Separating families of localic maps and localic embeddings

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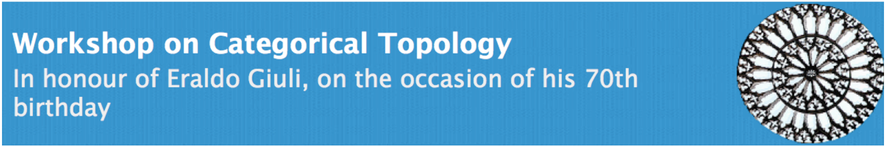
(*joint work with joint work with Luis Español and Tomasz Kubiak*)



Separating points from closed sets in Top

In **Top** – the category of all topological spaces – any embedding separates points from closed sets, i.e.

$$f : X \to Y \text{ embedding} \quad \Longrightarrow \quad \begin{array}{c}\text{for each closed } K \subseteq X \text{ and } x \in X \setminus K \\ f(x) \notin \overline{f(K)}\end{array}$$

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The converse implication holds if X is a T_0-space.

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The converse implication holds if X is a T_0-space.

Fact

If X is T_0-space and f separates points from closed sets, then f is a topological embedding.

Separating points from closed sets in Top

Now let:

$$f : X \to Y_f \quad \text{for all} \quad f \in F.$$

Definition. The family F separates points from closed sets if for each closed $K \subseteq X$ and $x \in X \setminus K$, we can find an $f \in F$ with $f(x) \notin \overline{f(K)}$.

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The Embedding Theorem.

If X is T_0-space and F separates points from closed sets, then

$$e : X \to \prod_{f \in F} Y_f$$

is a topological embedding where e is determined by $\pi_f e = f$.

R. Engelking, *General Topology*, Polish Sci. Publ., Warszawa, 1977.

Separating points from closed sets from spaces to locales

Avoiding points

A family F separates points from closed sets iff

$$K = \bigcap_{f \in F} f^{-1}\left(\overline{f(K)}\right) \quad \text{for all closed} \quad K \subseteq X.$$

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But $Y_f \setminus \overline{f\left(X \setminus U\right)} = \bigcup\{V \in \mathcal{O}Y_f : f^{-1}(V) \subseteq U\} = \left(f^{-1}\right)_*(U)$, where $\left(f^{-1}\right)_* : \mathcal{O}X \to \mathcal{O}Y_f$ is the right adjoint of the inverse image map $f^{-1} : \mathcal{O}Y_f \to \mathcal{O}X$.

Separating points from closed sets from spaces to locales

Reformulation

Let X, Y_f, and $f : X \to Y_f$ be in **Top** for all $f \in F$. The following are equivalent:

(1) (`Separating points from closed sets`) For each closed $K \subseteq X$ and $x \in X \setminus K$, there exists an $f \in F$ with $f(x) \notin \overline{f(K)}$.

(2) (`Localic formulation`)

$$U = \bigcup_{f \in F} f^{-1}\left(f^{-1}\right)_*(U) \quad \text{for all open} \quad U \subseteq X. \qquad (\mathrm{S}^{\mathbf{Top}})$$

Pointfree topology

the category of frames **Frm**

- The objects in **Frm** are *frames*, i.e.
  - ∗ complete lattices L in which
  - ∗ $a \wedge \bigvee_{i \in I} a_i = \bigvee \{a \wedge a_i : i \in I\}$ for all $a \in L$ and $\{a_i : i \in I\} \subseteq L$.

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- Morphisms, called *frame homomorphisms*, are those maps between frames h that preserve
  - $*$ arbitrary joins, $h(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} h(a_i)$, $h(0) = 0$,
  - $*$ finite meets, $h(a_1 \wedge a_2) = h(a_1) \wedge h(a_2)$, $h(1) = 1$.

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  - ∗ finite meets, $h(a_1 \wedge a_2) = h(a_1) \wedge h(a_2)$, $h(1) = 1$.
- Motivating example:
  - If X is a topological space, then its topology $\mathcal{O}X$ is a frame.
  - If $f : X \to Y$ in **Top**, then $f^{-1} : \mathcal{O}Y \to \mathcal{O}X$ in **Frm**.
  - $\mathcal{O} :$ **Top** $\to$ **Frm** is a contravariant functor with $X \longmapsto \mathcal{O}X$ and $X \xrightarrow{f} Y \longmapsto \mathcal{O}Y \xrightarrow{f^{-1}} \mathcal{O}X$.

Pointfree topology the dual category $\mathbf{Frm}^{op}$

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- Morphisms, called *localic maps*, are of course, just frame homomorphisms taken backwards.
- $\mathcal{O} : \mathbf{Top} \to \mathbf{Frm}^{op}$ is a covariant functor with $X \longmapsto \mathcal{O}X$ and $X \xrightarrow{f} Y \longmapsto \mathcal{O}X \xrightarrow{f^{-1}} \mathcal{O}Y$.

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Advantage: $\mathbf{Frm}^{op}$ can be thought of as a natural extension of (sober) spaces.

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Advantage: $\mathbf{Frm}^{op}$ can be thought of as a natural extension of (sober) spaces.

Disadvantage: Morphisms thought in this way may obscure the intuition.

Pointfree topology

the category of locales **Loc**

Since each frame homomorphism h preserves arbitrary joins, it has a (uniquely determined) right adjoint h_* that can be used as a representation of h running in the proper direction.

J. Picado and A. Pultr, *Locales treated mostly in a covariant way*, Textos de Matemática, Vol. 41, University of Coimbra, 2008.

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- Morphisms, called *localic maps*, are mappings between locales f that have left adjoints f^* preserving finite meets.
- Motivating example: $\mathcal{O} : \mathbf{Top} \to \mathbf{Loc}$ is a covariant functor with $X \longmapsto \mathcal{O}X$ and $X \xrightarrow{f} Y \longmapsto \mathcal{O}X \xrightarrow{(f^{-1})_*} \mathcal{O}Y$.

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Pointfree topology embeddings

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However, for our purposes it is enough to observe the following:

Fact

In **Loc** the extremal monomorphism coincide with the strong monomorphisms, and those are precisely the one-to-one localic maps.

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Fact

In **Loc** the extremal monomorphism coincide with the strong monomorphisms, and those are precisely the one-to-one localic maps.

Definition. A localic map $f : L \to M$ is called an embedding if it is one-to-one, equivalently, if

$$a = f^* f(a) \quad \text{for all } a \in L.$$

Separating families of localic maps

Definition. Let $f : L \to L_f$ be a localic map for each $f \in F$. We say that the family F is separating if

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Note. (1) One could write $\leq$ in place of $=$.

(2) If $F \subseteq G$ and F is separating, then so is G.

(3) If $f : L \to M$, then $\{f\}$ is separating iff f is an embedding.

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We are particularly interested in families of the form $F \subseteq \mathbf{Loc}(L, M)$ for a fixed frame M.

Separating families of localic maps a first example

The chain $\mathbf{3} = \{0 < t < 1\}$ is called the Sierpiński locale.

A subset $B \subseteq L$ a base of L if $L = \{\bigvee C : C \subseteq B\}$.

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Let $B \subseteq L$ be a base, then for each $a \in L$ there is a subset $B_a \subseteq B$ such that $a = \bigvee B_a$ and so,

$$\bigvee_{b \in B} \left(f_b^{\mathbf{3}}\right)^* f_b^{\mathbf{3}}(a) \geq \bigvee_{b \in B_a} \left(f_b^{\mathbf{3}}\right)^* f_b^{\mathbf{3}}(a) = \bigvee_{b \in B_a} b = a.$$

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Proposition

Let L be a locale and B a base. Then the family $\{f_b^{\mathbf{3}} : b \in B\} \subseteq \mathbf{Loc}(L, \mathbf{3})$ is separating.

Separating families of localic maps zero-dimensional locales

Let $\mathbf{4} = \{0, t, \neg t, 1\}$ be the four element Boolean algebra.

Given an $a \in L$, we denote by $\neg a = \bigvee\{b \in L : a \wedge b = 0\}$ the pseudo-complement of a.

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Then $\mathbf{Loc}(L, \mathbf{4}) = \{f_a^{\mathbf{4}} : a \in BL\}$.

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Proof. Let $a \in L$ and $B_a \subseteq B$ such that $a = \bigvee B_a$. Then

$$\bigvee_{b \in B} \left(f_b^{\mathbf{4}}\right)^* f_b^{\mathbf{4}}(a) \geq \bigvee_{b \in B_a} \left(f_b^{\mathbf{4}}\right)^* f_b^{\mathbf{4}}(a) = a.$$

Hence $\{f_b^{\mathbf{4}} : b \in B\} \subseteq \mathbf{Loc}(L, \mathbf{4})$ is separating. □

Separating families of localic maps completely regular locales

Being really inside. Given $a, b \in L$, one writes

$$b \prec\!\!\prec a$$

if there exists a family $\{c_r \in L : r \in \mathbb{Q} \cap [0,1]\}$ such that

$$b \leq c_r \leq a \quad \text{and} \quad \neg c_r \vee c_s = 1 \text{ if } r < s.$$

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A locale L is called completely regular if

$$a = \bigvee\{x \in L : x \prec\!\!\prec a\} \quad \text{for all } a \in L.$$

Separating families of localic maps completely regular locales

Lemma

Let L be completely regular and $B \subseteq L$ be a base. Then

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Proof. For each $x \in L$ and $B_x \subseteq B$ such that $x = \bigvee B_x$. Then

$$a = \bigvee_{x \prec\!\!\prec a} x = \bigvee_{x \prec\!\!\prec a} \bigvee_{b \in B_x} b \leq \bigvee_{b \in B, b \prec\!\!\prec a} b \leq a \quad \text{for all } a \in L. \quad \square$$

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Let $\mathbb{I}$ denote the localic unit interval.

Lemma

Let L be a locale and $b \prec\!\!\prec a$ in L. Then there is a localic map $f_{b,a} : L \to \mathbb{I}$ such that

$$(0, -) \leq f(\neg b) \quad \text{and} \quad (-, 1) \leq f(a).$$

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Let L be a completely regular locale and $B \subseteq L$ a base. Then $\{f_{c,b} : b, c \in B, c \prec\!\!\prec b\} \subseteq \mathbf{Loc}(L, \mathbb{I})$ is separating.

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Proof. For each $b, c \in B$ with $c \prec\!\!\prec b$ there is an $f_{c,b} \in \mathbf{Loc}(L, \mathbb{I})$ such that $(0, -) \leq f_{c,b}(\neg c)$ and $(-, 1) \leq f_{c,b}(b)$. Hence $(f_{c,b})^* (0, -) \leq \neg c$ and

$$c \leq \neg (f_{c,b})^* (0, -) \leq (f_{c,b})^* (-, 1) \leq (f_{c,b})^* f_{c,b}(b).$$

Let $a \in L$ and $B_a \subseteq B$ such that $a = \bigvee B_a$. We have that

$$a = \bigvee_{b \in B_a} \bigvee_{c \in B, c \prec\!\!\prec b} c \leq \bigvee_{b,c \in B, c \prec\!\!\prec b} (f_{c,b})^* f_{c,b}(b) \leq \bigvee_{b,c \in B, c \prec\!\!\prec b} (f_{c,b})^* f_{c,b}(a).$$

Hence$\{f_{c,b} : b, c \in B, c \prec\!\!\prec b\} \subseteq \mathbf{Loc}(L, \mathbb{I})$ is separating. □

The localic embedding theorem — products of locales

We denote by $\left(\pi_i : \bigoplus_{j\in J} L_j \to L_i\right)_{i\in J}$ the product of the system $\{L_i : i \in J\}$ in the category **Loc**.

Hence, for any family of localic maps $f_i : L \to L_i$ there is a unique localic map $f : L \to \bigoplus_{j\in J} L_j$ such that $f_j = \pi_j f$ for all $j \in J$:

$$\begin{array}{ccc} L & \xrightarrow{\ f_j\ } & L_j \\ {\scriptstyle f}\downarrow & \nearrow_{\pi_j} & \\ \bigoplus_{j\in J} L_j & & \end{array}$$

The localic product of κ copies of L is denoted L^κ.

A construction of $\bigoplus_{j\in J} L_j$ can be found in:

J. Picado and A. Pultr, *Locales treated mostly in a covariant way*, Textos de Matemática, Vol. 41, University of Coimbra, 2008.

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Proof. Let $e : L \to \bigoplus_{f \in F} L_f$ be determined by $f = \pi_f e$ for each $f \in F$. For any $a \in L$ we have

$$a = \bigvee_{f \in F} f^* f(a) = \bigvee_{f \in F} e^*(\pi_f)^* \pi_f e(a) \leq e^* e(a) \leq a.$$

Hence $e^* e = \mathrm{id}_L$, i.e. e is an embedding. □

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Hence $e^* e = \mathrm{id}_L$, i.e. e is an embedding. □

Corollary

Let L and M be locales. If $F \subseteq \mathbf{Loc}(L, M)$ is separating, then L embedds into $M^{|F|}$.

The localic embedding theorem — a first application

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Let L be a locale and B a base. Then L embedds into $\mathbf{3}^{|B|}$.

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Proof. Recall that the family $\{f_b^{\mathbf{3}} : b \in B\} \subseteq \mathbf{Loc}(L, \mathbf{3})$ is separating.

Hence there is an embedding of L into $\mathbf{3}^{|\{f_b^{\mathbf{3}} : b \in B\}|} = \mathbf{3}^{|B|}$. □

The localic embedding theorem a first application

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Let us say that a universal locale for a class of locales is a locale in this class in which every locale belonging to the class can be embedded as a sublocale.

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Let us say that a universal locale for a class of locales is a locale in this class in which every locale belonging to the class can be embedded as a sublocale.

Theorem

Let κ be an infinite cardinal. Then the localic product $\mathbf{3}^{\kappa}$ is universal for all the locales whose weight is smaller or equal than κ.

The localic embedding theorem zero-dimensional locales

Corollary

Let L be a zero-dimensional locale and $B \subseteq \mathcal{B}L$ a base. Then L embedds into $\mathbf{4}^{|B|}$.

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Let L be a zero-dimensional locale and $B \subseteq BL$ a base. Then L embedds into $\mathbf{4}^{|B|}$.

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Corollary

Let L be a frame and $\mathcal{S}(L)$ the locale of congruences on L. Then $\mathcal{S}(L)$ embedds into $\mathbf{4}^{|L|}$.

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Theorem

Let κ be an infinite cardinal. Then the localic product $\mathbf{4}^{\kappa}$ is universal for all the zero-dimensional locales whose weight is smaller or equal than κ.

The localic embedding theorem completely regular locales

Corollary

Let L be a completely regular and $B \subseteq L$ a base. Then L embedds into $\mathbb{I}^{|B \times B|}$.

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Proof. Recall that the family $\{f_{c,b} : b, c \in B, c \prec\!\!\prec b\} \subseteq \mathbf{Loc}(L, \mathbb{I})$ is separating.

Hence there is an embedding of L into $\mathbb{I}^{|\{f_{c,b}: b,c\in B, c\prec\!\!\prec b\}|} = \mathbb{I}^{|B\times B|}$. □

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Theorem

Let κ be an infinite cardinal. Then the localic product $\mathbb{I}^{\kappa}$ is universal for all the completely regular locales whose weight is smaller or equal than κ.