On extended real valued functions in pointfree topology

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VI Portuguese Category Seminar

July 23, 2009

UNIVERSIDADE DE COIMBRA

Background: the frame of reals $\mathfrak{L}(\mathbb{R})$

$$\mathfrak{L}(\mathbb{R}) = \mathrm{FRM}\langle\ (p,q)\ p,q \in \mathbb{Q}\ |$$

$$\begin{array}{ll} \text{(R1)} & (p,q) \wedge (r,s) = (p \vee r, q \wedge s) \\ \text{(R2)} & p \leq r < q \leq s \quad \Longrightarrow \quad (p,q) \vee (r,s) = (p,s) \\ \text{(R3)} & (p,q) = \bigvee\{(r,s) \mid p < r < s < q\} \\ \text{(R4)} & \bigvee\{(p,q) \mid p,q \in \mathbb{Q}\} = 1\rangle. \end{array}$$

B. Banaschewski and C. J. Mulvey,
Stone-Čech compactification of locales II,
J. Pure Appl. Algebra 33 (1984) 107–122.

B. Banaschewski,
The real numbers in pointfree topology,
Textos de Matemática, Série B, 12, Univ. de Coimbra, 1997.

Background: the commutative f-ring $\mathcal{R}L$

$$\mathcal{R}L = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)$$

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Algebraic operations

Let $\langle p, q\rangle = \{r \in \mathbb{Q} \mid p < r < q\}$, let $\diamond \in \{+, \cdot, \wedge, \vee\}$, and let

$$\langle r, s\rangle \diamond \langle t, u\rangle = \big\{x \diamond y \mid x \in \langle r, s\rangle \text{ and } y \in \langle t, u\rangle\big\}.$$

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$$(f_1 \diamond f_2)(p, q) \quad = \quad \bigvee\big\{f_1(r, s) \wedge f_2(t, u) \mid \langle r, s\rangle \diamond \langle t, u\rangle \subseteq \langle p, q\rangle\big\},$$

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These operations satisfy all the axioms in $\mathbb{Q}$ so that $(C(L), +, \cdot, \leq)$ becomes a commutative archimedean and strong f-ring with unit **1**.

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We also have the following descriptions of the partial order:

$$
\begin{aligned}
f_1 \leq f_2 \quad &\Leftrightarrow \quad f_1(p,-) \leq f_2(p,-) \quad \text{for all } p \in \mathbb{Q} \\
&\Leftrightarrow \quad f_2(-,q) \leq f_1(-,q) \quad \text{for all } q \in \mathbb{Q} \\
&\Leftrightarrow \quad f_1(r,-) \wedge f_2(-,r) = 0 \quad \text{for all } r \in \mathbb{Q} \\
&\Leftrightarrow \quad f_2(p,-) \vee f_1(-,q) = 1 \quad \text{for all } p < q \in \mathbb{Q}.
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$$(-,q) := \bigvee_{p\in\mathbb{Q}}(p,q) \qquad\qquad (p,-) := \bigvee_{q\in\mathbb{Q}}(p,q)$$

$$\mathfrak{L}_l(\mathbb{R}) = \langle(-,q)\mid q\in\mathbb{Q}\rangle \qquad\qquad \mathfrak{L}_u(\mathbb{R}) = \langle(p,-)\mid p\in\mathbb{Q}\rangle$$

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$$\text{(r3)}\quad (p, —) = \bigvee_{r>p} (r, —) \qquad \text{(r4)}\quad (—, q) = \bigvee_{s<q} (—, s)$$

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Y.-M. Li, G.-J. Wang,
Localic Katětov-Tong insertion theorem and localic Tietze extension theorem,
Comment. Math. Univ. Carolinae 38 (1997) 801–814.

Background: the frames of upper and lower reals $\mathfrak{L}_u(\mathbb{R})$ aand $\mathfrak{L}_l(\mathbb{R})$

(r1) $p \geq q \implies (p,-) \wedge (-,q) = 0$

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(With $(p,q) = (p,-) \wedge (-,q)$ one goes back to (R1)-(R4))

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$$\mathfrak{L}_u(\mathbb{R}) \quad = \quad \mathrm{FRM}\langle\{(p,-) \mid p \in \mathbb{Q},\ (r,-) \text{ satisfy (r3) and (r5)}\}\rangle$$

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(r4) $(-, q) = \bigvee_{s<q}(-, s)$

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$$\mathfrak{L}_l(\mathbb{R}) = \mathrm{FRM}\langle\{(-,r) \mid r \in \mathbb{Q},\ (-,r) \text{ satisfy (r4) and (r6)}\}\rangle$$

$$\mathrm{LSC}(L) = \{f \in \mathrm{F}(L) \mid f(\mathfrak{L}_u(\mathbb{R})) \subseteq \mathfrak{c}L\}$$

Background: the frames of upper and lower reals $\mathfrak{L}_u(\mathbb{R})$ aand $\mathfrak{L}_l(\mathbb{R})$

$$\text{(r1)}\quad p \geq q \implies (p,-) \wedge (-,q) = 0$$

$$\text{(r2)}\quad p < q \implies (p,-) \vee (-,q) = 1$$

$$\text{(r3)}\quad (p,-) = \bigvee_{r>p}(r,-) \qquad \text{(r4)}\quad (-,q) = \bigvee_{s<q}(-,s)$$

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Background: the frames of upper and lower reals $\mathfrak{L}_u(\mathbb{R})$ aand $\mathfrak{L}_l(\mathbb{R})$

$$\text{(r1)}\quad p \geq q \implies (p,—) \wedge (—,q) = 0$$

$$\text{(r2)}\quad p < q \implies (p,—) \vee (—,q) = 1$$

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$$\mathrm{C}(L) = \mathrm{LSC}(L) \cap \mathrm{USC}(L)$$

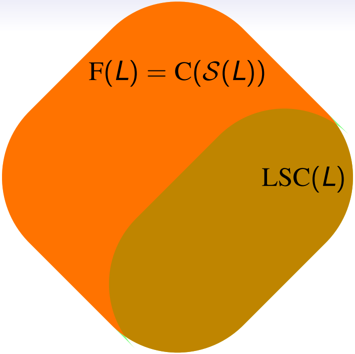
"The set $\mathrm{C}(X)$ of all continuous, real-valued functions on a topological space X will be provided with an algebraic structure and an order structure. Since their definitions do not involve continuity, we begin by imposing these structures on the collection $\mathbb{R}^X$ of all functions from X into the set $\mathbb{R}$ of real numbers. [...]

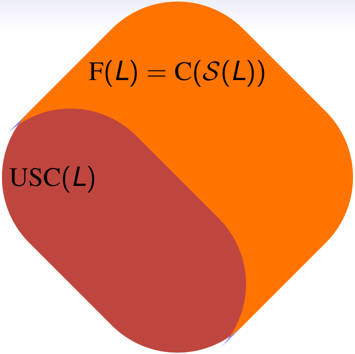
In fact, it is clear that $\mathbb{R}^X$ is a commutative ring with unity element (provided that X is non empty). [...]

Therefore $\mathrm{C}(X)$ is a commutative ring, a subring of $\mathbb{R}^X$."

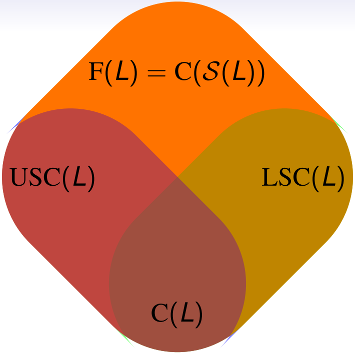
L. Gillman and M. Jerison,
Rings of Continuous Functions

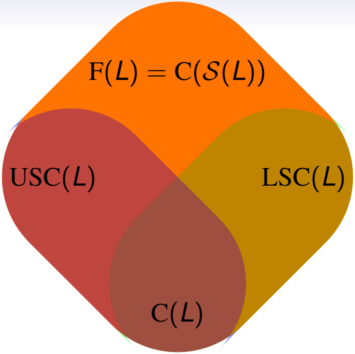
$$\mathrm{F}(L) = \mathrm{C}(\mathcal{S}(L))$$


F(L) = C(S(L))
LSC(L)

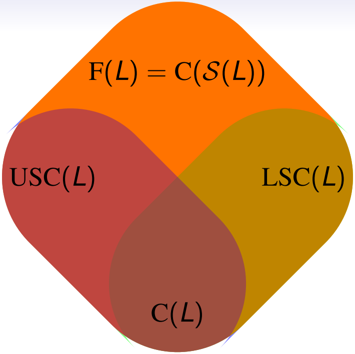


$\mathrm{F}(L) = \mathrm{C}(\mathcal{S}(L))$
$\mathrm{USC}(L)$


$\mathrm{F}(L) = \mathrm{C}(\mathcal{S}(L))$
$\mathrm{USC}(L)$
$\mathrm{LSC}(L)$
$\mathrm{C}(L)$



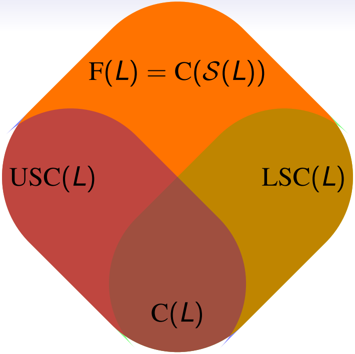
$(\mathrm{C}(L), +, \cdot, \leq)$ is a commutative f-ring, a subring of $(\mathrm{F}(L), +, \cdot, \leq)$.



$(\mathrm{C}(L), +, \cdot, \leq)$ is a commutative f-ring, a subring of $(\mathrm{F}(L), +, \cdot, \leq)$.

$$f \in \mathrm{USC}(L) \iff -f \in \mathrm{LSC}(L)$$

The posets $(\mathrm{USC}(L), \leq)$ and $(\mathrm{LSC}(L), \leq)$ are order isomorphic.



$(\mathrm{C}(L), +, \cdot, \leq)$ is a commutative f-ring, a subring of $(\mathrm{F}(L), +, \cdot, \leq)$.

Question

$$f \in \mathrm{USC}(L) \iff -f \in \mathrm{LSC}(L)$$

The posets $(\mathrm{USC}(L), \leq)$ and $(\mathrm{LSC}(L), \leq)$ are order isomorphic.

What else can be said about $(\mathrm{USC}(L), +, \cdot, \leq)$ and $(\mathrm{LSC}(L), +, \cdot, \leq)$?

Background: the frame of extended reals $\mathfrak{L}(\overline{\mathbb{R}})$

(r1) $p \geq q \implies (p, -) \wedge (-, q) = 0$

(r2) $p < q \implies (p, -) \vee (-, q) = 1$

(r3) $(p, -) = \bigvee_{r>p}(r, -)$ (r4) $(-, q) = \bigvee_{s<q}(-, s)$

(r5) $\bigvee_{p\in\mathbb{Q}}(p, -) = 1$ (r6) $\bigvee_{q\in\mathbb{Q}}(-, q) = 1$

$$\mathfrak{L}(\mathbb{R}) \quad = \quad \mathrm{FRM}\langle\{(p,-),(-,q)\ p,q \in \mathbb{Q},\ (r,-) \text{ satisfy } (\text{r1})\text{–}(\text{r6})\}\rangle$$

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$$\text{(r3)}\quad (p,—) = \bigvee_{r>p}(r,—) \qquad \text{(r4)}\quad (—,q) = \bigvee_{s<q}(—,s)$$

$$\mathfrak{L}(\overline{\mathbb{R}}) = \mathrm{FRM}\langle\{(p,—),(—,q)\ p,q \in \mathbb{Q},\ (r,—) \text{ satisfy } \text{(r1)–(r4)}\}\rangle$$

Background: the frame of extended reals $\mathfrak{L}(\overline{\mathbb{R}})$

(r3) $(p,-) = \bigvee_{r>p}(r,-)$

$$\mathfrak{L}_u(\overline{\mathbb{R}}) = \mathrm{FRM}\langle\{(p,-) \mid p \in \mathbb{Q},\ (r,-) \text{ satisfy (r3)}\}\rangle$$

Background: the frame of extended reals $\mathfrak{L}(\overline{\mathbb{R}})$

(r4) $(-, q) = \bigvee_{s<q} (-, s)$

$$\mathfrak{L}_l(\overline{\mathbb{R}}) \quad = \quad \mathrm{FRM}\langle \{(-, r) \mid r \in \mathbb{Q},\ (-, r) \text{ satisfy (r4)}\}\rangle$$

Background: the frame of extended reals $\mathfrak{L}(\overline{\mathbb{R}})$

$$
\begin{array}{ll}
\text{(r1)} & p \geq q \implies (p,-) \wedge (-,q) = 0 \\
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\text{(r3)} & (p,-) = \bigvee_{r>p} (r,-) \qquad \text{(r4)} \quad (-,q) = \bigvee_{s<q} (-,s)
\end{array}
$$

$$\mathfrak{L}(\overline{\mathbb{R}}) = \mathrm{FRM}\langle \{(p,-),(-,q)\ p,q \in \mathbb{Q},\ (r,-) \text{ satisfy (r1)–(r4)}\}\rangle$$

$$\mathfrak{L}_u(\overline{\mathbb{R}}) = \mathrm{FRM}\langle \{(p,-) \mid p \in \mathbb{Q},\ (r,-) \text{ satisfy (r3)}\}\rangle$$

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$$\overline{\mathrm{F}}(L) = \mathrm{Frm}(\mathfrak{L}(\overline{\mathbb{R}}), \mathcal{S}(L))$$

Background: the frame of extended reals $\mathfrak{L}(\overline{\mathbb{R}})$

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$$\mathfrak{L}(\overline{\mathbb{R}}) = \mathrm{FRM}\langle \{(p,-),(-,q)\ p,q \in \mathbb{Q},\ (r,-) \text{ satisfy } (\mathrm{r1})\text{–}(\mathrm{r4})\}\rangle$$

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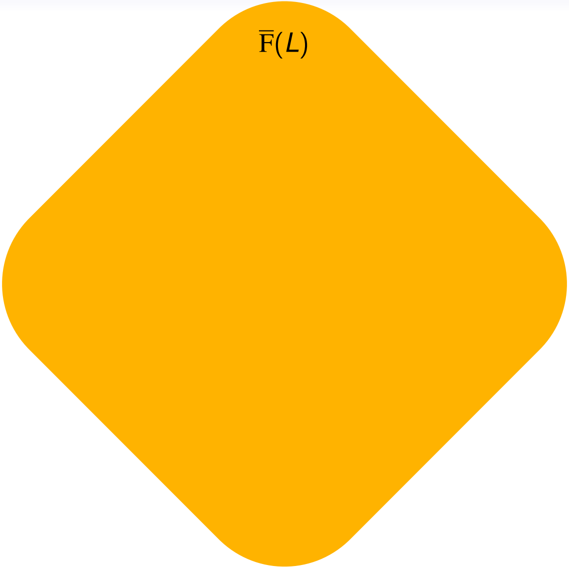
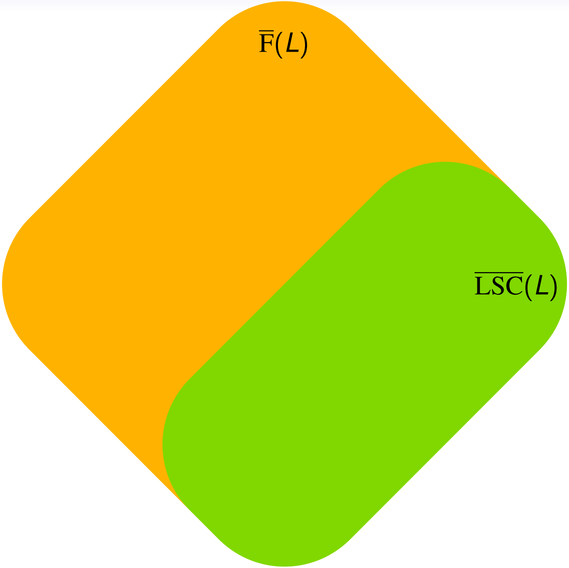
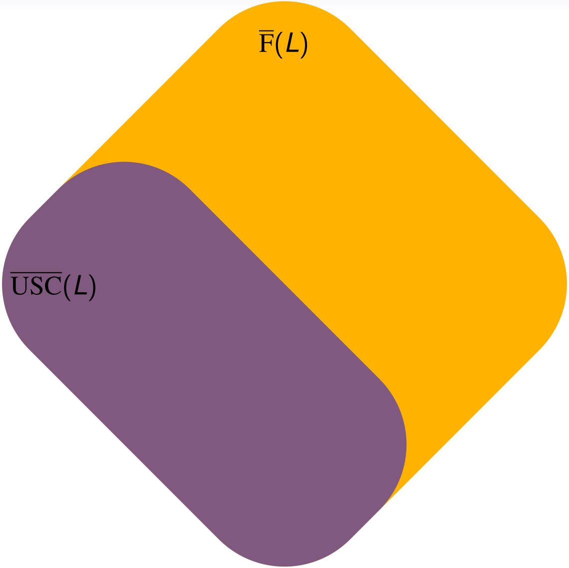
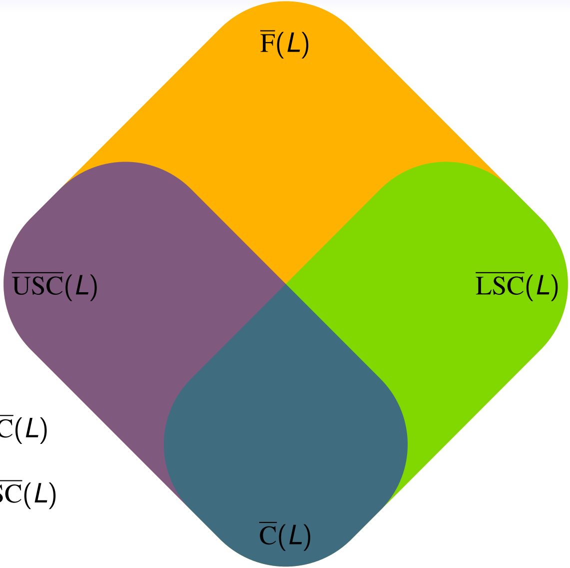
$$\overline{\mathrm{C}}(L) = \overline{\mathrm{LSC}}(L) \cap \overline{\mathrm{USC}}(L)$$

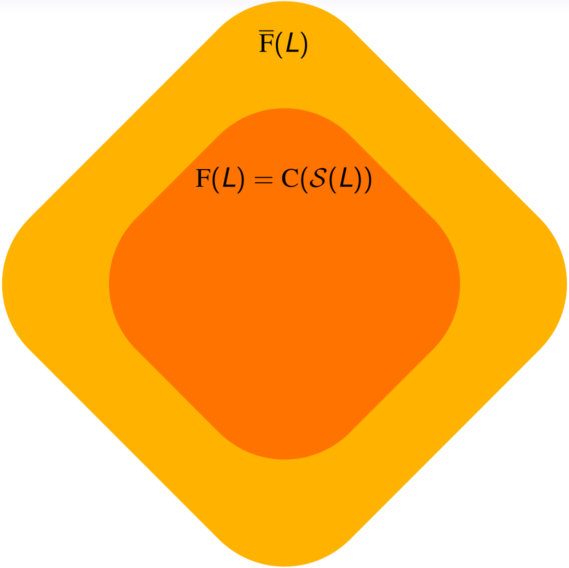

$\overline{\mathrm{F}}(L)$

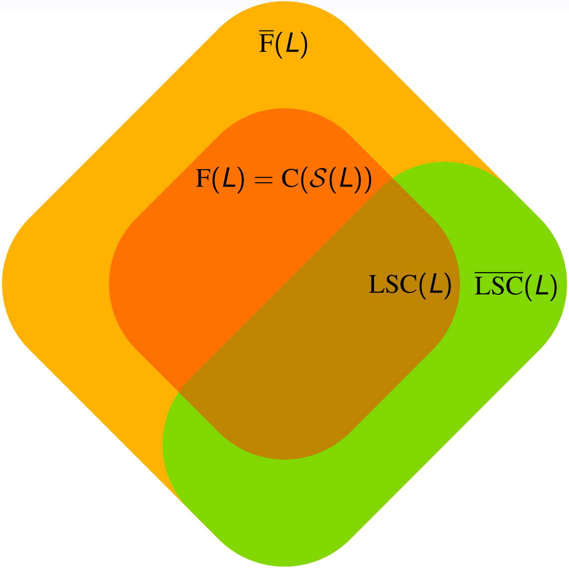

$\overline{\mathrm{F}}(L)$
$\overline{\mathrm{LSC}}(L)$

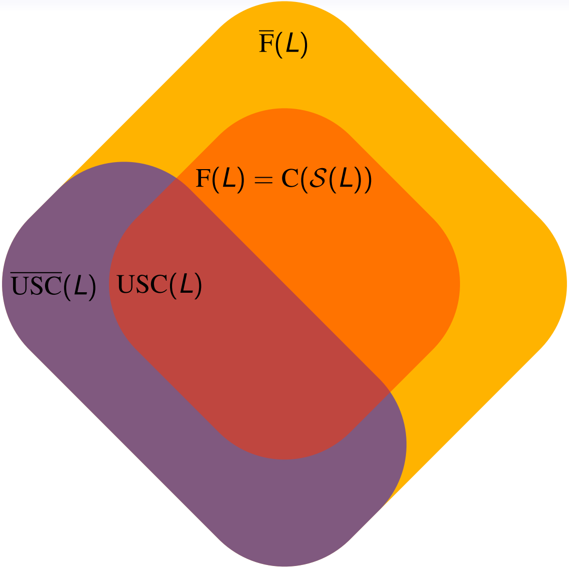

$\overline{\mathrm{F}}(L)$
$\overline{\mathrm{USC}}(L)$

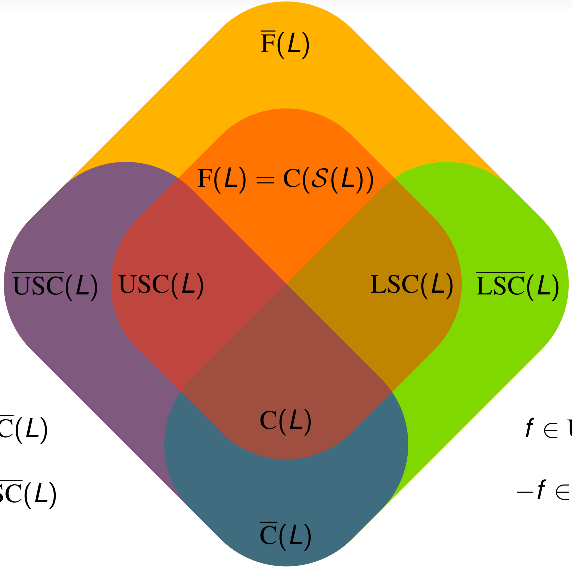


$$f \in \overline{\mathrm{USC}}(L) \Updownarrow -f \in \overline{\mathrm{LSC}}(L)$$


$\overline{\mathrm{F}}(L)$
$\mathrm{F}(L) = \mathrm{C}(\mathcal{S}(L))$


$\overline{\mathrm{F}}(L)$
$\mathrm{F}(L) = \mathrm{C}(\mathcal{S}(L))$
$\mathrm{LSC}(L)$
$\overline{\mathrm{LSC}}(L)$


$\overline{\mathrm{F}}(L)$
$\mathrm{F}(L) = \mathrm{C}(\mathcal{S}(L))$
$\overline{\mathrm{USC}}(L)$
$\mathrm{USC}(L)$



$$f \in \overline{\mathrm{USC}}(L) \Updownarrow -f \in \overline{\mathrm{LSC}}(L)$$

$$f \in \mathrm{USC}(L) \Updownarrow -f \in \mathrm{LSC}(L)$$

Background: generating frame homomorphisms by scales

Definition

A collection $\{c_r : r \in \mathbb{Q}\} \subseteq L$ is called an extended scale on L if

$$c_r \vee c_s^* = 1 \text{ whenever } r < s.$$

An extended scale is called a scale if

$$\bigvee\{c_r : r \in \mathbb{Q}\} = 1 = \bigvee\{c_r^* : r \in \mathbb{Q}\}.$$

Remark

An extended scale $\{c_r : r \in \mathbb{Q}\}$ in L is necessarily an antitone family.

Furthermore, if $\mathcal{C}$ consists of complemented elements, then $\mathcal{C}$ is an extended scale if and only if it is antitone.

Background: generating frame homomorphisms by scales

Lemma

Let $\mathcal{C} = \{c_r : r \in \mathbb{Q}\}$ be an extended scale in L and let

$$f(p, \text{—}) = \bigvee_{r>p} c_p \quad \text{and} \quad f(\text{—}, q) = \bigvee_{r<q} c_r^*$$

for all $r \in \mathbb{Q}$. Then the following hold:

(1) *The above two formulas determine an $f \in \overline{\mathrm{C}}(L)$;*

(2) *If $\mathcal{C}$ is a scale, then $f \in \mathrm{C}(L)$.*

Lemma

Let $f, g \in \overline{\mathrm{C}}(L)$ be generated by the extended scales $\{c_r : r \in \mathbb{Q}\}$ and $\{d_r : r \in \mathbb{Q}\}$, respectively. Then:

(1) *$f(r, \text{—}) \leq c_r \leq f(\text{—}, r)^*$ for all $r \in \mathbb{Q}$;*

(2) *$f \leq g$ if and only if $c_r \leq d_s$ whenever $r > s$ in $\mathbb{Q}$.*

Alternative description of algebraic operations

$(\mathrm{C}(L), +, \cdot, \leq)$ is a commutative f-ring with unit $\mathbf{1}$.

Algebraic operations

Let $\langle p, q\rangle = \{r \in \mathbb{Q} \mid p < r < q\}$, let $\diamond \in \{+, \cdot, \wedge, \vee\}$, and let

$$\langle r, s\rangle \diamond \langle t, u\rangle = \big\{x \diamond y \mid x \in \langle r, s\rangle \text{ and } y \in \langle t, u\rangle\big\}.$$

Given $f_1,\ f_2,\ f \in \mathrm{C}(L)$ and $r \in \mathbb{Q}$, we define

$$\begin{aligned}
(f_1 \diamond f_2)(p, q) &= \bigvee\big\{f_1(r, s) \wedge f_2(t, u) \mid \langle r, s\rangle \diamond \langle t, u\rangle \subseteq \langle p, q\rangle\big\},\\
(-f)(p, q) &= f(-q, -p),\\
\mathbf{r}(p, q) &= \begin{cases} 1 & \text{if } r \in \langle p, q\rangle,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Alternative description of algebraic operations

$(\mathrm{C}(L), +, \cdot, \leq)$ is a commutative f-ring with unit $\mathbf{1}$.

1. Constant real functions

For each $r \in \mathbb{Q}$ take $\mathcal{C}_r = \{c_p^r\}_{p \in \mathbb{Q}} \subseteq L$ with $c_p^r = \begin{cases} 0, & \text{if } r \leq p, \\ 1, & \text{if } p < r. \end{cases}$

Alternative description of algebraic operations

$(\mathrm{C}(L), +, \cdot, \leq)$ is a commutative f-ring with unit **1**.

1. Constant real functions

For each $r \in \mathbb{Q}$ take $\mathcal{C}_r = \{c_p^r\}_{p \in \mathbb{Q}} \subseteq L$ with $c_p^r = \begin{cases} 0, & \text{if } r \leq p, \\ 1, & \text{if } p < r. \end{cases}$

$\mathcal{C}_r$ is a scale in L. $\mathbf{r} \in \mathrm{C}(L)$ is the constant real function generated.

$$\mathbf{r}(p, -) = c_p^r = \begin{cases} 0, & \text{if } r \leq p, \\ 1, & \text{if } p < r, \end{cases} \quad \text{and} \quad \mathbf{r}(-, q) = \begin{cases} 1, & \text{if } r < q, \\ 0, & \text{if } q < r. \end{cases}$$

Alternative description of algebraic operations

$(\mathrm{C}(L), +, \cdot, \leq)$ is a commutative f-ring with unit $\mathbf{1}$.

1. Constant real functions

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$\mathcal{C}_r$ is a scale in L. $\mathbf{r} \in \mathrm{C}(L)$ is the constant real function generated.

$$\mathbf{r}(p, —) = c_p^r = \begin{cases} 0, & \text{if } r \leq p, \\ 1, & \text{if } p < r, \end{cases} \quad \text{and} \quad \mathbf{r}(—, q) = \begin{cases} 1, & \text{if } r < q, \\ 0, & \text{if } q < r. \end{cases}$$

2. Opposite real function

For each $f \in \mathrm{C}(L)$ take $\mathcal{C}_r = \{c_p^{-f}\}_{p \in \mathbb{Q}} \subseteq L$ with $c_p^{-f} = f(—, -p)$.

Alternative description of algebraic operations

$(\mathrm{C}(L), +, \cdot, \leq)$ is a commutative f-ring with unit **1**.

1. Constant real functions

For each $r \in \mathbb{Q}$ take $\mathcal{C}_r = \{c_p^r\}_{p \in \mathbb{Q}} \subseteq L$ with $c_p^r = \begin{cases} 0, & \text{if } r \leq p, \\ 1, & \text{if } p < r. \end{cases}$

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For each $f \in \mathrm{C}(L)$ take $\mathcal{C}_r = \{c_p^{-f}\}_{p \in \mathbb{Q}} \subseteq L$ with $c_p^{-f} = f(—, -p)$.

The opposite real function $-f \in \mathrm{C}(L)$ generated is given by.

$$-f(p, —) = c_p^{-f} = f(—, -p) \quad \text{and} \quad -f(—, q) = f(-q, —).$$

Alternative description of algebraic operations

3. Maximum

For each $f, g \in \mathrm{C}(L)$ take $\mathcal{C}_r = \{c_p^{f\vee g}\}_{p\in\mathbb{Q}} \subseteq L$ with

$$c_p^{f\vee g} = f(p,—) \vee g(p,—).$$

Alternative description of algebraic operations

3. Maximum

For each $f, g \in \mathrm{C}(L)$ take $\mathcal{C}_r = \{c_p^{f\vee g}\}_{p\in\mathbb{Q}} \subseteq L$ with

$$c_p^{f\vee g} = f(p,\text{—}) \vee g(p,\text{—}).$$

The maximum real function $f \vee g \in \mathrm{C}(L)$ generated is given by.

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Scales in $\mathcal{S}(L)$

A collection of sublocales $\{S_r : r \in \mathbb{Q}\} \subseteq \mathcal{S}(L)$ is called an extended scale on $\mathcal{S}(L)$ if $S_r \vee S_s^* = 1$ whenever $r < s$.
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Algebraic operations in $\overline{\mathrm{F}}(L)$

1. Constant real functions

Apart from the constant real functions $\mathbf{r} \in \mathrm{F}(L)$, we have in $\overline{\mathrm{F}}(L)$ the constant extended real functions $+\infty$ and $-\infty$ generated by the extended scales

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$$R_f^+ = \bigvee_{r \in \mathbb{Q}} f(-, r), \qquad R_f^- = \bigvee_{r \in \mathbb{Q}} f(r, -) \qquad \text{and} \qquad R_f = R_f^+ \wedge R_f^-.$$

Give, $f, g \in \overline{\mathrm{F}}(L)$, we say that f and g are sum compatible in case

- $(R_f^+ \vee R_g^-) \wedge (R_g^+ \vee R_f^-) = 1.$

Algebraic operations in $\overline{\mathrm{F}}(L)$

Things become more complicated in the other two cases left: the sum and the product.

This is quite natural if we think in the typical indeterminations

$$-\infty + \infty \qquad \text{and} \qquad 0 \cdot \infty.$$

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Algebraic operations in $\overline{\mathrm{F}}(L)$

5. Sum

Let $f, g \in \overline{\mathrm{F}}(L)$ be sum compatible. Take $\mathcal{C}_r = \{S_p^{f+g}\}_{p \in \mathbb{Q}} \subseteq L$ with

$$S_p^{f+g} = \bigvee_{r \in \mathbb{Q}} f(r, \text{—}) \wedge g(p - r, \text{—}).$$

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Algebraic operations in $\overline{\mathrm{F}}(L)$

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The sum real function $f+g \in \overline{\mathrm{F}}(L)$ generated is given by.

$$(f+g)(p,\text{—}) = \bigvee_{r\in\mathbb{Q}} f(r,\text{—}) \wedge g(p-r,\text{—})$$

and

$$(f+g)(\text{—},q) = \bigvee_{s\in\mathbb{Q}} f(\text{—},s) \wedge g(\text{—},q-s).$$

Algebraic operations in $\overline{\mathrm{F}}(L)$

Proposition

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Algebraic operations in $\overline{\mathrm{F}}(L)$

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Algebraic operations in $\overline{\mathrm{F}}(L)$

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Algebraic operations in $\overline{\mathrm{F}}(L)$

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(4) If $f, g \in \overline{\mathrm{C}}(L)$ then $f + g \in \overline{\mathrm{C}}(L)$.

Algebraic operations in $\overline{\mathrm{F}}(L)$

6. Product

Let $f \in \overline{\mathrm{F}}(L)$. We shall denote

$$\mathrm{coz}\,(f) = f((—, 0) \vee (0, —)).$$

Algebraic operations in $\overline{\mathrm{F}}(L)$

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