:

Addition. Let $A, B \in \mathbb{R}$ and define

$$A + B = \{p + q \in \mathbb{Q} : p \in A \text{ and } q \in B\}.$$

It is easy to check that $A + B$ is a Dedekind cut and the operation so defined extend that of $\mathbb{Q}$, i.e. $A_p + A_q = A_{p+q}$.

But in some other cases the situation is more complicated:

Opposite. Let $A \in \mathbb{R}$. How to define $-A$?

$$-A = \{q - p \in \mathbb{Q} : q < 0 \text{ and } p \notin A\}$$

Yet another look at Dedekind cuts

$\mathbb{R}$ as a complete ordered field

Sometimes the description of the algebraic operation is really easy:

Addition. Let $A, B \in \mathbb{R}$ and define

$$A + B = \{p + q \in \mathbb{Q} : p \in A \text{ and } q \in B\}.$$

It is easy to check that $A + B$ is a Dedekind cut and the operation so defined extend that of $\mathbb{Q}$, i.e. $A_p + A_q = A_{p+q}$.

But in some other cases the situation is more complicated:

Opposite. Let $A \in \mathbb{R}$. How to define $-A$?

$$-A = \{q - p \in \mathbb{Q} : q < 0 \text{ and } p \notin A\}$$

Now $-A$ is a Dedekind cut, the opposite of A.

Yet another look at Dedekind cuts

Indefinite cuts

Here it is in order to recall Dedekind's remark:

> *Every rational number produces one cut or, strictly speaking, two cuts, which, however, we shall not look as essentially different.*

Yet another look at Dedekind cuts

Indefinite cuts

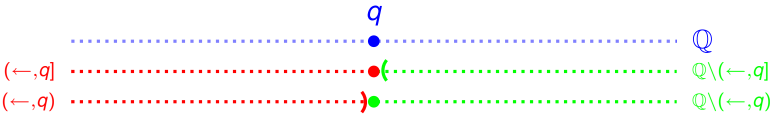
Here it is in order to recall Dedekind's remark:

> *Every rational number produces one cut or, strictly speaking, two cuts, which, however, we shall not look as essentially different.*

In other words, there are two cuts associated to each $q \in \mathbb{Q}$, namely,

$$((\leftarrow, q], \mathbb{Q} \setminus (\leftarrow, q]) \quad \text{and} \quad ((\leftarrow, q), \mathbb{Q} \setminus (\leftarrow, q)),$$

where $(\leftarrow, q] = \{p \in \mathbb{Q} \mid p \leq q\}$ and $(\leftarrow, q) = \{p \in \mathbb{Q} \mid p < q\}$.



Yet another look at Dedekind cuts

Indefinite cuts

We can simplify some of the descriptions of the algebraic operations by eliminating the last condition of Dedekind cuts:

Definition (Indefinite Dedekind cut)

An indefinite Dedekind cut is a subset $A \subseteq \mathbb{Q}$ such that

(D1) A is a down-set, i.e. if $p < q$ in $\mathbb{Q}$ and $q \in A$, then $p \in A$;

(D2) $\varnothing \neq A \neq \mathbb{Q}$.

Yet another look at Dedekind cuts

We can simplify some of the descriptions of the algebraic operations by eliminating the last condition of Dedekind cuts:

Definition (Indefinite Dedekind cut)

An indefinite Dedekind cut is a subset $A \subseteq \mathbb{Q}$ such that

(D1) A is a down-set, i.e. if $p < q$ in $\mathbb{Q}$ and $q \in A$, then $p \in A$;

(D2) $\varnothing \neq A \neq \mathbb{Q}$.

In other words, we will take into consideration now both subsets

$$(\leftarrow, q) \text{ and } (\leftarrow, q] \quad \text{for each } q \in \mathbb{Q}$$

as indefinite Dedekind cuts.

Yet another look at Dedekind cuts

Indefinite cuts

We can simplify some of the descriptions of the algebraic operations by eliminating the last condition of Dedekind cuts:

Definition (Indefinite Dedekind cut)

An indefinite Dedekind cut is a subset $A \subseteq \mathbb{Q}$ such that

(D1) A is a down-set, i.e. if $p < q$ in $\mathbb{Q}$ and $q \in A$, then $p \in A$;

(D2) $\varnothing \neq A \neq \mathbb{Q}$.

And we define an equivalence relation $\sim$ on the set of all indefinite Dedekind cuts by

$$A \sim B \iff \bigcup_{q \in A} (\leftarrow, q) = \bigcup_{q \in B} (\leftarrow, q).$$

The real numbers are precisely the equivalence classes w.r.t. $\sim$.

Yet another look at Dedekind cuts

Now we can provide a nicer description of the algebraic operations. For example:

Yet another look at Dedekind cuts

Now we can provide a nicer description of the algebraic operations. For example:

Opposite. Let A be an indefinite Dedekind cut and $[A] \in \mathbb{R}$. Then

$$-A = \{-p \in \mathbb{Q} : p \notin A\}$$

is an indefinite Dedekind cut and it is a representative of the opposite real number of A, i.e. $[-A] = -[A]$.

Yet another look at Dedekind cuts

Now we can provide a nicer description of the algebraic operations. For example:

Opposite. Let A be an indefinite Dedekind cut and $[A] \in \mathbb{R}$. Then

$$-A = \{-p \in \mathbb{Q} : p \notin A\}$$

is an indefinite Dedekind cut and it is a representative of the opposite real number of A, i.e. $[-A] = -[A]$.

This considerably simplifies the description of the different algebraic operations on $\mathbb{R}$. Compare with

$$-A = \{q - p \in \mathbb{Q} : q < 0 \text{ and } p \notin A\}$$

Yet another look at Dedekind cuts

Definition (Indefinite Dedekind cut)

An indefinite Dedekind cut is a subset $A \subseteq \mathbb{Q}$ such that

(D1) A is a down-set, i.e. if $p < q$ in $\mathbb{Q}$ and $q \in A$, then $p \in A$;

(D2) $\varnothing \neq A \neq \mathbb{Q}$.

Yet another look at Dedekind cuts

After identifying each subset $A \subseteq \mathbb{Q}$ with its characteristic function $\chi_A : \mathbb{Q} \to \mathbf{2}$ into the two-element lattice $\mathbf{2} = \{0, 1\}$ (given by $\chi_A(q) = 1$ iff $q \in A$) one has, equivalently:

Definition (Indefinite Dedekind cut)

An indefinite Dedekind cut is a subset $A \subseteq \mathbb{Q}$ such that

(D1) A is a down-set, i.e. if $p < q$ in $\mathbb{Q}$ and $q \in A$, then $p \in A$;

(D2) $\varnothing \neq A \neq \mathbb{Q}$.

Yet another look at Dedekind cuts

After identifying each subset $A \subseteq \mathbb{Q}$ with its characteristic function $\chi_A : \mathbb{Q} \to \mathbf{2}$ into the two-element lattice $\mathbf{2} = \{0, 1\}$ (given by $\chi_A(q) = 1$ iff $q \in A$) one has, equivalently:

Definition (Indefinite Dedekind cut)

An indefinite Dedekind cut is a function $\mathcal{S} : \mathbb{Q} \to \mathbf{2}$ such that

(D1) A is a down-set, i.e. if $p < q$ in $\mathbb{Q}$ and $q \in A$, then $p \in A$;

(D2) $\varnothing \neq A \neq \mathbb{Q}$.

Yet another look at Dedekind cuts

After identifying each subset $A \subseteq \mathbb{Q}$ with its characteristic function $\chi_A : \mathbb{Q} \to \mathbf{2}$ into the two-element lattice $\mathbf{2} = \{0, 1\}$ (given by $\chi_A(q) = 1$ iff $q \in A$) one has, equivalently:

Definition (Indefinite Dedekind cut)

An indefinite Dedekind cut is a function $\mathcal{S} : \mathbb{Q} \to \mathbf{2}$ such that

(D1) $\mathcal{S}$ is decreasing, i.e. $\mathcal{S}(q) \leq \mathcal{S}(p)$ whenever $p < q$;

(D2) $\varnothing \neq A \neq \mathbb{Q}$.

Yet another look at Dedekind cuts

After identifying each subset $A \subseteq \mathbb{Q}$ with its characteristic function $\chi_A : \mathbb{Q} \to \mathbf{2}$ into the two-element lattice $\mathbf{2} = \{0, 1\}$ (given by $\chi_A(q) = 1$ iff $q \in A$) one has, equivalently:

Definition (Indefinite Dedekind cut)

An indefinite Dedekind cut is a function $\mathcal{S} : \mathbb{Q} \to \mathbf{2}$ such that

(D1) $\mathcal{S}$ is decreasing, i.e. $\mathcal{S}(q) \leq \mathcal{S}(p)$ whenever $p < q$;

(D2) $\bigvee_{q \in \mathbb{Q}} \mathcal{S}(q) = 1$ and $\bigwedge_{q \in \mathbb{Q}} \mathcal{S}(q) = 0$.

Yet another look at Dedekind cuts

After identifying each subset $A \subseteq \mathbb{Q}$ with its characteristic function $\chi_A : \mathbb{Q} \to \mathbf{2}$ into the two-element lattice $\mathbf{2} = \{0, 1\}$ (given by $\chi_A(q) = 1$ iff $q \in A$) one has, equivalently:

Definition (Indefinite Dedekind cut)

An indefinite Dedekind cut is a function $\mathcal{S} : \mathbb{Q} \to \mathbf{2}$ such that

(D1) $\mathcal{S}$ is decreasing, i.e. $\mathcal{S}(q) \leq \mathcal{S}(p)$ whenever $p < q$;

(D2) $\bigvee_{q\in\mathbb{Q}} \mathcal{S}(q) = 1$ and $\bigwedge_{q\in\mathbb{Q}} \mathcal{S}(q) = 0$.

A Dedekind cut in the previous sense is an indefinite Dedekind cut if it is right continuous, i.e. if it satisfies the additional condition

(D3) $\mathcal{S}(q) = \bigvee_{p>q} \mathcal{S}(p)$ for each $q \in \mathbb{Q}$.

From indefinite Dedekind cuts to scales

Frames

We can now try to extend the previous notion by considering an arbitrary frame L instead of the two element lattice **2**.

Recall that a frame is a complete lattice L in which

$$a \wedge \bigvee B = \bigvee \{a \wedge b : b \in B\} \quad \text{for all } a \in L \text{ and } B \subseteq L.$$

From indefinite Dedekind cuts to scales

Frames

We can now try to extend the previous notion by considering an arbitrary frame L instead of the two element lattice **2**.

Recall that a frame is a complete lattice L in which

$$a \wedge \bigvee B = \bigvee \{a \wedge b : b \in B\} \quad \text{for all } a \in L \text{ and } B \subseteq L.$$

The most familiar examples of frames are:

(a) the two element lattice **2** (and, more generally, any complete chain),

(b) the topology $\mathcal{O}X$ of a topological space $(X, \mathcal{O}X)$, and

(c) the complete Boolean algebras.

From indefinite Dedekind cuts to scales

Frames

Being a Heyting algebra, each frame L has the implication $\rightarrow$ satisfying

$$a \wedge b \leq c \text{ iff } a \leq b \rightarrow c.$$

The pseudocomplement of an $a \in L$ is

$$a^* = a \rightarrow 0 = \bigvee\{b \in L : a \wedge b = 0\}.$$

Given $a, b \in L$, we denote by $\prec$ the relation defined by

$$a \prec b \quad \text{iff} \quad a^* \vee b = 1.$$

From indefinite Dedekind cuts to scales

Frames

Being a Heyting algebra, each frame L has the implication $\rightarrow$ satisfying

$$a \wedge b \leq c \text{ iff } a \leq b \rightarrow c.$$

The pseudocomplement of an $a \in L$ is

$$a^* = a \rightarrow 0 = \bigvee \{b \in L : a \wedge b = 0\}.$$

Given $a, b \in L$, we denote by $\prec$ the relation defined by

$$a \prec b \quad \text{iff} \quad a^* \vee b = 1.$$

In particular, when $L = \mathcal{O}X$ for some topological space X, one has $U^* = \operatorname{Int}(X \setminus U)$ and $U \prec V$ iff $\operatorname{Cl} U \subseteq V$ for each $U, V \in \mathcal{O}X$.

Also, in a Boolean algebra, the pseudocomplement is a complement and $a \prec b$ iff $a \leq b$.

From indefinite Dedekind cuts to scales

Scales on frames

Definition (Indefinite Dedekind cuts)

An indefinite Dedekind cut is a function $\mathcal{S} : \mathbb{Q} \to \mathbf{2}$ such that

(D1) $\mathcal{S}$ is decreasing, i.e. $\mathcal{S}(q) \leq \mathcal{S}(p)$ whenever $p < q$;

(D2) $\bigvee_{q \in \mathbb{Q}} \mathcal{S}(q) = 1$ and $\bigwedge_{q \in \mathbb{Q}} \mathcal{S}(q) = 0$.

From indefinite Dedekind cuts to scales

Definition (Scales on a frame)

Let L be a frame. A scale is a function $s : \mathbb{Q} \to L$ satisfying

(S1) $s(q) \prec s(p)$ whenever $p < q$;

(S2) $\bigvee_{q \in \mathbb{Q}} s(q) = 1 = \bigvee_{q \in \mathbb{Q}} s(q)^*$.

From indefinite Dedekind cuts to scales

Scales on frames

Definition (Scales on a frame)

Let L be a frame. A scale is a family $(s_q \mid q \in \mathbb{Q})$ in L satisfying

(S1) $s_q \prec s_p$ whenever $p < q$;

(S2) $\bigvee_{q\in\mathbb{Q}} s_q = 1 = \bigvee_{q\in\mathbb{Q}} s_q^*$.

From indefinite Dedekind cuts to scales

Scales on frames

Definition (Scales on a frame)

Let L be a frame. A scale is a family $(s_q \mid q \in \mathbb{Q})$ in L satisfying

(S1) $s_q \prec s_p$ whenever $p < q$;

(S2) $\bigvee_{q \in \mathbb{Q}} s_q = 1 = \bigvee_{q \in \mathbb{Q}} s_q^*$.

• $\underline{L = \mathbf{2}}$

A scale on **2** is just an indefinite Dedekind cut.

From indefinite Dedekind cuts to scales

Scales on frames

Definition (Scales on a frame)

Let L be a frame. A scale is a family $(s_q \mid q \in \mathbb{Q})$ in L satisfying

(S1) $s_q \prec s_p$ whenever $p < q$;

(S2) $\bigvee_{q\in\mathbb{Q}} s_q = 1 = \bigvee_{q\in\mathbb{Q}} s_q^*$.

- $\underline{L = \mathcal{O}X}$ for some topological space $(X, \mathcal{O}X)$.

A scale on $\mathcal{O}X$ is a function $\mathcal{S} : \mathbb{Q} \to \mathcal{O}X$ satisfying

(S1) $U_q \prec U_p$ whenever $p < q$, i.e. $\mathrm{Cl}\, U_q \subseteq U_p$ whenever $p < q$;

(S2) $\bigcup_{q\in\mathbb{Q}} U_q = X$ and $\bigcap_{q\in\mathbb{Q}} U_q = \varnothing$.

We shall also refer to them as scales of open subsets.

From indefinite Dedekind cuts to scales

Scales on frames

Definition (Scales on a frame)

Let L be a frame. A scale is a family $(s_q \mid q \in \mathbb{Q})$ in L satisfying

(S1) $s_q \prec s_p$ whenever $p < q$;

(S2) $\bigvee_{q\in\mathbb{Q}} s_q = 1 = \bigvee_{q\in\mathbb{Q}} s_q^*$.

- $\underline{L = \mathfrak{D}X}$ for some set X.

A scale on $\mathfrak{D}X$ is a function $\mathcal{S} : \mathbb{Q} \to \mathfrak{D}X$ satisfying

(S1) $S_q \prec S_p$ whenever $p < q$, i.e. $S_q \subseteq S_p$ whenever $p < q$;

(S2) $\bigcup_{q\in\mathbb{Q}} S_q = X$ and $\bigcap_{q\in\mathbb{Q}} S_q = \varnothing$.

We shall denote by $\mathrm{Scale}(X)$ the collection of all scales on X.

From indefinite Dedekind cuts to scales

Scales on frames

Definition (Scales on a frame)

Let L be a frame. A scale is a family $(s_q \mid q \in \mathbb{Q})$ in L satisfying

(S1) $s_q \prec s_p$ whenever $p < q$;

(S2) $\bigvee_{q\in\mathbb{Q}} s_q = 1 = \bigvee_{q\in\mathbb{Q}} s_q^*$.

• $\underline{L = \mathfrak{D}X}$ for some set X.

A scale on $\mathfrak{D}X$ is a function $\mathcal{S} : \mathbb{Q} \to \mathfrak{D}X$ satisfying

(S1) $S_q \prec S_p$ whenever $p < q$, i.e. $S_q \subseteq S_p$ whenever $p < q$;

(S2) $\bigcup_{q\in\mathbb{Q}} S_q = X$ and $\bigcap_{q\in\mathbb{Q}} S_q = \varnothing$.

We shall denote by $\mathrm{Scale}(X)$ the collection of all scales on X. From now on we will only consider scales of this type, but we want to emphasize that the same techniques could be applied to work with scales of open sets.

Scales and real functions

Some binary relations in $\mathrm{Scale}(X)$

We will consider three binary relations $\leq$, $\preceq$ and $\sim$ on $\mathrm{Scale}(X)$:

$$\mathcal{S} \leq \mathcal{T} \iff S_q \subseteq T_q \quad \text{for each } q \in \mathbb{Q}$$

$$\mathcal{S} \preceq \mathcal{T} \iff S_q \subseteq T_p \quad \text{for each } p < q \in \mathbb{Q}$$

Scales and real functions

Some binary relations in $\mathrm{Scale}(X)$

We will consider three binary relations $\leq$, $\preceq$ and $\sim$ on $\mathrm{Scale}(X)$:

$$\mathcal{S} \leq \mathcal{T} \iff S_q \subseteq T_q \quad \text{for each } q \in \mathbb{Q}$$

$$\mathcal{S} \preceq \mathcal{T} \iff S_q \subseteq T_p \quad \text{for each } p < q \in \mathbb{Q}$$

Clearly enough we have that $\mathcal{S} \leq \mathcal{T}$ implies that $\mathcal{S} \preceq \mathcal{T}$.

$\leq$ is a partial order while $\preceq$ is only a preorder.

Scales and real functions

Some binary relations in $\mathrm{Scale}(X)$

We will consider three binary relations $\leq$, $\preceq$ and $\sim$ on $\mathrm{Scale}(X)$:

$$\mathcal{S} \leq \mathcal{T} \iff S_q \subseteq T_q \quad \text{for each } q \in \mathbb{Q}$$

$$\mathcal{S} \preceq \mathcal{T} \iff S_q \subseteq T_p \quad \text{for each } p < q \in \mathbb{Q}$$

Clearly enough we have that $\mathcal{S} \leq \mathcal{T}$ implies that $\mathcal{S} \preceq \mathcal{T}$.

$\leq$ is a partial order while $\preceq$ is only a preorder.

We can use the preorder $\preceq$ to define an equivalence relation $\sim$:

$$\mathcal{S} \sim \mathcal{T} \iff \mathcal{S} \preceq \mathcal{T} \text{ and } \mathcal{T} \preceq \mathcal{S}.$$

Scales and real functions

Some binary relations in $\mathrm{Scale}(X)$

We will consider three binary relations $\leq$, $\preceq$ and $\sim$ on $\mathrm{Scale}(X)$:

$$\mathcal{S} \leq \mathcal{T} \iff S_q \subseteq T_q \quad \text{for each } q \in \mathbb{Q}$$

$$\mathcal{S} \preceq \mathcal{T} \iff S_q \subseteq T_p \quad \text{for each } p < q \in \mathbb{Q}$$

Clearly enough we have that $\mathcal{S} \leq \mathcal{T}$ implies that $\mathcal{S} \preceq \mathcal{T}$.

$\leq$ is a partial order while $\preceq$ is only a preorder.

We can use the preorder $\preceq$ to define an equivalence relation $\sim$:

$$\mathcal{S} \sim \mathcal{T} \iff \mathcal{S} \preceq \mathcal{T} \text{ and } \mathcal{T} \preceq \mathcal{S}.$$

This relation, determines a partial order on $\mathrm{Scale}(X)/\sim$:

$$[\mathcal{S}] \preceq [\mathcal{T}] \iff \mathcal{S} \preceq \mathcal{T}.$$

By the construction of $\sim$, the corresponding relation is indeed well-defined and it yields a partially ordered set $(\mathrm{Scale}(X)/\sim, \preceq)$.

Scales and real functions

The real function generated by a scale

Notation

Given $f : X \to \mathbb{R}$ and $q \in \mathbb{Q}$, we write $[f \geq q] = \{x \in X \mid q \leq f(x)\}$ and $[f > q] = \{x \in X \mid q < f(x)\}$.

Scales and real functions

The real function generated by a scale

Notation

Given $f : X \to \mathbb{R}$ and $q \in \mathbb{Q}$, we write $[f \geq q] = \{x \in X \mid q \leq f(x)\}$ and $[f > q] = \{x \in X \mid q < f(x)\}$.

Proposition

Let X be a set and $\mathcal{S} = (S_q \mid q \in \mathbb{Q})$ a scale on X. Then

$$f_{\mathcal{S}}(x) = \bigvee\{q \in \mathbb{Q} \mid x \in S_q\}$$

determines a unique function $f_{\mathcal{S}} : X \to \mathbb{R}$ such that

$$[f_{\mathcal{S}} > q] \subseteq S_q \subseteq [f_{\mathcal{S}} \geq q] \quad \text{for each } q \in \mathbb{Q}.$$

Scales and real functions

The real function generated by a scale

Definition

Let $\mathcal{S} = (S_q \mid q \in \mathbb{Q})$ be a scale in X. The function $f_{\mathcal{S}} : X \to \mathbb{R}$ defined by

$$f_{\mathcal{S}}(x) = \bigvee\{q \in \mathbb{Q} \mid x \in S_q\}$$

for each $x \in X$, is said to be the real function generated by $\mathcal{S}$.

Scales and real functions

The real function generated by a scale

Definition

Let $\mathcal{S} = (S_q \mid q \in \mathbb{Q})$ be a scale in X. The function $f_{\mathcal{S}} : X \to \mathbb{R}$ defined by

$$f_{\mathcal{S}}(x) = \bigvee\{q \in \mathbb{Q} \mid x \in S_q\}$$

for each $x \in X$, is said to be the real function generated by $\mathcal{S}$.

Proposition

Let $\mathcal{S}$ and $\mathcal{T}$ be two scales on X generating real functions $f_{\mathcal{S}}$ and $f_{\mathcal{T}}$, respectively. Then

$$\mathcal{S} \preceq \mathcal{T} \iff f_{\mathcal{S}} \leq f_{\mathcal{T}};$$

consequently,

$$\mathcal{S} \sim \mathcal{T} \iff f_{\mathcal{S}} = f_{\mathcal{T}}.$$

Scales and real functions

Scales generating a given real function

Lemma

Let X be a set, $\mathcal{S} = (S_q \mid q \in \mathbb{Q})$ a scale on X and

$$\mathcal{S}^{min} \equiv \left(S_q^{min} = \bigcup_{p>q} S_p \mid q \in \mathbb{Q}\right) \text{ and}$$

$$\mathcal{S}^{max} \equiv \left(S_q^{max} = \bigcap_{p<q} S_p \mid q \in \mathbb{Q}\right).$$

Then:

(1) $\mathcal{S}^{min}$ and $\mathcal{S}^{max}$ are scales on X.

(2) $\mathcal{S}^{min} \leq \mathcal{S} \leq \mathcal{S}^{max}$ and $\mathcal{S}^{min} \sim \mathcal{S} \sim \mathcal{S}^{max}$.

(3) If $\mathcal{T} \sim \mathcal{S}$, then $\mathcal{S}^{min} \leq \mathcal{T} \leq \mathcal{S}^{max}$.

(4) If $\mathcal{T} \sim \mathcal{S}$, then $\mathcal{T}^{min} = \mathcal{S}^{min}$ and $\mathcal{T}^{max} = \mathcal{S}^{max}$.

(5) $\mathcal{S}^{min} = \{[f_\mathcal{S} > q] \mid q \in \mathbb{Q}\}$ and $\mathcal{S}^{max} = \{[f_\mathcal{S} \geq q] \mid q \in \mathbb{Q}\}$.

Scales and real functions

Scales generating a given real function

Now we can characterize the equivalence class of a given scale as an interval in the partially ordered set $(\mathrm{Scale}(X), \leq)$:

Proposition

Let X be a set and $\mathcal{S} = (S_q \mid q \in \mathbb{Q})$ a scale on X. Then

$$[\mathcal{S}] = \{\mathcal{T} \mid \mathcal{S}^{\min} \leq \mathcal{T} \leq \mathcal{S}^{\max}\}.$$

Scales and real functions

Scales generating a given real function

Now we can characterize the equivalence class of a given scale as an interval in the partially ordered set $(\mathrm{Scale}(X), \leq)$:

Proposition

Let X be a set and $\mathcal{S} = (S_q \mid q \in \mathbb{Q})$ a scale on X. Then

$$[\mathcal{S}] = \{\mathcal{T} \mid \mathcal{S}^{\mathrm{min}} \leq \mathcal{T} \leq \mathcal{S}^{\mathrm{max}}\}.$$

Proposition

Let $f : X \to \mathbb{R}$, $\mathcal{S}_f^{min} = \{[f > q] \mid q \in \mathbb{Q}\}$ and $\mathcal{S}_f^{max} = \{[f \geq q] \mid q \in \mathbb{Q}\}$

(1) $\mathcal{S}_f^{min}$ and $\mathcal{S}_f^{max}$ are scales generating f.

Scales and real functions

Scales generating a given real function

Now we can characterize the equivalence class of a given scale as an interval in the partially ordered set $(\mathrm{Scale}(X), \leq)$:

Proposition

Let X be a set and $\mathcal{S} = (S_q \mid q \in \mathbb{Q})$ a scale on X. Then

$$[\mathcal{S}] = \{\mathcal{T} \mid \mathcal{S}^{\min} \leq \mathcal{T} \leq \mathcal{S}^{\max}\}.$$

Proposition

Let $f : X \to \mathbb{R}$, $\mathcal{S}_f^{min} = \{[f > q] \mid q \in \mathbb{Q}\}$ and $\mathcal{S}_f^{max} = \{[f \geq q] \mid q \in \mathbb{Q}\}$

(1) $\mathcal{S}_f^{min}$ and $\mathcal{S}_f^{max}$ are scales generating f.

(2) If $\mathcal{S}$ is a scale that generates f, then $\mathcal{S}^{min} = \mathcal{S}_f^{min}$ and $\mathcal{S}^{max} = \mathcal{S}_f^{max}$.

Scales and real functions

Scales generating a given real function

Now we can characterize the equivalence class of a given scale as an interval in the partially ordered set $(\mathrm{Scale}(X), \leq)$:

Proposition

Let X be a set and $\mathcal{S} = (S_q \mid q \in \mathbb{Q})$ a scale on X. Then

$$[\mathcal{S}] = \{\mathcal{T} \mid \mathcal{S}^{\min} \leq \mathcal{T} \leq \mathcal{S}^{\max}\}.$$

Proposition

Let $f : X \to \mathbb{R}$, $\mathcal{S}_f^{min} = \{[f > q] \mid q \in \mathbb{Q}\}$ and $\mathcal{S}_f^{max} = \{[f \geq q] \mid q \in \mathbb{Q}\}$

(1) $\mathcal{S}_f^{min}$ and $\mathcal{S}_f^{max}$ are scales generating f.

(2) If $\mathcal{S}$ is a scale that generates f, then $\mathcal{S}^{min} = \mathcal{S}_f^{min}$ and $\mathcal{S}^{max} = \mathcal{S}_f^{max}$.

(3) $\mathcal{S}$ is a scale that generates f if and only if $\mathcal{S}_f^{min} \leq \mathcal{S} \leq \mathcal{S}_f^{max}$.

Scales and real functions

Scales generating a given real function

Now we can characterize the equivalence class of a given scale as an interval in the partially ordered set $(\mathrm{Scale}(X), \leq)$:

Proposition

Let X be a set and $\mathcal{S} = (S_q \mid q \in \mathbb{Q})$ a scale on X. Then

$$[\mathcal{S}] = \{\mathcal{T} \mid \mathcal{S}^{\min} \leq \mathcal{T} \leq \mathcal{S}^{\max}\}.$$

Proposition

Let $f : X \to \mathbb{R}$, $\mathcal{S}_f^{min} = \{[f > q] \mid q \in \mathbb{Q}\}$ and $\mathcal{S}_f^{max} = \{[f \geq q] \mid q \in \mathbb{Q}\}$

(1) $\mathcal{S}_f^{min}$ and $\mathcal{S}_f^{max}$ are scales generating f.

(2) If $\mathcal{S}$ is a scale that generates f, then $\mathcal{S}^{min} = \mathcal{S}_f^{min}$ and $\mathcal{S}^{max} = \mathcal{S}_f^{max}$.

(3) $\mathcal{S}$ is a scale that generates f if and only if $\mathcal{S}_f^{min} \leq \mathcal{S} \leq \mathcal{S}_f^{max}$.

(4) The collection of all scales that generate f is $[\mathcal{S}_f^{min}] = [\mathcal{S}_f^{max}]$.

Scales and real functions

Scales generating a given real function

We can now establish the desired correspondence:

Proposition

Let X be a set. There exists an order isomorphism between the partially ordered sets $(\mathrm{F}(X), \leq)$ of real functions on X and $(\mathrm{Scale}(X)/\sim, \preceq)$.

In fact, this correspondence is more than an order isomorphism.

As we will see in what follows it can be used to express the algebraic operations between real functions purely in terms of scales.

Furthermore, when the space is enriched with some additional structure (e.g. a topology or a preorder) the real functions preserving the structure ((semi)continuous functions or increasing functions, respectively) can be characterized by mean of scales.

Algebraic operations

In what follows we will try to show how one can deal with the usual algebraic operations in terms of scales, without constructing the corresponding real functions.

Algebraic operations

In what follows we will try to show how one can deal with the usual algebraic operations in terms of scales, without constructing the corresponding real functions.

Opposite scale

Given a scale $\mathcal{S}$ on X, define

$$-\mathcal{S} = (X \setminus S_{-q} \mid q \in \mathbb{Q}).$$

(1) $-\mathcal{S}$ is a scale on X;

(2) $f_{-\mathcal{S}} = -f_{\mathcal{S}}$.

Algebraic operations

In what follows we will try to show how one can deal with the usual algebraic operations in terms of scales, without constructing the corresponding real functions.

Finite joins and meets

Given two scales $\mathcal{S}$ and $\mathcal{T}$ on X, we write

$$\mathcal{S} \vee \mathcal{T} = (S_q \cup T_q \mid q \in \mathbb{Q}) \quad \text{and} \quad \mathcal{S} \wedge \mathcal{T} = (S_q \cap T_q \mid q \in \mathbb{Q}).$$

(1) $\mathcal{S} \vee \mathcal{T} = \mathcal{T} \vee \mathcal{S}$ is a scale on X;

(2) $f_{\mathcal{S} \vee \mathcal{T}} = f_{\mathcal{S}} \vee f_{\mathcal{T}}$ and $f_{\mathcal{S} \wedge \mathcal{T}} = f_{\mathcal{S}} \wedge f_{\mathcal{T}}$.

Algebraic operations

In what follows we will try to show how one can deal with the usual algebraic operations in terms of scales, without constructing the corresponding real functions.

Arbitrary joins and meets

Given a family of scales $\{\mathcal{S}^i\}_{i\in I}$ on X, we define

$$\bigvee_{i\in I}\mathcal{S}^i = \left(\bigcup_{i\in I} S^i_q \mid q\in\mathbb{Q}\right) \quad \text{and} \quad \bigwedge_{i\in I}\mathcal{S}^i = \left(\bigcap_{i\in I} S^i_q \mid q\in\mathbb{Q}\right).$$

If $\bigcap_{q\in\mathbb{Q}}\bigcup_{i\in I} S^i_q = \varnothing$, then we have that:

(1) $\bigvee_{i\in I}\mathcal{S}^i$ is a scale on X;

(2) $f_{\bigvee_{i\in I}\mathcal{S}^i} = \bigvee_{i\in I} f_{\mathcal{S}^i}$.

Algebraic operations

In what follows we will try to show how one can deal with the usual algebraic operations in terms of scales, without constructing the corresponding real functions.

Arbitrary joins and meets

Given a family of scales $\{\mathcal{S}^i\}_{i\in I}$ on X, we define

$$\bigvee_{i\in I}\mathcal{S}^i = \left(\bigcup_{i\in I} S^i_q \mid q\in\mathbb{Q}\right) \quad \text{and} \quad \bigwedge_{i\in I}\mathcal{S}^i = \left(\bigcap_{i\in I} S^i_q \mid q\in\mathbb{Q}\right).$$

Dually, if $\bigcup_{q\in\mathbb{Q}}\bigcap_{i\in I} S^i_q = X$ we have that:

(1) $\bigwedge_{i\in I}\mathcal{S}^i = -\left(\bigvee_{i\in I} -\mathcal{S}^i\right)$ is a scale on X;

(2) $f_{\bigwedge_{i\in I}\mathcal{S}^i} = \bigwedge_{i\in I} f_{\mathcal{S}^i}$.

Algebraic operations

In what follows we will try to show how one can deal with the usual algebraic operations in terms of scales, without constructing the corresponding real functions.

Product with a scalar

Given $r \in \mathbb{R}$ such that $r > 0$ and a scale $\mathcal{S}$ on X, we define

$$r \cdot \mathcal{S} = \left(\bigcup_{p<r} S_{\frac{q}{p}} \mid q \in \mathbb{Q}\right).$$

(1) $r \cdot \mathcal{S}$ is a scale on X;

(2) $f_{r\cdot\mathcal{S}} = r \cdot f_{\mathcal{S}}$.

Algebraic operations

In what follows we will try to show how one can deal with the usual algebraic operations in terms of scales, without constructing the corresponding real functions.

Sum and difference

Given two scales $\mathcal{S}$ and $\mathcal{T}$ on X, we define

$$\mathcal{S}+\mathcal{T} = \left(\bigcup_{p\in\mathbb{Q}} S_p \cap T_{q-p} \mid q \in \mathbb{Q}\right) \text{ and } \mathcal{S}-\mathcal{T} = \left(\bigcup_{p\in\mathbb{Q}} S_p \setminus T_{p-q} \mid q \in \mathbb{Q}\right).$$

(1) $\mathcal{S}+\mathcal{T} = \mathcal{T}+\mathcal{S}$ is a scale on X;

(2) $f_{\mathcal{S}+\mathcal{T}} = f_{\mathcal{S}} + f_{\mathcal{T}}$.

Algebraic operations

In what follows we will try to show how one can deal with the usual algebraic operations in terms of scales, without constructing the corresponding real functions.

Product

Given two scales $\mathcal{S}$ and $\mathcal{T}$ on X such that $\mathcal{S}^{\mathbf{0}} \preceq \mathcal{S}, \mathcal{T}$, we define

$$\mathcal{S} \cdot \mathcal{T} = \left(\bigcup_{0<p} S_p \cap T_{\frac{q}{p}} \mid q \in \mathbb{Q} \right).$$

(1) $\mathcal{S} \cdot \mathcal{T} = \mathcal{T} + \mathcal{S}$ is a scale on X;

(2) $f_{\mathcal{S}+\mathcal{T}} = f_{\mathcal{S}} + f_{\mathcal{T}}$.

Semicontinuous real functions and scales

Let $(X, \mathcal{O}X)$ be a topological space and $f : X \to \mathbb{R}$. Then

(1) f is lower semicontinuous if $[f > q] \in \mathcal{O}X$ for each $q \in \mathbb{Q}$;

(2) f is upper semicontinuous if $[f < q] \in \mathcal{O}X$ for each $q \in \mathbb{Q}$;

(3) f is continuous if it is both lower and upper semicontinuous.

Semicontinuous real functions and scales

Let $(X, \mathcal{O}X)$ be a topological space and $f : X \to \mathbb{R}$. Then

(1) f is lower semicontinuous if $[f > q] \in \mathcal{O}X$ for each $q \in \mathbb{Q}$;

(2) f is upper semicontinuous if $[f < q] \in \mathcal{O}X$ for each $q \in \mathbb{Q}$;

(3) f is continuous if it is both lower and upper semicontinuous.

Semicontinuous scales

Let $\mathcal{S}$ be a scale on $(X, \mathcal{O}X)$ and $f_{\mathcal{S}}$ the real function generated by $\mathcal{S}$:

(1) $f_{\mathcal{S}}$ is lower semicontinuous iff $S_q \subseteq \operatorname{Int} S_p$ whenever $p < q \in \mathbb{Q}$;

(2) $f_{\mathcal{S}}$ is upper semicontinuous iff $\operatorname{Cl} S_q \subseteq S_p$ whenever $p < q \in \mathbb{Q}$;

(3) $f_{\mathcal{S}}$ is continuous iff $\operatorname{Cl} S_q \subseteq \operatorname{Int} S_p$ whenever $p < q \in \mathbb{Q}$.

Representability of preorders through scales

Let $(X, \mathcal{O}X, \mathcal{R})$ be a topological preordered space, i.e. a topological space $(X, \mathcal{O}X)$ endowed with a preorder $\mathcal{R}$ (a reflexive and transitive relation).

The asymmetric part $\mathcal{P}$ of $\mathcal{R}$ is defined for each $x, y \in X$ as

$$x\mathcal{P}y \text{ if and only if } x\mathcal{R}y \text{ and not } y\mathcal{R}x.$$

A subset A of $(X, \mathcal{R})$ is said to be increasing if $x\mathcal{R}y$ together with $x \in A$ imply $y \in A$.

For a subset A of X we write $i(A) = \{y \in X \mid \exists x \in A \text{ such that } x\mathcal{R}y\}$ to denote the smallest increasing subset of X containing A.

Representability of preorders through scales

A function $f : (X, \mathcal{R}) \to (\mathbb{R}, \leq)$ is increasing if $f(x) \leq f(y)$ whenever $x\mathcal{R}y$, stricly increasing if $f(x) < f(y)$ whenever $x\mathcal{P}y$ and it is a preorder embedding in case $f(x) \leq f(y)$ if and only if $x\mathcal{R}y$.

A preorder $\mathcal{R}$ on X is said to be *representable* if there exists a preorder embedding (also called "utility function") $f : (X, \mathcal{R}) \to (\mathbb{R}, \leq)$.

Representability of preorders through scales

A function $f : (X, \mathcal{R}) \to (\mathbb{R}, \leq)$ is increasing if $f(x) \leq f(y)$ whenever $x\mathcal{R}y$, stricly increasing if $f(x) < f(y)$ whenever $x\mathcal{P}y$ and it is a preorder embedding in case $f(x) \leq f(y)$ if and only if $x\mathcal{R}y$.

A preorder $\mathcal{R}$ on X is said to be *representable* if there exists a preorder embedding (also called "utility function") $f : (X, \mathcal{R}) \to (\mathbb{R}, \leq)$.

Order-preserving scales

Let $\mathcal{S}$ be a scale on $(X, \mathcal{R})$ and $f_{\mathcal{S}}$ the real function generated by $\mathcal{S}$:

(1) $f_{\mathcal{S}}$ is increasing iff $i(S_q) \subseteq S_p$ whenever $p < q \in \mathbb{Q}$;

(2) $f_{\mathcal{S}}$ is stricly increasing iff for each $x, y \in X$ with $x\mathcal{P}y$ there exist $p < q \in \mathbb{Q}$ such that $x \in S_p$ and $y \notin S_q$;

(3) $f_{\mathcal{S}}$ is preorder embedding iff it is both increasing and strictly increasing.