On lattice-valued frames: the completely distributive case

Javier Gutiérrez García

(*joint work with Ulrich Höhle and M. Angeles de Prada Vicente*)

Chengdu, April 6, 2010

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Dedicated to Professor Liu on his 70^{th} birthday


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Origin of the work

- 29th Linz Seminar on Fuzzy Set Theory:

  "*Foundations of Lattice-Valued Mathematics with Applications to Algebra and Topology*"

  Linz, (Austria), February 4 to 9, (2008)

  Attendants:
  Rodabaugh, Zhang, Höhle, Kubiak, Šostak, de Prada Vicente,...

A. Pultr, S.E. Rodabaugh,
Category theoretic aspects of chain-valued frames:
Part I: Categorical and presheaf theoretic foundations,
Part II: Applications to lattice-valued topology,
Fuzzy Sets and Systems 159 (2008) 501–528 and 529–558.

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- Visit of Ulrich Höhle to Bilbao, April 4 to 11, (2008)

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Introduction

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- L-valued (Fuzzy) topology

Pointfree topology

$$(X, \mathcal{O}X) \rightsquigarrow (\mathcal{O}X, \subseteq)$$

Motivation

$$(X, \mathcal{O}X) \rightsquigarrow (\mathcal{O}X, \subseteq) \qquad A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i)$$

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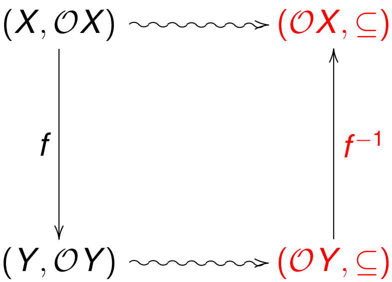

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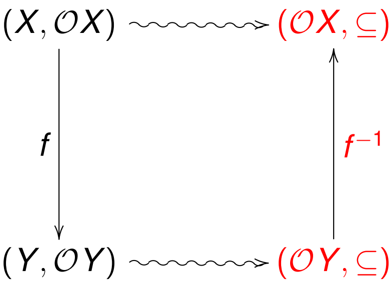
f^{-1} preserves $\bigcup$ and $\cap$



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- $\mathcal{O}$: **Top** $\to$ **Frm** is a contravariant functor with $X \longmapsto \mathcal{O}X$ and $X \xrightarrow{f} Y \longmapsto \mathcal{O}Y \xrightarrow{f^{-1}} \mathcal{O}X$.

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Advantage: **Loc** can be thought of as a natural extension of (sober) spaces.

Disadvantage: Morphisms thought in this way may obscure the intuition.

Pointfree Topology, Pointless Topology, Frame Theory, Locale Theory. . .

Pointfree Topology, Pointless Topology, Frame Theory, Locale Theory...

"*Locales not only "capture" or "model" the lattice theoretical behaviour of topological spaces, more importantly when we work in a universe where choice principles are not allowed, it is locales, not spaces, which provide the right context in which to do topology.*"

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"*As an illustration, we look at the topological and the localic versions of Stone's representation theorem for distributive lattices.*

. . .

Hence, in absence of Prime Ideal Theorem, we can prove Stone's representation theorem in the context of locales, but no longer in the context of spaces. "

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D. Zhang, Y. Liu, *L-fuzzy version of Stone's representation theorem for distributive lattices*, Fuzzy Sets and Systems 76 (1995) 259-270.

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An element $p \in L \setminus \{1\}$ is called prime if for each $\alpha, \beta \in L$ with

$$\alpha \wedge \beta \leq p \quad \Longrightarrow \quad \alpha \leq p \text{ or } \beta \leq p.$$

We denote by $\mathrm{Spec}\, L$ the spectrum of L, i.e. the set of all prime elements of L.

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The functor Spec assigns to each frame L its spectrum $\mathrm{Spec}\, L$, endowed with the hull-kernel topology whose open sets are

$$\Delta_L(\alpha) = \{p \in \mathrm{Spec}\, L : \alpha \not\leq p\} = \mathrm{Spec}\, L \setminus {\uparrow}\alpha \quad \text{for } \alpha \in L.$$

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Recall that a topological space X is sober if the only prime opens are those of the form $X \setminus \overline{\{x\}}$ for some $x \in X$ and a frame L is spatial if L is generated by its prime elements, i.e. if

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Pointfree topology — spatial frames and sober spaces

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The categories **Sob** of sober topological spaces and **SpatFrm** of spatial frames are dual under the restrictions of the functors $\mathcal{O}$ and Spec.

$$\textbf{Sob} \sim \textbf{SpatFrm}$$

L-valued topology

With L a complete lattice and X a set, L^X is the complete lattice of all maps from X to L, called L-sets, in which

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An L-valued topological space (shortly, an L-topological space) is a pair (X, τ) consisting of a set X and a subset τ of L^X (the L-valued topology or L-topology on the set X) closed under finite meets and arbitrary joins.

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Given two L-topological spaces $(X, \tau), (Y, \sigma)$ a map $f : X \to Y$ is an L-continuous map if the correspondence $f^{-1}(b)$ maps σ into τ. The resulting category will be denoted by **L-Top**.

“*It is a natural and interesting question whether or not it is possible to establish a category to play the same role with respect to a given notion of fuzzy topology as that locales play for topological spaces.*”

D. Zhang, Y. Liu, *L-fuzzy version of Stone’s representation theorem for distributive lattices*, Fuzzy Sets and Systems 76 (1995) 259-270.

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Hence we can construct the functor $L\mathcal{O} : L\text{-}\mathbf{Top} \to \mathbf{Frm}$

$$L\mathcal{O}(X, \tau) = \tau$$
$$L\mathcal{O}\big(f : (X, \tau) \to (Y, \sigma)\big) = f^{-1} : \sigma \to \tau.$$

Rodabaugh's approach

$L\mathcal{O}$ has a right adjoint called the *L*-spectrum functor $L\mathrm{Spec}$.

S.E. Rodabaugh, *Point set lattice theoretic topology*, Fuzzy Sets and Systems 40 (1991) 296–345.

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This is defined by introducing the notion of *L*-point of a frame *A*, which are defined to be frame homomorphisms from *A* to *L*.

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$L\mathcal{O}$ has a right adjoint called the L-spectrum functor $L\mathrm{Spec}$.

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$$L\text{-}\mathbf{Top} \underset{L\mathrm{Spec}}{\overset{\mathcal{O}L}{\rightleftarrows}} \mathbf{Frm}$$

with right unit Ψ_L and left unit Φ_L.

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Let us denote by L-**SpatFrm** the full subcategory of L-spatial frames (frames for which Φ_L is injective) and L-**Sob** the full subcategory of L-sober spaces (L-topological spaces for which Ψ_L is bijective).

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The categories **Sob** and **L-SpatFrm** are dual under the restrictions of the functors $L\mathcal{O}$ and $L\mathrm{Spec}$.

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The approach is not completely satisfactory...
For example the fuzzy real line fails to be L-sober.

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Liu and Zhang's approach

Let L be a frame and (X, τ) a stratified L-topological space.

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Let L be a frame and (X, τ) a stratified L-topological space.

Then L can be embedded in a natural way into τ:

$$i_X : L \to \tau, \qquad i_X(\alpha) = \underline{\alpha} \quad \text{for each } \alpha \in L.$$

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If $f : (X,\tau) \to (Y,\sigma)$ is an L-continuous map between two stratified L-topological spaces, then $f^{-1} : \sigma \to \tau$ is a frame homomorphism such that

$$i_Y = f^{-1} \circ i_X.$$

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An *L-fuzzy locale* is defined to be a pair (A, i_A), where A is a frame and $i_A : L \to A$ is a frame homomorphism.

A continuous map between two L-fuzzy locales $f : (A, i_A) \to (B, i_B)$ is a frame homomorphism $f : B \to A$ such that

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The category consisting of L-fuzzy locales as objects and continuous maps as morphism is called the category of L-fuzzy locales, which is easily checked to be the comma category $\mathbf{Loc} \downarrow L$.

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Chain-valued frames

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- Motivation: the iota functor ι_L

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Let L be a complete lattice and X be a set. For a fixed $\alpha \in L \setminus \{1\}$ and let $a \in L^X$, we denote

$$[a \not\leq \alpha] = \{x \in X : a(x) \not\leq \alpha\}.$$

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Now, given an L-topology τ on X, we consider the topology

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This defines a functor ι_L : L-Top $\to$ Top by

$$\iota_L(X,\tau) = (X, \iota_L(\tau)), \qquad \iota_L(h) = h.$$

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We can consider the system of frame homomorphisms

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The above discussion means that A is, up to frame isomorphism, an L-topology on X.

Chain valued frames

"*The notion of chain-valued frame, is introduced to be an abstraction of the distinctive properties of the system of level mappings from an L-topology* τ *into* $\iota_L(\tau)$. *These conditions, when L is a complete chain, were taken as axioms* (F0), (F1) *and* (F2) *in order to define L-frames and the associated category L*-Frm."

- A. Pultr, S.E. Rodabaugh, *Lattice-valued frames, functor categories, and classes of sober spaces*, in: Topological and Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets,
  Kluwer Academic Publishers, 2003, pp. 153–187, (Chapter 6).
- A. Pultr, S.E. Rodabaugh, *Category theoretic aspects of chain-valued frames: Part I: Categorical and presheaf theoretic foundations, Part II: Applications to lattice-valued topology*, Fuzzy Sets and Systems 159 (2008) 501–528 and 529–558.

Chain valued frames

Let L be a complete chain. An L-frame A is a system

$$\left(\varphi_\alpha^A : A^{\mathfrak{u}} \to A^{\mathfrak{l}}\ \alpha \in L \setminus \{1\}\right)$$

of frame morphisms – $A^{\mathfrak{u}}$ is the upper frame and $A^{\mathfrak{l}}$ is the lower frame – satisfying each of these conditions:

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The resulting category, with composition and identities component-wise in Frm, is denoted by *L*-Frm.

Chain valued frames

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"*During the preparation of the Volume, U. Höhle communicated to the authors of Chapter 6 that a complete chain is really only needed for its meet-irreducibles, and that for spatial L one also has meet-irreducibles which suffice for the constructions of Chapter 6.*"

U. Höhle and S.E. Rodabaugh, *Weakening the requirement that L be a complete chain*, in: Topological and Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Kluwer Academic Publishers, 2003, pp. 189–197, (Chapter 7).

The completely distributive case

The completely distributive case

- Completely distributive lattices

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- Lattice-valued frames for CD lattices

Completely distributive lattices

Completely distributive lattices

Given $\alpha, \beta \in L$, we say that α is way below β, in symbols $\alpha \ll \beta$, if and only if

$$\left.\begin{array}{l} S \subseteq L \text{ and} \\ \beta \leq \bigvee S \end{array}\right\} \Longrightarrow \text{ there exist } \gamma_1, \ldots \gamma_n \in S \text{ such that } \alpha \leq \vee_{i=1}^{n} \gamma_i.$$

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Recall that L is continuous if and only if the way-below relation is approximating, i.e., if and only if

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Let $\alpha, \beta, \gamma, \delta \in L$, then:

(1) $\alpha \ll \beta$ implies $\alpha \leq \beta$.

(2) $\alpha \leq \beta \ll \gamma \leq \delta$ implies $\alpha \ll \delta$.

(3) If L is continuous, then $\alpha \ll \beta$ implies $\alpha \ll \gamma \ll \beta$ for some $\gamma \in L$

Completely distributive lattices

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Then we have that L^{op} is alertcontinuous if and only if it satisfies

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The following properties of the binary relation $\lll$ will be needed:

(1) $\alpha \lll \beta$ implies $\alpha \leq \beta$.

(2) $\alpha \leq \beta \lll \gamma \leq \delta$ implies $\alpha \lll \delta$.

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A lattice is called completely distributive iff it is complete and for any familiy $\{x_{j,k} : j \in J, k \in K(j)\}$ in L the identity

$$\bigwedge_{j \in J} \bigvee_{k \in K(j)} x_{j,k} = \bigvee_{f \in M} \bigwedge_{j \in J} x_{j,f(j)} \tag{CD}$$

holds, where M is the set of choice functions defined on J with values $f(j) \in K(j)$.

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We recall now the following result:

Let L be a complete lattice. Then the following are equivalent:

(1) L is completely distributive.

(2) L is a spatial frame and L^{op} is continuous.

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Lattice-valued frames for CD lattices

The ι_L functor — completely distributive lattices

Let L be a completely distributive lattice. The mapping $\iota_p : \tau \to \iota_L(\tau)$ is a frame morphism for each $p \in \mathrm{Spec}\, L$ (this is not true in general if p fails to be prime). Consider the system of frame morphisms

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(F0)' Since $p = \bigwedge \left(\twoheaduparrow p \cap \operatorname{Spec} L\right)$ for each $p \in \operatorname{Spec} L$, we have that

$$[f \not\leq p] = \bigcup_{q \in \twoheaduparrow p \cap \operatorname{Spec} L} [f \not\leq q] \qquad \text{for each } p \in \operatorname{Spec} L \text{ and } f \in L^X.$$

Consequently, for each $p \in \operatorname{Spec} L$,

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(F1) Since L is a spatial frame then $\{\iota_p(f) : f \in \tau, p \in \mathrm{Spec}\, L\}$ is a subbase of $\iota_L(\tau)$. Indeed, for each $\alpha \in L$ we have $\alpha = \bigwedge (\uparrow\alpha \cap \mathrm{Spec}\, L)$ and so

$$\iota_\alpha(f) = [f \not\leq \alpha] = \bigcup_{p \in \uparrow\alpha \cap \mathrm{Spec}\, L} [f \not\leq p] = \bigcup_{p \in \uparrow\alpha \cap \mathrm{Spec}\, L} \iota_p(f).$$

Hence,

$$\iota_L(\tau) = \Big\langle \bigcup_{p \in \mathrm{Spec}\, L} \iota_p(\tau) \Big\rangle.$$

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(F2) Since L is a spatial frame, for each distinct $a, b \in \tau$ there exists $x \in X$ such that $a(x) \neq b(x)$, hence there exists $p \in \mathrm{Spec}\, L$ such that either $f(x) \leq p$ and $g(x) \not\leq p$ or $a(x) \not\leq p$ and $b(x) \leq p$ and so $[a \not\leq p] \neq [b \not\leq p]$. It follows that

$$\text{if } a \neq b \in \tau \text{ then } \iota_p(a) \neq \iota_p(b) \text{ for some } p \in \mathrm{Spec}\, L.$$

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In a complete chain L any non-empty subset is downdirected and $\operatorname{Spec} L = L \setminus \{1\}$ and hence $\left(\varphi_p^A : A^{\mathfrak{u}} \to A^{\mathfrak{l}} \mid p \in \operatorname{Spec} L\right)$ is a system of frame morphisms, then axiom (F0) can be equivalently stated as:

(F0) $\varphi^A_{\bigwedge S} = \bigvee_{s \in S} \varphi^A_s$ for each downdirected $\varnothing \neq S \subseteq \operatorname{Spec} L$.

A subset S of a complete lattice is downdirected if it is non-empty and for any $\alpha, \beta \in S$ there is some $\gamma \in S$ such that $\gamma \leq \alpha \wedge \beta$.

In a complete chain L any non-empty subset is downdirected and $\mathrm{Spec}\, L = L \setminus \{1\}$ and hence $\left(\varphi_p^A : A^{\mathfrak{u}} \to A^{\mathfrak{l}} \mid p \in \mathrm{Spec}\, L\right)$ is a system of frame morphisms, then axiom (F0) can be equivalently stated as:

(F0) $\varphi^A_{\bigwedge S} = \bigvee_{s \in S} \varphi^A_s$ for each downdirected $\varnothing \neq S \subseteq \mathrm{Spec}\, L$.

For a completely distributive lattice L the following are equivalent

(F0) $\varphi^A_{\bigwedge S} = \bigvee_{s \in S} \varphi^A_s$ for each downdirected $\varnothing \neq S \subseteq \mathrm{Spec}\, L$.

(F0)' $\varphi^A_p = \bigvee_{q \in \twoheaduparrow p \cap \mathrm{Spec}\, L} \varphi^A_q$ for each $p \in \mathrm{Spec}\, L$.

L-valued frames

Let L be a completely distributive lattice. An L-frame A is a system

$$\left(\varphi_p^A : A^{\mathfrak{u}} \to A^{\mathfrak{l}}\ p \in \operatorname{Spec} L\right)$$

of frame morphisms – $A^{\mathfrak{u}}$ is the upper frame and $A^{\mathfrak{l}}$ is the lower frame – satisfying each of these conditions:

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(F1) $A^{\mathfrak{l}} = \langle \bigcup_{p \in \operatorname{Spec} L} \varphi_p^A(A^{\mathfrak{u}}) \rangle$. (collectionwise extremally epimorphic)

(F2) If $a \neq b$ in $A^{\mathfrak{u}}$ then $\varphi_p^A(a) \neq \varphi_p^A(b)$ for some $p \in \operatorname{Spec} L$. (collectionwise monomorphic)

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An L-frame morphism $h : A \to B$ is an ordered pair of frame homomorphisms

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The resulting category, with composition and identities component-wise in Frm, is denoted by L-Frm.

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In particular if we denote by $\mathfrak{F}_0$ and $\mathfrak{F}_1$ the categories in which the objects are frame morphisms for which only (F0) (resp. (F0) and (F1)) is (resp. are) satisfied.Then

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- $\mathfrak{F}_1$ is complete and cocomplete.
- L-Frm is complete and cocomplete.

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We have already specified an answer by proving that the condition of a complete chain can be relaxed to a completely distributive lattice and that the completeness and cocompleteness of $\mathfrak{F}_0$, $\mathfrak{F}_1$ and L-Frm are still satisfied.

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In this context the natural question arises whether weakening of complete distributivity is still possible. As an answer to this question we show that complete distributivity is necessary for the property that for every L-topological space (X, τ) the system

$$\left(\iota_p : \tau \to \iota_L(\tau) \mid p \in \mathrm{Spec}\, L\right)$$

of frame homomorphisms ι_p satisfies (F0) and (F2).

Let L be a frame, (X, τ) an L-topological space and

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the system of frame morphisms determined by the ι-functor. Then:

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- If $(\iota_p)_{p \in \operatorname{Spec} L}$ satisfies axiom (F0) for each (X, τ), then

$$p = \bigwedge \big(\twoheaduparrow p \cap \operatorname{Spec} L\big) \quad \text{for each } p \in \operatorname{Spec} L.$$

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- If $(\iota_p)_{p \in \mathrm{Spec}\, L}$ satisfies axiom (F2) for each (X, τ), then L is spatial.
- If $(\iota_p)_{p \in \mathrm{Spec}\, L}$ is an L-frame for each (X, τ), then L is a completely distributive lattice.

Eskerrik asko!

¡Muchas gracias!

谢谢你