Localic analogues of general insertion and extension theorems for real-valued functions

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Motivation

Topological Extension Theorem (Mrówka).

Let X be a topological space, $S \subseteq X$ and $f : S \to \mathbb{R}$ be a bounded continuous function. TFAE:

(1) f has a continuous extension to the whole of X.

(2) If $r > s \in \mathbb{Q}$, then $[f \geq r]$ and $[f \leq s]$ are completely separated in X.

(A and B are said to be completely separated in X if there is a continuous $f : X \to [0, 1]$ such that $f = 0$ on A and $f = 1$ on B).

S. Mrówka
On some approximation theorems
Nieuw Archief voor Wiskunde, (3) 16 (1968) 94–111.

Motivation

Topological Insertion Theorem (Blair-Lane).

Let X be a topological space and let $f, g : X \to \mathbb{R}$. TFAE:

(1) There exists a continuous $h : X \to \mathbb{R}$ such that $f \leq h \leq g$.

(2) If $r > s \in \mathbb{Q}$, then $[f \geq r]$ and $[g \leq s]$ are completely separated.

R.L. Blair
Extensions of Lebesgue sets and of real valued functions
Czechoslovak Math. J., 31 (1981) 63–74.

E.P. Lane
Insertion of a continuous function
Topology Proc., 4 (1979) 463–478.

Background: the sublocale lattice $\mathcal{S}(L)$

Frm locale L, subobject lattice: is a CO-FRAME

$\mathcal{S}L$ = the dual FRAME

for each $a \in L$

$\mathfrak{c}(a)$: closed
$\mathfrak{o}(a)$: open

} *complemented*

$$\bigvee_{i \in I} \mathfrak{c}(a_i) = \mathfrak{c}\Big(\bigvee_{i \in I} a_i\Big)$$

$$\mathfrak{c}(a) \wedge \mathfrak{c}(b) = \mathfrak{c}(a \wedge b)$$

subframe $\mathfrak{c}L := \{\mathfrak{c}(a) : a \in L\} \simeq L$

$$\bigwedge_{i \in I} \mathfrak{o}(a_i) = \mathfrak{o}\Big(\bigvee_{i \in I} a_i\Big)$$

$$\mathfrak{o}(a) \vee \mathfrak{o}(b) = \mathfrak{o}(a \wedge b)$$

subframe $\mathfrak{o}L := \langle\{\mathfrak{o}(a) : a \in L\}\rangle$

(the geometric motivation reads backwards)

Background: closure and interior of a sublocale

Let L be a frame and $S \subset L$ a sublocale.

The closure of S:

$$\overline{S} = \bigvee\{\mathfrak{c}(a) : \mathfrak{c}(a) \leq S\} = \mathfrak{c}(\bigwedge S) = \uparrow \bigwedge S$$

$$\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$$

The interior of S:

$$S^\circ = \bigwedge\{\mathfrak{o}(a) : S \leq \mathfrak{o}(a)\}.$$

$$\mathfrak{c}(a)^\circ = \mathfrak{o}(a^*)$$

Background: the frame of reals $\mathfrak{L}(\mathbb{R})$

$\mathfrak{L}(\mathbb{R}) = \mathrm{FRM}\langle\ (p, q)\ p, q \in \mathbb{Q} \mid$

$$\begin{aligned}
&\text{(R1)} \quad (p, q) \wedge (r, s) = (p \vee r, q \wedge s) \\
&\text{(R2)} \quad p \le r < q \le s \Rightarrow (p, q) \vee (r, s) = (p, s) \\
&\text{(R3)} \quad (p, q) = \bigvee \{(r, s) \mid p < r < s < q\} \\
&\text{(R4)} \quad \bigvee \{(p, q) \,|\, p, q \in \mathbb{Q}\} = 1 \rangle.
\end{aligned}$$

$$\boxed{(-, q) := \bigvee_{p \in \mathbb{Q}} (p, q)} \qquad \boxed{(p, -) := \bigvee_{q \in \mathbb{Q}} (p, q)}$$

$$\mathfrak{L}_l(\mathbb{R}) = \langle (-, q) \mid q \in \mathbb{Q} \rangle \qquad \mathfrak{L}_u(\mathbb{R}) = \langle (p, -) \mid p \in \mathbb{Q} \rangle$$

Background: localic real-valued functions

- $f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}L$ **general** $\mathrm{F}(L)$
- $f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}L$ **usc** $\mathrm{USC}(L)$
  s. t. $f(\mathfrak{L}_l(\mathbb{R})) \subseteq \mathfrak{c}L$
- $f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}L$ **lsc** $\mathrm{LSC}(L)$
  s. t. $f(\mathfrak{L}_u(\mathbb{R})) \subseteq \mathfrak{c}L$
- $f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}L$ **continuous** $\mathrm{C}(L)$
  s. t. $f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}L$

$$f \le g \equiv f(p,-) \le g(p,-) \quad \forall p \in \mathbb{Q} \iff f(-,q) \ge g(-,q) \quad \forall q \in \mathbb{Q}$$

Scales on $\mathcal{S}L$

A collection of sublocales $\mathcal{C} = \{S_r : r \in \mathbb{Q}\} \subseteq \mathcal{S}L$ is a scale on $\mathcal{S}L$ if

- $S_r \vee S_s^* = 1$ whenever $r < s$.
- $\bigvee \mathcal{C} = 1 = \bigvee \mathcal{C}^*$.

If $\mathcal{C} = \{S_r : r \in \mathbb{Q}\} \subseteq \mathcal{S}L$ is a scale on $\mathcal{S}L$ then there exists a unique $f \in \mathrm{F}(L)$ such that for all $r \in \mathbb{Q}$

(i) $f(r, -) = \bigvee_{s>r} S_s, \quad f(-, r) = \bigvee_{s<r} S_s^*$ and

(ii) $f(r, -) \leq S_r \leq f(-, r)^*$.

f is the localic real valued function generated by $\mathcal{C}$.

Given $f \in \mathrm{F}(L)$, both $\{f(r, -) : r \in \mathbb{Q}\}$ and $\{f(-, r)^* : r \in \mathbb{Q}\}$ are scales that generate f.

Scales on $\mathcal{S}L$ (continued)

Proposition

Let $f, g \in \mathrm{F}(L)$ be generated be the scales $\mathcal{C} = \{S_r : r \in Q\}$ and $\mathcal{D} = \{T_r : r \in Q\}$, respectively. Then:

$$f \leq g \quad \text{if and only if} \quad S_r \leq T_s \quad \text{whenever } r > s.$$

Proposition

Let $f \in \mathrm{F}(L)$ be generated by the scale $\mathcal{C} = \{S_r : r \in Q\}$. Then:

(1) $f \in \mathrm{USC}(L)$ if and only if $S_r \leq \overline{S_s}$ whenever $r > s$;

(2) $f \in \mathrm{USC}(L)$ if and only if $S_r{}^\circ \leq S_s$ whenever $r > s$;

(3) $f \in \overline{\mathrm{C}}(L)$ if and only if $S_r{}^\circ \leq \overline{S - s}$ whenever $r > s$.

Background: Katětov relation

Let $(M, \leq)$ be a complete lattice. A binary relation ϱ on M is a Katětov relation if and only if for all $x, y, z, x_1, x_2, y_1, y_2 \in M$ the following hold:

(P1) $x \,\varrho\, y \Rightarrow x \leq y$.

(P2) $x_2 \leq x_1 \,\varrho\, y_1 \leq y_2 \Rightarrow x_2 \,\varrho\, y_2$.

(P3) $x_1 \,\varrho\, y$ and $x_2 \,\varrho\, y \Rightarrow (x_1 \vee x_2) \,\varrho\, y$.

(P4) $x \,\varrho\, y_1$ and $x \,\varrho\, y_2 \Rightarrow x \,\varrho\, (y_1 \wedge y_2)$.

(P5) $x \,\varrho\, y \Rightarrow x \,\varrho\, z \,\varrho\, y$ for some $z \in M$. (Interpolation Property)

(Such a relation has various names in the literature: quasi-proximity relation, subordination. . .)

M. Katětov
On real-valued functions in topological spaces
Fund. Math., 38 (1951) 85–91; Correction, Fund Math. 40 (1953) 139–142.

Katětov lemmas

Lemma (Katětov)

Let ϱ be a Katětov relation on M and $A, B \subset M$ countable such that

$$\left(\bigvee A\right) \varrho\, b \quad \text{and} \quad a\, \varrho \left(\bigwedge B\right) \quad \text{for all } a \in A, b \in B,$$

then there is a $c \in M$ such that $a \varrho c \varrho b$ for all $a \in A$ and $b \in B$.

Lemma (Katětov)

Let ϱ be an Katětov relation on M and $\{a_r\}_{r\in\mathbb{Q}}, \{b_r\}_{r\in\mathbb{Q}} \subset M$ such that

$$r > s \quad \Longrightarrow \quad a_r \leq a_s,\ b_r \leq b_s \ \text{and}\ a_r \varrho b_s.$$

Then there is $\{c_r\}_{r\in\mathbb{Q}} \subseteq K$ such that

$$r > p > q > s \quad \Longrightarrow \quad a_r \varrho c_p \varrho c_q \varrho b_s.$$

Katětov relations on $\mathcal{S}L$

We are particularly interested in considering Katětov relations on the frame $\mathcal{S}L$.

Given a frame L, a Katětov relation ϱ in $\mathcal{S}L$ is said to be strong, if

$$S \,\varrho\, T \quad \Longrightarrow \quad S^\circ \leq T \text{ and } S \leq \overline{T}.$$

Examples

Given $S, T \in \mathcal{S}L$ we write

(1) $S \prec T \iff S^\circ \leq f(-,1)^* \leq f(0,-) \leq \overline{T}$ for some $f \in \mathrm{C}(L)$.

$\prec$ is a strong Katětov relation.

(2) $S \Subset T \iff S^\circ \leq \overline{T}$.

$\Subset$ is a strong Katětov relation iff L is normal.

The insertion result

Theorem

Let L be a frame. Let $f, g \in \mathrm{F}(L)$ be two localic real functions on L. If there exists a strong Katětov relation ϱ on $\mathcal{S}L$ such that

$$f(r,—)\,\varrho\,g(s,—) \text{ whenever } r > s,$$

then there exists an $h \in \mathrm{C}(L)$ such that $f \leq h \leq g$.

Proof:

(1) Apply Katětov Lemma with $a_r = f(r,—)$ and $b_r = g(r,—)$ to obtain a countable family $\{S_r\}_{r\in\mathbb{Q}} \subset \mathcal{S}L$ such that

$$r > p > q > s \quad \Longrightarrow \quad f(r,—)\,\varrho\,S_p\,\varrho\,S_q\,\varrho\,g(s,—).$$

(2) $\mathcal{C} = \{S_r : r \in Q\}$ is a scale on $\mathcal{S}L$ and the real-valued function h generated by $\mathcal{C}$ satisfies

$$f \leq h \leq g \quad \text{and} \quad h \in \mathrm{C}(L). \quad \square$$

Katětov-Tong Theorem

Let $S, T \in \mathcal{S}L$ we write

$$S \Subset T \iff S^\circ \leq \overline{T}.$$

Theorem (Localic Katětov-Tong)

Let L be a frame. Then the following are equivalent:

(1) *L is normal.*

(2) $\Subset$ *is a strong Katětov relation*

(3) *If $f \in \mathrm{USC}(L)$, $G \in \mathrm{LSC}(L)$, and $f \leq g$, then there exists $h \in \mathrm{C}(L)$ such that $f \leq h \leq g$.*

Localic Insertion Theorem

Let $S, T \in \mathcal{S}L$ we write

$$S \prec T \iff S^\circ \leq f(-,1)^* \leq f(0,-) \leq \overline{T} \quad \text{for some } f \in \mathrm{C}(L).$$

Definition

Two sublocales S and T in L are said to be completely separated if

$$f(s,-) \leq S \quad \text{and} \quad f(-,t) \leq T \quad \text{for some } f \in \mathrm{C}(L).$$

Localic Insertion Theorem (Blair-Lane).

Let L be a frame and let $f, g \in \mathrm{F}(L)$. TFAE:

(1) There exists $h \in \mathrm{C}(L)$ such that $f \leq h \leq g$.

(2) If $r > s$, then $f(r,-) \prec g(s,-)$.

(3) If $r > s$, then $f(-,r)$ and $g(s,-)$ are completely separated.

(4) If $r > s$, then $f(r,-)^*$ and $g(-,s)^*$ are completely separated.

Extension results: Localic Tietze

Given a frame L, we shall denote

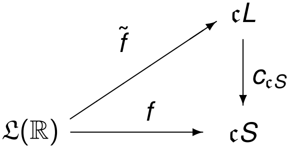
$$\mathrm{F}^*(L) = \{f \in \mathrm{F}(L) : \mathbf{0} \leq f \leq \mathbf{1}\} = \{f \in \mathrm{F}(L) : f((-,0) \vee (1,-)) = 0\}$$

and

$$\mathrm{C}^*(L) = \{f \in \mathrm{C}(L) : \mathbf{0} \leq f \leq \mathbf{1}\} = \{f \in \mathrm{C}(L) : f((-,0) \vee (1,-)) = 0\}$$

Theorem (Localic Tietze)

Let L be a normal frame, S a closed sublocale in L and $f \in \mathrm{C}^*(S)$. Then there exists an extension of f to the whole L, i.e. there exists $\widetilde{f} \in \mathrm{C}^*(L)$ such that $c_{\mathfrak{c}S} \circ \widetilde{f} = f$.

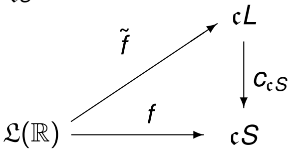


Extension results: Localic Extension Theorem

Localic Extension Theorem (Mrówka)

Let L be a frame, S a complemented sublocale in L and $f \in \mathrm{C}^*(S)$. Then the following are equivalent:

(1) There exists an extension of f to the whole L, i.e. there exists $\widetilde{f} \in \mathrm{C}^*(L)$ such that $c_{\mathfrak{c}S} \circ \widetilde{f} = f$.



(2) If $r > s$, then $f(r, —)$ and $f(—, s)$ are completely separated in L.

Extension results: Localic Extension Theorem (proof)

Proof.

(1) $\Longrightarrow$ (2) is the easy part.

(2) $\Longrightarrow$ (1): Let f_1 and g_2 be generated, respectively, by the scales $\mathcal{C} = \{S_p : p \in \mathbb{Q}\}$ and $\mathcal{D} = \{T_q : q \in \mathbb{Q}\}$ where

$$S_p = \begin{cases} 0(= L), & \text{if } p \geq 1; \\ f(p, \text{—}), & \text{if } 0 \leq p < 1; \\ 1(= \{1\}), & \text{if } p < 0 \end{cases} \quad T_q = \begin{cases} 0(= L), & \text{if } q \geq 0; \\ f(\text{—}, -q), & \text{if } -1 \leq q < 0; \\ 1(= \{1\}), & \text{if } q < -1. \end{cases}$$

Then f_1 and $f_2 = -g_2$ belong to $\mathrm{C}^*(L)$ and if $r > s$ then $f_2(r, \text{—})$ and $f_1(\text{—}, s)$ are completely separated in L.

It follows from the Localic Insertion Theorem that there exists $h \in \mathrm{C}(L)$ such that $f_2 \leq h \leq f_1$.

The real valued function $h \in \mathrm{C}^*(L)$ is the desired extension of f. □