Real valued functions in Pointfree Topology

Javier Gutiérrez García

Department of Mathematics, University of the Basque Country, SPAIN

– joint work with Tomasz Kubiak (Poznan) and Jorge Picado (Coimbra)

eman ta zabal zazu

Universidad del País Vasco
Euskal Herriko Unibertsitatea

UAM

UNIVERSIDADE DE COIMBRA

International Conference on Topology and Its Applications, ICTA 2009
HACETTEPE UNIVERSITY

> *“The aim of these notes is to show how various facts in classical topology connected with the real numbers have their counterparts, if not actually their logical antecedents, in pointfree topology, that is, in the setting of frames and their homomorphisms.*
>
> *. . . the treatment here will specifically concentrate on the pointfree version of continuous real functions which arises from it.”*

B. Banaschewski,
The real numbers in pointfree topology,
Textos de Matemática, Série B, 12, Univ. de Coimbra, 1997.

> *“The set $\mathrm{C}(X)$ of all continuous, real-valued functions on a topological space X will be provided with an algebraic structure and an order structure. Since their definitions do not involve continuity, we begin by imposing these structures on the collection $\mathbb{R}^X$ of all functions from X into the set $\mathbb{R}$ of real numbers. [...]*
>
> *In fact, it is clear that $\mathbb{R}^X$ is a commutative ring with unity element (provided that X is non empty). [...]*
>
> *Therefore $\mathrm{C}(X)$ is a commutative ring, a subring of $\mathbb{R}^X$.”*

L. Gillman and M. Jerison,
Rings of Continuous Functions

Motivation: Katětov-Tong Theorem

Urysohn's Lemma.

Let X be a topological space. TFAE:

(1) X is normal.

(2) For every disjoint closed sets F and G, there exists a continuous $h : X \to [0, 1]$ such that $h(F) = \{0\}$ and $h(G) = \{1\}$.

(3) For every closed set F and open set U such that $F \subseteq U$, there exists a continuous $h : X \to \mathbb{R}$ such that $\chi_F \leq h \leq \chi_U$.

Question

Let X be a topological space and let $f, g : X \to \mathbb{R}$ be such that $f \in \mathrm{USC}(X)$, $g \in \mathrm{LSC}(X)$ and $f \leq g$.

Does there exists a continuous $h \in \mathrm{C}(X)$ such that $f \leq h \leq g$

Motivation: Katětov-Tong Theorem

Urysohn's Lemma.

Let X be a topological space. TFAE:

(1) X is normal.

(2) For every disjoint closed sets F and G, there exists a continuous $h : X \to [0, 1]$ such that $h(F) = \{0\}$ and $h(G) = \{1\}$.

(3) For every closed set F and open set U such that $F \subseteq U$, there exists a continuous $h : X \to \mathbb{R}$ such that $\chi_F \leq h \leq \chi_U$.

Question

Let X be a topological space and let $f, g : X \to \mathbb{R}$ be such that $f \in \mathrm{USC}(X)$, $g \in \mathrm{LSC}(X)$ and $f \leq g$.

Does there exists a continuous $h \in \mathrm{C}(X)$ such that $f \leq h \leq g$

Motivation: Katětov-Tong Theorem

Urysohn's Lemma.

Let X be a topological space. TFAE:

(1) X is normal.

(2) For every disjoint closed sets F and G, there exists a continuous $h : X \to [0, 1]$ such that $h(F) = \{0\}$ and $h(G) = \{1\}$.

(3) For every closed set F and open set U such that $F \subseteq U$, there exists a continuous $h : X \to \mathbb{R}$ such that $\chi_F \leq h \leq \chi_U$.

Question

Let X be a topological space and let $f, g : X \to \mathbb{R}$ be such that $f \in \mathrm{USC}(X)$, $g \in \mathrm{LSC}(X)$ and $f \leq g$.

Does there exists a continuous $h \in \mathrm{C}(X)$ such that $f \leq h \leq g$

Motivation: Katětov-Tong Theorem

Urysohn's Lemma.

Let X be a topological space. TFAE:

(1) X is normal.

(2) For every disjoint closed sets F and G, there exists a continuous $h : X \to [0,1]$ such that $h(F) = \{0\}$ and $h(G) = \{1\}$.

(3) For every closed set F and open set U such that $F \subseteq U$, there exists a continuous $h : X \to \mathbb{R}$ such that $\chi_F \leq h \leq \chi_U$.

Question

Let X be a topological space and let $f, g : X \to \mathbb{R}$ be such that $f \in \mathrm{USC}(X)$, $g \in \mathrm{LSC}(X)$ and $f \leq g$.

Does there exists a continuous $h \in \mathrm{C}(X)$ such that $f \leq h \leq g$

Motivation: Katětov-Tong Theorem

Question

Let X be a topological space and let $f, g : X \to \mathbb{R}$ be such that $f \in \mathrm{USC}(X)$, $g \in \mathrm{LSC}(X)$ and $f \leq g$.

Does there exists a continuous $h \in \mathrm{C}(X)$ such that $f \leq h \leq g$

Answer

Yes, if X is METRIC [Hahn, 1917]

Yes, if X is PARACOMPACT [Dieudonné, 1944]

Yes, if X is NORMAL [Katětov-Tong, 1948]

Motivation: Katětov-Tong Theorem

Question

Let X be a topological space and let $f, g : X \to \mathbb{R}$ be such that $f \in \mathrm{USC}(X)$, $g \in \mathrm{LSC}(X)$ and $f \leq g$.

Does there exists a continuous $h \in \mathrm{C}(X)$ such that $f \leq h \leq g$

Answer

Yes, if X is METRIC [Hahn, 1917]

Yes, if X is PARACOMPACT [Dieudonné, 1944]

Yes, if X is NORMAL [Katětov-Tong, 1948]

Motivation: Katětov-Tong Theorem

Question

Let X be a topological space and let $f, g : X \to \mathbb{R}$ be such that $f \in \mathrm{USC}(X)$, $g \in \mathrm{LSC}(X)$ and $f \leq g$.

Does there exists a continuous $h \in \mathrm{C}(X)$ such that $f \leq h \leq g$

Answer

Yes, if X is METRIC [Hahn, 1917]

Yes, if X is PARACOMPACT [Dieudonné, 1944]

Yes, if X is NORMAL [Katětov-Tong, 1948]

Motivation: Katětov-Tong Theorem

Katětov-Tong Insertion Theorem.

Let X be a topological space and let $f, g : X \to \mathbb{R}$. TFAE:

(1) X is normal.

(2) For every $f \in \mathrm{USC}(X)$ and every $g \in \mathrm{LSC}(X)$ with $f \leq g$, there exists a continuous $h \in \mathrm{C}(X)$ such that $f \leq h \leq g$.

M. Katětov,
On real-valued functions in topological spaces,
Fund. Math. 38 (1951) 85-91; correction 40 (1953) 203-205.

H. Tong,
Some characterizations of normal and perfectly normal spaces,
Duke Math. J. 19 (1952) 289-292.

Motivation: Stone Insertion Theorem

Stone Insertion Theorem.

Let X be a topological space and let $f, g : X \to \mathbb{R}$. TFAE:

(1) X is extremally disconnected (any two disjoint open sets in X have disjoint closures).

(2) For every $f \in \mathrm{LSC}(X)$ and every $g \in \mathrm{USC}(X)$ with $f \leq g$, there exists a continuous $h \in \mathrm{C}(X)$ such that $f \leq h \leq g$.

M.H. Stone,
Boundedness properties in function-lattices,
Canad. J. Math. 1 (1949) 176–186.

Motivation: Dowker Insertion Theorem

Dowker Insertion Theorem.

Let X be a topological space and let $f, g : X \to \mathbb{R}$. TFAE:

(1) X is normal and countably paracompact.

(2) For every $f \in \mathrm{USC}(X)$ and every $g \in \mathrm{LSC}(X)$ with $f < g$, there exists a continuous $h \in \mathrm{C}(X)$ such that $f < h < g$.

C. H. Dowker,
On countably paracompact spaces,
Canad. J. Math. 3 (1951) 219–224.

Motivation: Michael Insertion Theorem

Michael Insertion Theorem.

Let X be a topological space and let $f, g : X \to \mathbb{R}$. TFAE:

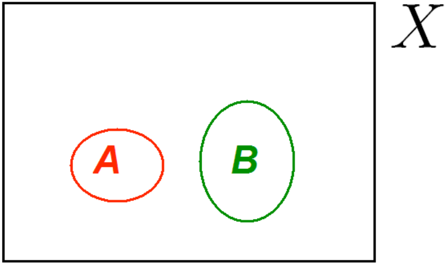
(1) X is perfectly normal (every two disjoint closed sets can be precisely separated by a continuous real valued function).

(2) For every $f \in \mathrm{USC}(X)$ and every $g \in \mathrm{LSC}(X)$ with $f \leq g$, there exists a continuous $h \in \mathrm{C}(X)$ such that $f \leq h \leq g$ and $f(x) < h(x) < g(x)$ whenever $f(x) < g(x)$.

E. Michael,
Continuous selections I,,
Ann. of Math. 63 (1956) 361–382.

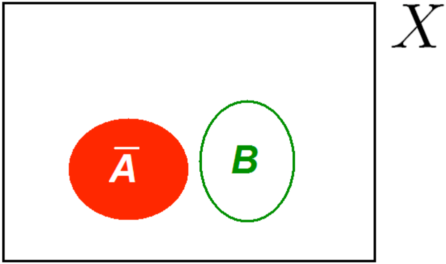
Motivation: Kubiak Insertion Theorem

A topological space X is completely normal if for every pair of subsets A and B of X which are separated (i.e. $\overline{A} \cap B = \varnothing = A \cap \overline{B}$) there are disjoint open sets containing A and B respectively.



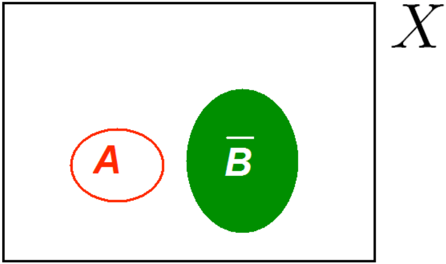
Motivation: Kubiak Insertion Theorem

A topological space X is completely normal if for every pair of subsets A and B of X which are separated (i.e. $\overline{A} \cap B = \varnothing = A \cap \overline{B}$) there are disjoint open sets containing A and B respectively.



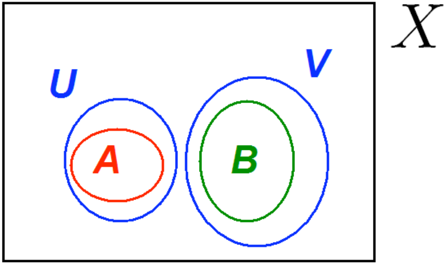
Motivation: Kubiak Insertion Theorem

A topological space X is completely normal if for every pair of subsets A and B of X which are separated (i.e. $\overline{A} \cap B = \varnothing = A \cap \overline{B}$) there are disjoint open sets containing A and B respectively.



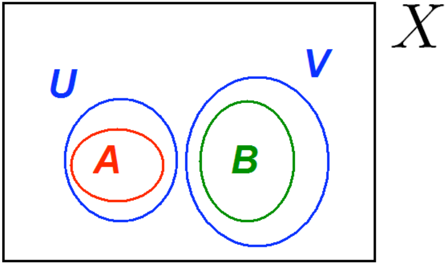
Motivation: Kubiak Insertion Theorem

A topological space X is completely normal if for every pair of subsets A and B of X which are separated (i.e. $\overline{A} \cap B = \varnothing = A \cap \overline{B}$) there are disjoint open sets containing A and B respectively.



Motivation: Kubiak Insertion Theorem

A topological space X is completely normal if for every pair of subsets A and B of X which are separated (i.e. $\overline{A} \cap B = \varnothing = A \cap \overline{B}$) there are disjoint open sets containing A and B respectively.



(A standard exercise is to show that this is equivalent to hereditary normality.)

Motivation: Kubiak Insertion Theorem

Kubiak Insertion Theorem.

Let X be a topological space and let $f, g : X \to \mathbb{R}$. TFAE:

(1) X is completely normal.

(2) If $\overline{A} \subseteq B$ and $A \subseteq \overset{\circ}{B}$, then there exists an open set U such that $A \subseteq U \subseteq \overline{U} \subseteq B$.

(3) If $f^{-} \leq g$ and $f \leq g^{\circ}$, then there exists a lower semicontinuous $h : X \to \mathbb{R}$ such that $f \leq h \leq h^{-} \leq g$
(where f^{-} denotes the upper regularization of f and g° denotes the lower regularization of g).

Motivation: Kubiak Insertion Theorem

Kubiak Insertion Theorem.

Let X be a topological space and let $f, g : X \to \mathbb{R}$. TFAE:

(1) X is completely normal.

(2) If $\overline{A} \subseteq B$ and $A \subseteq \overset{\circ}{B}$, then there exists an open set U such that $A \subseteq U \subseteq \overline{U} \subseteq B$.

(3) If $f^- \leq g$ and $f \leq g^\circ$, then there exists a lower semicontinuous $h : X \to \mathbb{R}$ such that $f \leq h \leq h^- \leq g$ (where f^- denotes the upper regularization of f and g° denotes the lower regularization of g).

T. Kubiak,
A strengthening of the Katětov-Tong insertion theorem,
Comment. Math. Univ. Carolinae 34 (1993) 357–362.

> *"The aim of these notes is to show how various facts in classical topology connected with the real numbers have their counterparts, if not actually their logical antecedents, in pointfree topology, that is, in the setting of frames and their homomorphisms.*
>
> *. . . the treatment here will specifically concentrate on the pointfree version of continuous real functions which arises from it."*

Our intention in this talk is to extend this study to the case of general real valued functions (paying particular attention to the semicontinuous ones) in the setting of pointfree topology.

B. Banaschewski,
The real numbers in pointfree topology,
Textos de Matemática, Série B, 12, Univ. de Coimbra, 1997.

> *“The aim of these notes is to show how various facts in classical topology connected with the real numbers have their counterparts, if not actually their logical antecedents, in pointfree topology, that is, in the setting of frames and their homomorphisms.*
>
> *. . . the treatment here will specifically concentrate on the pointfree version of continuous real functions which arises from it.”*

Our intention in this talk is to extend this study to the case of general real valued functions (paying particular attention to the semicontinuous ones) in the setting of pointfree topology.

B. Banaschewski,
The real numbers in pointfree topology,
Textos de Matemática, Série B, 12, Univ. de Coimbra, 1997.

Pointfree topology

$$\begin{tikzcd}(X, \mathcal{O}X) \arrow[r, squiggly] & (\mathcal{O}X, \subseteq)\end{tikzcd}$$

$$A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i)$$

f^{-1} preserves $\bigcup$ and $\cap$

$$(Y, \mathcal{O}Y) \qquad (\mathcal{O}Y, \subseteq)$$

$$\begin{tikzcd}\boxed{\text{TOPOLOGY}} \arrow[r, squiggly, "\text{Abstraction}"] & \boxed{\text{POINTFREE TOPOLOGY}}\end{tikzcd}$$

Pointfree topology

$$(X, \mathcal{O}X) \rightsquigarrow (\mathcal{O}X, \subseteq)$$

$$A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i)$$

f^{-1} preserves $\bigcup$ and $\cap$

$$(Y, \mathcal{O}Y) \qquad (\mathcal{O}Y, \subseteq)$$

$$\boxed{\text{TOPOLOGY}} \overset{\text{Abstraction}}{\rightsquigarrow} \boxed{\text{POINTFREE TOPOLOGY}}$$

Pointfree topology

$$\begin{tikzcd} (X,\mathcal{O}X) \arrow[r, squiggly] \arrow[d, "f"'] & (\mathcal{O}X,\subseteq) \\ (Y,\mathcal{O}Y) & (\mathcal{O}Y,\subseteq) \end{tikzcd}$$

$$A\cap\bigcup_{i\in I}B_i=\bigcup_{i\in I}(A\cap B_i)$$

f^{-1} preserves $\bigcup$ and $\cap$

$$\begin{tikzcd} \boxed{\text{TOPOLOGY}} \arrow[r, squiggly, "\text{Abstraction}"] & \boxed{\text{POINTFREE TOPOLOGY}} \end{tikzcd}$$

Pointfree topology

$$\begin{tikzcd}
(X, \mathcal{O}X) \arrow[r, squiggly] \arrow[d, "f"'] & (\mathcal{O}X, \subseteq) \\
(Y, \mathcal{O}Y) \arrow[r, squiggly] & (\mathcal{O}Y, \subseteq)
\end{tikzcd}$$

$$A \cap \bigcup_{i\in I} B_i = \bigcup_{i\in I} (A \cap B_i)$$

f^{-1} preserves $\bigcup$ and $\cap$

$$\begin{tikzcd}
\boxed{\text{TOPOLOGY}} \arrow[r, squiggly, "\text{Abstraction}"] & \boxed{\text{POINTFREE TOPOLOGY}}
\end{tikzcd}$$

Pointfree topology

$$\begin{tikzcd}
(X, \mathcal{O}X) \arrow[r, rightsquigarrow] \arrow[d, "f"'] & (\mathcal{O}X, \subseteq) \\
(Y, \mathcal{O}Y) \arrow[r, rightsquigarrow] & (\mathcal{O}Y, \subseteq) \arrow[u, "f^{-1}"']
\end{tikzcd}$$

$$A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i)$$

f^{-1} preserves $\bigcup$ and $\cap$

$$\begin{tikzcd}
\boxed{\text{TOPOLOGY}} \arrow[r, rightsquigarrow, "\text{Abstraction}"] & \boxed{\text{POINTFREE TOPOLOGY}}
\end{tikzcd}$$

Pointfree topology

$$\begin{tikzcd}
(X, \mathcal{O}X) \arrow[r, rightsquigarrow] \arrow[d, "f"'] & (\mathcal{O}X, \subseteq) \\
(Y, \mathcal{O}Y) \arrow[r, rightsquigarrow] & (\mathcal{O}Y, \subseteq) \arrow[u, "f^{-1}"']
\end{tikzcd}$$

$$A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i)$$

f^{-1} preserves $\bigcup$ and $\cap$

$$\begin{tikzcd}
\boxed{\text{TOPOLOGY}} \arrow[r, rightsquigarrow, "\text{Abstraction}"] & \boxed{\text{POINTFREE TOPOLOGY}}
\end{tikzcd}$$

Pointfree topology

$$\begin{tikzcd}
(X,\mathcal{O}X) \arrow[r, rightsquigarrow] \arrow[d, "f"'] & (\mathcal{O}X,\subseteq) \\
(Y,\mathcal{O}Y) \arrow[r, rightsquigarrow] & (\mathcal{O}Y,\subseteq) \arrow[u, "f^{-1}"']
\end{tikzcd}$$

$$A \cap \bigcup_{i\in I} B_i = \bigcup_{i\in I}(A \cap B_i)$$

f^{-1} preserves $\bigcup$ and $\cap$

TOPOLOGY $\overset{\text{Abstraction}}{\rightsquigarrow}$ POINTFREE TOPOLOGY

Pointfree topology

$$\begin{tikzcd}
(X, \mathcal{O}X) \arrow[r, rightsquigarrow] \arrow[d, "f"'] & \color{red}(\mathcal{O}X, \subseteq) \\
(Y, \mathcal{O}Y) \arrow[r, rightsquigarrow] & \color{red}(\mathcal{O}Y, \subseteq) \arrow[u, "\color{red}f^{-1}"']
\end{tikzcd}$$

$$A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i)$$

f^{-1} preserves $\bigcup$ and $\cap$

$$\begin{tikzcd}
\boxed{\text{TOPOLOGY}} \arrow[r, rightsquigarrow, "\text{Abstraction}"] & \boxed{\text{POINTFREE TOPOLOGY}}
\end{tikzcd}$$

Pointfree topology

B. Banaschewski,
The real numbers in pointfree topology,
Textos de Matemática, Série B, 12, Univ. de Coimbra, 1997.

P. T. Johnstone,
Stone Spaces,
Cambridge Univ. Press, Cambridge, 1982.

P. T. Johnstone,
The point of pointless topology.
Bull. Amer. Math. Soc. 8 (1983) 41-53.

Pointfree topology

the category of frames Frm

- The objects in Frm are *frames*, i.e.
  - complete lattices L in which
  - $a \wedge \bigvee_{i\in I} a_i = \bigvee \{a \wedge a_i : i \in I\}$ for all $a \in L$ and $\{a_i : i \in I\} \subseteq L$.
- Morphisms, called *frame homomorphisms*, are those maps between frames h that preserve
  - arbitrary joins,

$$h\Big(\bigvee_{i\in I} a_i\Big) = \bigvee_{i\in I} h(a_i), \quad h(0) = 0,$$

  - finite meets,

$$h(a_1 \wedge a_2) = h(a_1) \wedge h(a_2), \quad h(1) = 1.$$

Pointfree topology

the category of frames $\boxed{\text{Frm}}$

- The objects in Frm are *frames*, i.e.
    - complete lattices L in which
    - $a \wedge \bigvee_{i\in I} a_i = \bigvee\{a \wedge a_i : i \in I\}$ for all $a \in L$ and $\{a_i : i \in I\} \subseteq L$.
- Morphisms, called *frame homomorphisms*, are those maps between frames h that preserve
    - arbitrary joins,

$$h\Big(\bigvee_{i\in I} a_i\Big) = \bigvee_{i\in I} h(a_i), \quad h(0) = 0,$$

    - finite meets,

$$h(a_1 \wedge a_2) = h(a_1) \wedge h(a_2), \quad h(1) = 1.$$

Pointfree topology

the category of frames $\boxed{\text{Frm}}$

- The objects in Frm are *frames*, i.e.
    - complete lattices L in which
    - $a \wedge \bigvee_{i\in I} a_i = \bigvee\{a \wedge a_i : i \in I\}$ for all $a \in L$ and $\{a_i : i \in I\} \subseteq L$.
- Morphisms, called *frame homomorphisms*, are those maps between frames h that preserve
    - arbitrary joins,

$$h\Big(\bigvee_{i\in I} a_i\Big) = \bigvee_{i\in I} h(a_i), \quad h(0) = 0,$$

    - finite meets,

$$h(a_1 \wedge a_2) = h(a_1) \wedge h(a_2), \quad h(1) = 1.$$

Pointfree topology

the category of frames $\boxed{\mathsf{Frm}}$

- The objects in Frm are *frames*, i.e.
    - ∗ complete lattices L in which
    - ∗ $a \wedge \bigvee_{i \in I} a_i = \bigvee \{a \wedge a_i : i \in I\}$ for all $a \in L$ and $\{a_i : i \in I\} \subseteq L$.
- Morphisms, called *frame homomorphisms*, are those maps between frames h that preserve
    - ∗ arbitrary joins,

$$h(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} h(a_i), \quad h(0) = 0,$$

    - ∗ finite meets,

$$h(a_1 \wedge a_2) = h(a_1) \wedge h(a_2), \quad h(1) = 1.$$

Pointfree topology

the category of frames $\boxed{\text{Frm}}$

- The objects in Frm are *frames*, i.e.
    - ∗ complete lattices L in which
    - ∗ $a \wedge \bigvee_{i \in I} a_i = \bigvee \{a \wedge a_i : i \in I\}$ for all $a \in L$ and $\{a_i : i \in I\} \subseteq L$.
- Morphisms, called *frame homomorphisms*, are those maps between frames h that preserve
    - ∗ arbitrary joins,

$$h\Big(\bigvee_{i \in I} a_i\Big) = \bigvee_{i \in I} h(a_i), \quad h(0) = 0,$$

    - ∗ finite meets,

$$h(a_1 \wedge a_2) = h(a_1) \wedge h(a_2), \quad h(1) = 1.$$

Pointfree topology

the category of frames $\boxed{\text{Frm}}$

- The objects in Frm are *frames*, i.e.
  - complete lattices L in which
  - $a \wedge \bigvee_{i\in I} a_i = \bigvee\{a \wedge a_i : i \in I\}$ for all $a \in L$ and $\{a_i : i \in I\} \subseteq L$.
- Morphisms, called *frame homomorphisms*, are those maps between frames h that preserve
  - arbitrary joins,

$$h\Big(\bigvee_{i\in I} a_i\Big) = \bigvee_{i\in I} h(a_i), \quad h(0) = 0,$$

  - finite meets,

$$h(a_1 \wedge a_2) = h(a_1) \wedge h(a_2), \quad h(1) = 1.$$

Pointfree topology

the category of frames $\boxed{\mathrm{Frm}}$

Being a Heyting algebra, each frame L has the Heyting operation $\rightarrow$ satisfying $a \wedge b \leq c$ iff $a \leq b \rightarrow c$.

The *pseudocomplement* of $a \in L$ is

$$a^* = a \rightarrow 0 = \bigvee\{b \in L : a \wedge b = 0\}.$$

When a is complemented, a^* is its complement and we denote it by the usual notation $\neg a$.

The set of all morphisms from L into M is denoted by

$$\boxed{\mathrm{Frm}(L, M)}$$

Pointfree topology

the category of frames $\boxed{\mathsf{Frm}}$

Being a Heyting algebra, each frame L has the Heyting operation $\to$ satisfying $a \wedge b \leq c$ iff $a \leq b \to c$.

The *pseudocomplement* of $a \in L$ is

$$a^* = a \to 0 = \bigvee\{b \in L : a \wedge b = 0\}.$$

When a is complemented, a^* is its complement and we denote it by the usual notation $\neg a$.

The set of all morphisms from L into M is denoted by

$$\boxed{\mathsf{Frm}(L, M)}$$

Pointfree topology

the category of frames $\boxed{\mathrm{Frm}}$

Being a Heyting algebra, each frame L has the Heyting operation $\rightarrow$ satisfying $a \wedge b \leq c$ iff $a \leq b \rightarrow c$.

The *pseudocomplement* of $a \in L$ is

$$a^* = a \rightarrow 0 = \bigvee \{b \in L : a \wedge b = 0\}.$$

When a is complemented, a^* is its complement and we denote it by the usual notation $\neg a$.

The set of all morphisms from L into M is denoted by

$$\boxed{\mathrm{Frm}(L, M)}$$

Pointfree topology

the category of frames $\boxed{\mathsf{Frm}}$

Being a Heyting algebra, each frame L has the Heyting operation $\to$ satisfying $a \wedge b \leq c$ iff $a \leq b \to c$.

The *pseudocomplement* of $a \in L$ is

$$a^* = a \to 0 = \bigvee \{ b \in L : a \wedge b = 0 \}.$$

When a is complemented, a^* is its complement and we denote it by the usual notation $\neg a$.

The set of all morphisms from L into M is denoted by

$$\boxed{\mathsf{Frm}(L, M)}$$

Pointfree topology

the category of frames $\boxed{\mathsf{Frm}}$

Motivating example: the lattice $\mathcal{O}X$ of all open subsets of a space X is a frame and if $f : X \to Y$ is a map, then $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ defined by $\mathcal{O}f(U) = f^{-1}(U)$ is a frame homomorphism.

Consequently we have a contravariant functor

$$\begin{tikzcd}\mathsf{Top} \arrow[r, "\mathcal{O}"] & \mathsf{Frm}\end{tikzcd}$$

There is a functor in the opposite direction, the spectrum functor

$$\begin{tikzcd}\mathsf{Top} & \mathsf{Frm} \arrow[l, "\Sigma"']\end{tikzcd}$$

which assigns to each frame L its spectrum $\Sigma L = \mathsf{Frm}(L, \mathbf{2} = \{0 < 1\})$, with open sets $\Sigma_a = \{\xi \in \Sigma L : \xi(a) = 1\}$.

Pointfree topology

the category of frames $\boxed{\text{Frm}}$

Motivating example: the lattice $\mathcal{O}X$ of all open subsets of a space X is a frame and if $f : X \to Y$ is a map, then $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ defined by $\mathcal{O}f(U) = f^{-1}(U)$ is a frame homomorphism.

Consequently we have a contravariant functor

$$\begin{tikzcd}\text{Top} \arrow[rr, "\mathcal{O}"] & & \text{Frm}\end{tikzcd}$$

There is a functor in the opposite direction, the spectrum functor

$$\begin{tikzcd}\text{Top} & & \text{Frm} \arrow[ll, "\Sigma"']\end{tikzcd}$$

which assigns to each frame L its spectrum $\Sigma L = \text{Frm}(L, \mathbf{2} = \{0 < 1\})$, with open sets $\Sigma_a = \{\xi \in \Sigma L : \xi(a) = 1\}$.

Pointfree topology

the category of frames $\boxed{\mathrm{Frm}}$

Motivating example: the lattice $\mathcal{O}X$ of all open subsets of a space X is a frame and if $f : X \to Y$ is a map, then $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ defined by $\mathcal{O}f(U) = f^{-1}(U)$ is a frame homomorphism.

Consequently we have a contravariant functor

$$\begin{tikzcd}\mathrm{Top} \arrow[r, "\mathcal{O}"] & \mathrm{Frm}\end{tikzcd}$$

There is a functor in the opposite direction, the spectrum functor

$$\begin{tikzcd}\mathrm{Top} & \mathrm{Frm} \arrow[l, "\Sigma"']\end{tikzcd}$$

which assigns to each frame L its spectrum $\Sigma L = \mathrm{Frm}(L, \mathbf{2} = \{0 < 1\})$, with open sets $\Sigma_a = \{\xi \in \Sigma L : \xi(a) = 1\}$.

Pointfree topology

the category of frames Frm

Motivating example: the lattice $\mathcal{O}X$ of all open subsets of a space X is a frame and if $f : X \to Y$ is a map, then $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ defined by $\mathcal{O}f(U) = f^{-1}(U)$ is a frame homomorphism.

Consequently we have a contravariant functor

$$\begin{tikzcd}\mathsf{Top} \arrow[rr, "\mathcal{O}"] & & \mathsf{Frm}\end{tikzcd}$$

There is a functor in the opposite direction, the spectrum functor

$$\begin{tikzcd}\mathsf{Top} & & \mathsf{Frm} \arrow[ll, "\Sigma"']\end{tikzcd}$$

which assigns to each frame L its spectrum $\Sigma L = \mathsf{Frm}(L, \mathbf{2} = \{0 < 1\})$, with open sets $\Sigma_a = \{\xi \in \Sigma L : \xi(a) = 1\}$.

Pointfree topology

the category of frames $\boxed{\mathbf{Frm}}$

We have two contravariant functors:

$$\boxed{\begin{tikzcd} \mathbf{Top} \arrow[r, shift left, "\mathcal{O}"] & \mathbf{Frm} \arrow[l, shift left, "\Sigma"] \end{tikzcd}}$$

which form a dual adjunction.

That is, there are adjunction maps

$$\boxed{\eta_L : L \to \mathcal{O}\Sigma L, \qquad \eta_L(a) = \Sigma_a \quad (a \in L)}$$

and

$$\boxed{\varepsilon_X : X \to \Sigma\mathcal{O}X, \qquad \varepsilon_X(x) = \hat{x},\ \hat{x}(U) \text{ iff } x \in U \quad (x \in X)}$$

natural in L and X respectively.

Pointfree topology

the category of frames Frm

We have two contravariant functors:

$$\begin{tikzcd} \mathbf{Top} \arrow[r, shift left, "\mathcal{O}"] & \mathbf{Frm} \arrow[l, shift left, "\Sigma"] \end{tikzcd}$$

which form a dual adjunction.

That is, there are adjunction maps

$$\boxed{\eta_L : L \to \mathcal{O}\Sigma L, \qquad \eta_L(a) = \Sigma_a \quad (a \in L)}$$

and

$$\boxed{\varepsilon_X : X \to \Sigma\mathcal{O}X, \qquad \varepsilon_X(x) = \hat{x},\ \hat{x}(U) \text{ iff } x \in U \quad (x \in X)}$$

natural in L and X respectively.

Pointfree topology

the category of frames $\boxed{\mathsf{Frm}}$

We have two contravariant functors:

$$\boxed{\begin{tikzcd} \mathsf{Top} \arrow[r, shift left, "\mathcal{O}"] & \mathsf{Frm} \arrow[l, shift left, "\Sigma"] \end{tikzcd}}$$

which form a dual adjunction.

That is, there are adjunction maps

$$\boxed{\eta_L : L \to \mathcal{O}\Sigma L, \qquad \eta_L(a) = \Sigma_a \quad (a \in L)}$$

and

$$\boxed{\varepsilon_X : X \to \Sigma\mathcal{O}X, \qquad \varepsilon_X(x) = \hat{x},\ \hat{x}(U) \text{ iff } x \in U \quad (x \in X)}$$

natural in L and X respectively.

Pointfree topology

the category of frames $\boxed{\text{Frm}}$

We have two contravariant functors:

$$\boxed{\begin{tikzcd} \text{Top} \arrow[r, shift left, "\mathcal{O}"] & \text{Frm} \arrow[l, shift left, "\Sigma"] \end{tikzcd}}$$

which form a dual adjunction.

That is, there are adjunction maps

$$\boxed{\eta_L : L \to \mathcal{O}\Sigma L, \qquad \eta_L(a) = \Sigma_a \quad (a \in L)}$$

and

$$\boxed{\varepsilon_X : X \to \Sigma\mathcal{O}X, \qquad \varepsilon_X(x) = \hat{x},\ \hat{x}(U) \text{ iff } x \in U \quad (x \in X)}$$

natural in L and X respectively.

Pointfree topology

the category of frames $\boxed{\text{Frm}}$

Frames L for which η_L is an isomorphism are called spatial, and η_L is then the reflection map from L to spatial frames.

On the other hand, spaces for which ε_X is an homeomorphism are called sober, and by general principles, the full subcategory Sob of Top given by this spaces is then dually equivalent to the full subcategory SpFrm of Frm given by the spatial frames.

$$\begin{tikzcd}
\text{Sob} \arrow[r, shift left, "\mathcal{O}"] & \text{SpFrm} \arrow[l, shift left, "\Sigma"]
\end{tikzcd}$$

Note that we also have a natural equivalence

$$\boxed{\text{Top}(X, \Sigma L) \simeq \text{Frm}(L, \mathcal{O}X)}$$

Pointfree topology

the category of frames $\boxed{\mathsf{Frm}}$

Frames L for which η_L is an isomorphism are called spatial, and η_L is then the reflection map from L to spatial frames.

On the other hand, spaces for which ε_X is an homeomorphism are called sober, and by general principles, the full subcategory Sob of Top given by this spaces is then dually equivalent to the full subcategory SpFrm of Frm given by the spatial frames.

$$\begin{tikzcd}
\mathsf{Sob} \arrow[r, shift left, "\mathcal{O}"] & \mathsf{SpFrm} \arrow[l, shift left, "\Sigma"]
\end{tikzcd}$$

Note that we also have a natural equivalence

$$\boxed{\mathsf{Top}(X, \Sigma L) \simeq \mathsf{Frm}(L, \mathcal{O}X)}$$

Pointfree topology

the category of frames $\boxed{\text{Frm}}$

Frames L for which η_L is an isomorphism are called spatial, and η_L is then the reflection map from L to spatial frames.

On the other hand, spaces for which ε_X is an homeomorphism are called sober, and by general principles, the full subcategory Sob of Top given by this spaces is then dually equivalent to the full subcategory SpFrm of Frm given by the spatial frames.

$$\begin{tikzcd}
\text{Sob} \arrow[r, shift left, "\mathcal{O}"] & \text{SpFrm} \arrow[l, shift left, "\Sigma"]
\end{tikzcd}$$

Note that we also have a natural equivalence

$$\boxed{\text{Top}(X, \Sigma L) \simeq \text{Frm}(L, \mathcal{O}X)}$$

Pointfree topology

the frame of reals

The fact that Frm is an algebraic category (in particular, one has free frames and quotient frames) permits a procedure familiar from traditional algebra, namely, the definition of a frame by *generators and relations*: take the quotient of the free frame on the given set of generators modulo the congruence generated by the pairs (u, v) for the given relations $u = v$.

So, in the context of pointfree topology the frame of reals may be introduced independent of any notion of real number:

The *frame of reals* is the frame $\mathfrak{L}(\mathbb{R})$ generated by all ordered pairs (p, q), where $p, q \in \mathbb{Q}$, subject to the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$

(R2) $p \leq r < q \leq s \Rightarrow (p, q) \vee (r, s) = (p, s)$

(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$.

(R4) $\bigvee\{(p, q) \big| p, q \in \mathbb{Q}\} = 1$.

Pointfree topology

the frame of reals

The fact that Frm is an algebraic category (in particular, one has free frames and quotient frames) permits a procedure familiar from traditional algebra, namely, the definition of a frame by *generators and relations*: take the quotient of the free frame on the given set of generators modulo the congruence generated by the pairs (u, v) for the given relations $u = v$.

So, in the context of pointfree topology the frame of reals may be introduced independent of any notion of real number:

The *frame of reals* is the frame $\mathfrak{L}(\mathbb{R})$ generated by all ordered pairs (p, q), where $p, q \in \mathbb{Q}$, subject to the following relations:

$$\begin{array}{ll}
\text{(R1)} & (p,q) \wedge (r,s) = (p \vee r, q \wedge s) \\
\text{(R2)} & p \leq r < q \leq s \Rightarrow (p,q) \vee (r,s) = (p,s) \\
\text{(R3)} & (p,q) = \bigvee \{(r,s) \mid p < r < s < q\}. \\
\text{(R4)} & \bigvee \{(p,q) \big| p,q \in \mathbb{Q}\} = 1.
\end{array}$$

Pointfree topology

the frame of reals

The fact that Frm is an algebraic category (in particular, one has free frames and quotient frames) permits a procedure familiar from traditional algebra, namely, the definition of a frame by *generators and relations*: take the quotient of the free frame on the given set of generators modulo the congruence generated by the pairs (u, v) for the given relations $u = v$.

So, in the context of pointfree topology the frame of reals may be introduced independent of any notion of real number:

The *frame of reals* is the frame $\mathfrak{L}(\mathbb{R})$ generated by all ordered pairs (p, q), where $p, q \in \mathbb{Q}$, subject to the following relations:

(R1) $(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$

(R2) $p \leq r < q \leq s \Rightarrow (p,q) \vee (r,s) = (p,s)$

(R3) $(p,q) = \bigvee\{(r,s) \mid p < r < s < q\}$.

(R4) $\bigvee\{(p,q) \big| p,q \in \mathbb{Q}\} = 1$.

Pointfree topology

the frame of reals

The fact that Frm is an algebraic category (in particular, one has free frames and quotient frames) permits a procedure familiar from traditional algebra, namely, the definition of a frame by *generators and relations*: take the quotient of the free frame on the given set of generators modulo the congruence generated by the pairs (u, v) for the given relations $u = v$.

So, in the context of pointfree topology the frame of reals may be introduced independent of any notion of real number:

The *frame of reals* is the frame $\mathfrak{L}(\mathbb{R})$ generated by all ordered pairs (p, q), where $p, q \in \mathbb{Q}$, subject to the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$

(R2) $p \leq r < q \leq s \Rightarrow (p, q) \vee (r, s) = (p, s)$

(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$.

(R4) $\bigvee\{(p, q) \big| p, q \in \mathbb{Q}\} = 1$.

Pointfree topology

the frame of reals

The fact that Frm is an algebraic category (in particular, one has free frames and quotient frames) permits a procedure familiar from traditional algebra, namely, the definition of a frame by *generators and relations*: take the quotient of the free frame on the given set of generators modulo the congruence generated by the pairs (u, v) for the given relations $u = v$.

So, in the context of pointfree topology the frame of reals may be introduced independent of any notion of real number:

The *frame of reals* is the frame $\mathfrak{L}(\mathbb{R})$ generated by all ordered pairs (p, q), where $p, q \in \mathbb{Q}$, subject to the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$

(R2) $p \leq r < q \leq s \Rightarrow (p, q) \vee (r, s) = (p, s)$

(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$.

(R4) $\bigvee\{(p, q) \big| p, q \in \mathbb{Q}\} = 1$.

Pointfree topology

the frame of reals

The fact that Frm is an algebraic category (in particular, one has free frames and quotient frames) permits a procedure familiar from traditional algebra, namely, the definition of a frame by *generators and relations*: take the quotient of the free frame on the given set of generators modulo the congruence generated by the pairs (u, v) for the given relations $u = v$.

So, in the context of pointfree topology the frame of reals may be introduced independent of any notion of real number:

The *frame of reals* is the frame $\mathfrak{L}(\mathbb{R})$ generated by all ordered pairs (p, q), where $p, q \in \mathbb{Q}$, subject to the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$

(R2) $p \leq r < q \leq s \Rightarrow (p, q) \vee (r, s) = (p, s)$

(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$.

(R4) $\bigvee\{(p, q) \big| p, q \in \mathbb{Q}\} = 1$.

Pointfree topology

the frame of reals

The fact that Frm is an algebraic category (in particular, one has free frames and quotient frames) permits a procedure familiar from traditional algebra, namely, the definition of a frame by *generators and relations*: take the quotient of the free frame on the given set of generators modulo the congruence generated by the pairs (u, v) for the given relations $u = v$.

So, in the context of pointfree topology the frame of reals may be introduced independent of any notion of real number:

The *frame of reals* is the frame $\mathfrak{L}(\mathbb{R})$ generated by all ordered pairs (p, q), where $p, q \in \mathbb{Q}$, subject to the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$

(R2) $p \leq r < q \leq s \Rightarrow (p, q) \vee (r, s) = (p, s)$

(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$.

(R4) $\bigvee\{(p, q) \big| p, q \in \mathbb{Q}\} = 1$.

Pointfree topology

the frame of reals

The spectrum of $\mathfrak{L}(\mathbb{R})$ is homeomorphic to the space $\mathbb{R}$ of extended reals endowed with the euclidean topology.

Consequently, the space $\mathbb{R}$ could be defined as $\Sigma\mathfrak{L}(\mathbb{R})$ since the latter construct requires no previous knowledge of $\mathbb{R}$.

P.T. Johnstone,
Stone Spaces,
Cambridge Univ. Press, 1982.

B. Banaschewski,
The real numbers in pointfree topology,
Textos de Matemática, Série B, 12, Univ. de Coimbra, 1997.

Pointfree topology

the frame of reals

The spectrum of $\mathfrak{L}(\mathbb{R})$ is homeomorphic to the space $\mathbb{R}$ of extended reals endowed with the euclidean topology.

Consequently, the space $\mathbb{R}$ could be defined as $\Sigma\mathfrak{L}(\mathbb{R})$ since the latter construct requires no previous knowledge of $\mathbb{R}$.

P.T. Johnstone,
Stone Spaces,
Cambridge Univ. Press, 1982.

B. Banaschewski,
The real numbers in pointfree topology,
Textos de Matemática, Série B, 12, Univ. de Coimbra, 1997.

Pointfree topology

the frame of reals

The spectrum of $\mathfrak{L}(\mathbb{R})$ is homeomorphic to the space $\mathbb{R}$ of extended reals endowed with the euclidean topology.

Consequently, the space $\mathbb{R}$ could be defined as $\Sigma\mathfrak{L}(\mathbb{R})$ since the latter construct requires no previous knowledge of $\mathbb{R}$.

P.T. Johnstone,
Stone Spaces,
Cambridge Univ. Press, 1982.

B. Banaschewski,
The real numbers in pointfree topology,
Textos de Matemática, Série B, 12, Univ. de Coimbra, 1997.

Pointfree topology

the frame of reals

The spectrum of $\mathfrak{L}(\mathbb{R})$ is homeomorphic to the space $\mathbb{R}$ of extended reals endowed with the euclidean topology.

Consequently, the space $\mathbb{R}$ could be defined as $\Sigma\mathfrak{L}(\mathbb{R})$ since the latter construct requires no previous knowledge of $\mathbb{R}$.

P.T. Johnstone,
Stone Spaces,
Cambridge Univ. Press, 1982.

B. Banaschewski,
The real numbers in pointfree topology,
Textos de Matemática, Série B, 12, Univ. de Coimbra, 1997.

Pointfree topology

continuous real functions

Combining the natural isomorphism

$$\boxed{\mathsf{Top}(X, \Sigma L) \simeq \mathsf{Frm}(L, \mathcal{O}X)}$$

for $\mathfrak{L}(\mathbb{R})$ with the homeomorphism $\Sigma\mathfrak{L}(\mathbb{R}) \simeq \mathbb{R}$ one obtains

$$\boxed{\mathsf{Top}(X, \mathbb{R}) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{O}X)}$$

Hence it is conceptually justified to adopt the following:

Definition

A continuous real function on L is a frame homomorphism $\mathfrak{L}(\mathbb{R}) \to L$.

Pointfree topology

continuous real functions

Combining the natural isomorphism

$$\boxed{\mathsf{Top}(X, \Sigma\mathfrak{L}(\mathbb{R})) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{O}X)}$$

for $\mathfrak{L}(\mathbb{R})$ with the homeomorphism $\Sigma\mathfrak{L}(\mathbb{R}) \simeq \mathbb{R}$ one obtains

$$\boxed{\mathsf{Top}(X, \mathbb{R}) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{O}X)}$$

Hence it is conceptually justified to adopt the following:

Definition

A continuous real function on L is a frame homomorphism $\mathfrak{L}(\mathbb{R}) \to L$.

Pointfree topology continuous real functions

Combining the natural isomorphism

$$\boxed{\mathsf{Top}(X, \Sigma\mathfrak{L}(\mathbb{R})) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{O}X)}$$

for $\mathfrak{L}(\mathbb{R})$ with the homeomorphism $\Sigma\mathfrak{L}(\mathbb{R}) \simeq \mathbb{R}$ one obtains

$$\boxed{\mathsf{Top}(X, \mathbb{R}) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{O}X)}$$

Hence it is conceptually justified to adopt the following:

Definition

A continuous real function on L is a frame homomorphism $\mathfrak{L}(\mathbb{R}) \to L$.

Pointfree topology

continuous real functions

Combining the natural isomorphism

$$\boxed{\mathrm{Top}(X, \Sigma\mathfrak{L}(\mathbb{R})) \simeq \mathrm{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{O}X)}$$

for $\mathfrak{L}(\mathbb{R})$ with the homeomorphism $\Sigma\mathfrak{L}(\mathbb{R}) \simeq \mathbb{R}$ one obtains

$$\boxed{\mathrm{Top}(X, \mathbb{R}) \simeq \mathrm{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{O}X)}$$

Hence it is conceptually justified to adopt the following:

Definition

A continuous real function on L is a frame homomorphism $\mathfrak{L}(\mathbb{R}) \to L$.

Pointfree topology

continuous real functions

Combining the natural isomorphism

$$\boxed{\mathrm{Top}(X, \Sigma\mathfrak{L}(\mathbb{R})) \simeq \mathrm{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{O}X)}$$

for $\mathfrak{L}(\mathbb{R})$ with the homeomorphism $\Sigma\mathfrak{L}(\mathbb{R}) \simeq \mathbb{R}$ one obtains

$$\boxed{\mathrm{Top}(X, \mathbb{R}) \simeq \mathrm{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{O}X)}$$

Hence it is conceptually justified to adopt the following:

Definition

A continuous real function on L is a frame homomorphism $\mathfrak{L}(\mathbb{R}) \rightarrow L$.

Pointfree topology

continuous real functions

We shall denote by $\mathrm{c}(L)$ the set of all continuous real functions on L:

$$\boxed{\mathrm{c}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)}$$

Algebraic operations

Let $\langle p, q\rangle = \{r \in \mathbb{Q} : p < r < q\}$, let $\diamond \in \{+, \cdot, \max, \min\}$, and let

$$\langle r, s\rangle \diamond \langle t, u\rangle = \{x \diamond y : x \in \langle r, s\rangle \text{ and } y \in \langle t, u\rangle\}.$$

Given f_1, f_2, $f \in \mathrm{c}(L)$ and $r \in \mathbb{Q}$, we define

$$\begin{aligned}
(f_1 \diamond f_2)(p, q) &= \bigvee\{f_1(r, s) \wedge f_2(t, u) : \langle r, s\rangle \diamond \langle t, u\rangle \subseteq \langle p, q\rangle\}, \\
(-f)(p, q) &= f(-q, -p), \\
\mathbf{r}(p, q) &= \begin{cases} 1 & \text{if } r \in \langle p, q\rangle, \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Pointfree topology

continuous real functions

We shall denote by $\mathrm{c}(L)$ the set of all continuous real functions on L:

$$\boxed{\mathrm{c}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)}$$

Algebraic operations

Let $\langle p, q\rangle = \{r \in \mathbb{Q} : p < r < q\}$, let $\diamond \in \{+, \cdot, \max, \min\}$, and let

$$\langle r, s\rangle \diamond \langle t, u\rangle = \big\{x \diamond y : x \in \langle r, s\rangle \text{ and } y \in \langle t, u\rangle\big\}.$$

Given $f_1,\ f_2,\ f \in \mathrm{c}(L)$ and $r \in \mathbb{Q}$, we define

$$\begin{aligned}
(f_1 \diamond f_2)(p, q) &= \bigvee\big\{f_1(r, s) \wedge f_2(t, u) : \langle r, s\rangle \diamond \langle t, u\rangle \subseteq \langle p, q\rangle\big\},\\
(-f)(p, q) &= f(-q, -p),\\
\mathbf{r}(p, q) &= \begin{cases} 1 & \text{if } r \in \langle p, q\rangle,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Pointfree topology

continuous real functions

We shall denote by $\mathrm{c}(L)$ the set of all continuous real functions on L:

$$\boxed{\mathrm{c}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)}$$

Algebraic operations

Let $\langle p, q\rangle = \{r \in \mathbb{Q} : p < r < q\}$, let $\diamond \in \{+, \cdot, \max, \min\}$, and let

$$\langle r, s\rangle \diamond \langle t, u\rangle = \big\{x \diamond y : x \in \langle r, s\rangle \text{ and } y \in \langle t, u\rangle\big\}.$$

Given $f_1,\ f_2,\ f \in \mathrm{c}(L)$ and $r \in \mathbb{Q}$, we define

$$\begin{aligned}
(f_1 \diamond f_2)(p, q) &= \bigvee\big\{f_1(r, s) \wedge f_2(t, u) : \langle r, s\rangle \diamond \langle t, u\rangle \subseteq \langle p, q\rangle\big\},\\
(-f)(p, q) &= f(-q, -p),\\
\mathbf{r}(p, q) &= \begin{cases} 1 & \text{if } r \in \langle p, q\rangle,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Pointfree topology

continuous real functions

We shall denote by $\mathrm{c}(L)$ the set of all continuous real functions on L:

$$\boxed{\mathrm{c}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)}$$

Algebraic operations

Let $\langle p, q\rangle = \{r \in \mathbb{Q} : p < r < q\}$, let $\diamond \in \{+, \cdot, \max, \min\}$, and let

$$\langle r, s\rangle \diamond \langle t, u\rangle = \big\{x \diamond y : x \in \langle r, s\rangle \text{ and } y \in \langle t, u\rangle\big\}.$$

Given f_1, f_2, $f \in \mathrm{c}(L)$ and $r \in \mathbb{Q}$, we define

$$\begin{aligned}
(f_1 \diamond f_2)(p, q) &= \bigvee\big\{f_1(r, s) \wedge f_2(t, u) : \langle r, s\rangle \diamond \langle t, u\rangle \subseteq \langle p, q\rangle\big\}, \\
(-f)(p, q) &= f(-q, -p), \\
\mathbf{r}(p, q) &= \begin{cases} 1 & \text{if } r \in \langle p, q\rangle, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Pointfree topology

continuous real functions

We shall denote by $\mathrm{c}(L)$ the set of all continuous real functions on L:

$$\boxed{\mathrm{c}(L) = \mathrm{Frm}(\mathfrak{L}(\mathbb{R}), L)}$$

Algebraic operations

Let $\langle p, q\rangle = \{r \in \mathbb{Q} : p < r < q\}$, let $\diamond \in \{+, \cdot, \max, \min\}$, and let

$$\langle r, s\rangle \diamond \langle t, u\rangle = \big\{x \diamond y : x \in \langle r, s\rangle \text{ and } y \in \langle t, u\rangle\big\}.$$

Given f_1, f_2, $f \in \mathrm{c}(L)$ and $r \in \mathbb{Q}$, we define

$$\begin{aligned} (f_1 \diamond f_2)(p, q) &= \bigvee\big\{f_1(r, s) \wedge f_2(t, u) : \langle r, s\rangle \diamond \langle t, u\rangle \subseteq \langle p, q\rangle\big\}, \\ (-f)(p, q) &= f(-q, -p), \\ \mathbf{r}(p, q) &= \begin{cases} 1 & \text{if } r \in \langle p, q\rangle, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Pointfree topology

continuous real functions

We shall denote by $\mathrm{c}(L)$ the set of all continuous real functions on L:

$$\boxed{\mathrm{c}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)}$$

Algebraic operations

Let $\langle p, q\rangle = \{r \in \mathbb{Q} : p < r < q\}$, let $\diamond \in \{+, \cdot, \max, \min\}$, and let

$$\langle r, s\rangle \diamond \langle t, u\rangle = \big\{x \diamond y : x \in \langle r, s\rangle \text{ and } y \in \langle t, u\rangle\big\}.$$

Given f_1, f_2, $f \in \mathrm{c}(L)$ and $r \in \mathbb{Q}$, we define

$$\begin{aligned} (f_1 \diamond f_2)(p, q) &= \bigvee \big\{f_1(r, s) \wedge f_2(t, u) : \langle r, s\rangle \diamond \langle t, u\rangle \subseteq \langle p, q\rangle\big\}, \\ (-f)(p, q) &= f(-q, -p), \\ \mathbf{r}(p, q) &= \begin{cases} 1 & \text{if } r \in \langle p, q\rangle, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Pointfree topology

continuous real functions

These operations satisfy all the lattice-ordered ring axioms in $\mathbb{Q}$ so that $(\mathrm{c}(L), +, \cdot, \leq)$ becomes a lattice-ordered ring with unit **1**.

We also have the following descriptions of the partial order:

$$\begin{aligned} f_1 \leq f_2 \quad &\Leftrightarrow \quad f_1(p,-) \leq f_2(p,-) \quad \text{for all } p \in \mathbb{Q} \\ &\Leftrightarrow \quad f_2(-,q) \leq f_1(-,q) \quad \text{for all } q \in \mathbb{Q} \\ &\Leftrightarrow \quad f_1(r,-) \wedge f_2(-,r) = 0 \quad \text{for all } r \in \mathbb{Q} \\ &\Leftrightarrow \quad f_2(p,-) \vee f_1(-,q) = 1 \quad \text{for all } p < q \in \mathbb{Q}. \end{aligned}$$

B. Banaschewski,
The real numbers in pointfree topology
Textos de Matemática, Série B, 12, Univ. de Coimbra, 1997.

Pointfree topology

continuous real functions

These operations satisfy all the lattice-ordered ring axioms in $\mathbb{Q}$ so that $(\mathrm{c}(L), +, \cdot, \leq)$ becomes a lattice-ordered ring with unit **1**.

We also have the following descriptions of the partial order:

$$\begin{aligned} f_1 \leq f_2 \quad &\Leftrightarrow \quad f_1(p,-) \leq f_2(p,-) \quad \text{for all } p \in \mathbb{Q} \\ &\Leftrightarrow \quad f_2(-,q) \leq f_1(-,q) \quad \text{for all } q \in \mathbb{Q} \\ &\Leftrightarrow \quad f_1(r,-) \wedge f_2(-,r) = 0 \quad \text{for all } r \in \mathbb{Q} \\ &\Leftrightarrow \quad f_2(p,-) \vee f_1(-,q) = 1 \quad \text{for all } p < q \in \mathbb{Q}. \end{aligned}$$

B. Banaschewski,
The real numbers in pointfree topology
Textos de Matemática, Série B, 12, Univ. de Coimbra, 1997.

Semicontinuity

semicontinuous real functions

Let $\mathfrak{L}_l(\mathbb{R})$ and $\mathfrak{L}_u(\mathbb{R})$ denote the subframes generated by elements:

$$(-,q) := \bigvee_{p\in\mathbb{Q}} (p,q) \quad \text{and} \quad (p,-) := \bigvee_{q\in\mathbb{Q}} (p,q).$$

One is tempted to follow the lines of the previous definition:

Definition

(1) An upper semicontinuous real function on L is a frame homomorphism $\mathfrak{L}_l(\mathbb{R}) \to L$.

(2) A lower semicontinuous real function on L is a frame homomorphism $\mathfrak{L}_u(\mathbb{R}) \to L$.

Y.-M. Li and G.-J. Wang,
Localic Katětov-Tong insertion theorem and localic Tietze extension theorem,
Comment. Math. Univ. Carolinae, 38 (1997) 801–814.

Semicontinuity

semicontinuous real functions

Let $\mathfrak{L}_l(\mathbb{R})$ and $\mathfrak{L}_u(\mathbb{R})$ denote the subframes generated by elements:

$$(-,q) := \bigvee_{p\in\mathbb{Q}} (p,q) \quad \text{and} \quad (p,-) := \bigvee_{q\in\mathbb{Q}} (p,q).$$

One is tempted to follow the lines of the previous definition:

Definition

(1) An upper semicontinuous real function on L is a frame homomorphism $\mathfrak{L}_l(\mathbb{R}) \to L$.

(2) A lower semicontinuous real function on L is a frame homomorphism $\mathfrak{L}_u(\mathbb{R}) \to L$.

Y.-M. Li and G.-J. Wang,
Localic Katětov-Tong insertion theorem and localic Tietze extension theorem,
Comment. Math. Univ. Carolinae, 38 (1997) 801–814.

Semicontinuity

semicontinuous real functions

Let $\mathfrak{L}_l(\mathbb{R})$ and $\mathfrak{L}_u(\mathbb{R})$ denote the subframes generated by elements:

$$(-,q) := \bigvee_{p\in\mathbb{Q}} (p,q) \quad \text{and} \quad (p,-) := \bigvee_{q\in\mathbb{Q}} (p,q).$$

One is tempted to follow the lines of the previous definition:

Definition

(1) An upper semicontinuous real function on L is a frame homomorphism $\mathfrak{L}_l(\mathbb{R}) \to L$.

(2) A lower semicontinuous real function on L is a frame homomorphism $\mathfrak{L}_u(\mathbb{R}) \to L$.

Y.-M. Li and G.-J. Wang,
Localic Katětov-Tong insertion theorem and localic Tietze extension theorem,
Comment. Math. Univ. Carolinae, 38 (1997) 801–814.

Semicontinuity

semicontinuous real functions

Let $\mathfrak{L}_l(\mathbb{R})$ and $\mathfrak{L}_u(\mathbb{R})$ denote the subframes generated by elements:

$$(-,q) := \bigvee_{p\in\mathbb{Q}} (p,q) \quad \text{and} \quad (p,-) := \bigvee_{q\in\mathbb{Q}} (p,q).$$

One is tempted to follow the lines of the previous definition:

Definition

(1) An upper semicontinuous real function on L is a frame homomorphism $\mathfrak{L}_l(\mathbb{R}) \to L$.

(2) A lower semicontinuous real function on L is a frame homomorphism $\mathfrak{L}_u(\mathbb{R}) \to L$.

Y.-M. Li and G.-J. Wang,
Localic Katětov-Tong insertion theorem and localic Tietze extension theorem,
Comment. Math. Univ. Carolinae, 38 (1997) 801–814.

Semicontinuity

semicontinuous real functions

But things become more complicated because $(\mathbb{R}, \mathcal{T}_l)$ fails to be sober. Indeed, the spectrum $\Sigma\mathfrak{L}_l(\mathbb{R})$ of $\mathfrak{L}_l(\mathbb{R})$ is homeomorphic to the space $(\mathbb{R}_{-\infty} = \mathbb{R} \cup \{-\infty\}, \mathcal{T}_l)$. Hence

$$\boxed{\mathrm{Top}(X, (\mathbb{R}, \mathcal{T}_l)) \subset \mathrm{Top}(X, (\mathbb{R}_{-\infty}, \mathcal{T}_l)) \simeq \mathrm{Frm}(\mathfrak{L}_l(\mathbb{R}), \mathcal{O}X).}$$

The frame homomorphisms $f \in \mathrm{Frm}(\mathfrak{L}_l(\mathbb{R}), \mathcal{O}X)$ corresponding to continue maps in $\mathrm{Top}(X, (\mathbb{R}, \mathcal{T}_l))$ are precisely those satisfying the additional condition:

$$\boxed{\bigvee_{q \in \mathbb{Q}} \mathfrak{o}(f(-, q)) = 1}$$

J. Gutiérrez García and J. Picado
On the algebraic representation of semicontinuity
Journal of Pure and Applied Algebra, 210 (2007) 299–306.

Semicontinuity

semicontinuous real functions

But things become more complicated because $(\mathbb{R},\mathcal{T}_l)$ fails to be sober. Indeed, the spectrum $\Sigma\mathfrak{L}_l(\mathbb{R})$ of $\mathfrak{L}_l(\mathbb{R})$ is homeomorphic to the space $(\mathbb{R}_{-\infty}=\mathbb{R}\cup\{-\infty\},\mathcal{T}_l)$. Hence

$$\boxed{\mathrm{Top}(X,(\mathbb{R},\mathcal{T}_l)) \subset \mathrm{Top}(X,(\mathbb{R}_{-\infty},\mathcal{T}_l)) \simeq \mathrm{Frm}(\mathfrak{L}_l(\mathbb{R}),\mathcal{O}X).}$$

The frame homomorphisms $f\in\mathrm{Frm}(\mathfrak{L}_l(\mathbb{R}),\mathcal{O}X)$ corresponding to continue maps in $\mathrm{Top}(X,(\mathbb{R},\mathcal{T}_l))$ are precisely those satisfying the additional condition:

$$\boxed{\bigvee_{q\in\mathbb{Q}} \mathfrak{o}(f(-,q)) = 1}$$

J. Gutiérrez García and J. Picado
On the algebraic representation of semicontinuity
Journal of Pure and Applied Algebra, 210 (2007) 299–306.

Semicontinuity

semicontinuous real functions

But things become more complicated because $(\mathbb{R}, \mathcal{T}_l)$ fails to be sober. Indeed, the spectrum $\Sigma\mathfrak{L}_l(\mathbb{R})$ of $\mathfrak{L}_l(\mathbb{R})$ is homeomorphic to the space $(\mathbb{R}_{-\infty} = \mathbb{R} \cup \{-\infty\}, \mathcal{T}_l)$. Hence

$$\boxed{\mathrm{Top}(X, (\mathbb{R}, \mathcal{T}_l)) \subset \mathrm{Top}(X, (\mathbb{R}_{-\infty}, \mathcal{T}_l)) \simeq \mathrm{Frm}(\mathfrak{L}_l(\mathbb{R}), \mathcal{O}X).}$$

The frame homomorphisms $f \in \mathrm{Frm}(\mathfrak{L}_l(\mathbb{R}), \mathcal{O}X)$ corresponding to continue maps in $\mathrm{Top}(X, (\mathbb{R}, \mathcal{T}_l))$ are precisely those satisfying the additional condition:

$$\boxed{\bigvee_{q \in \mathbb{Q}} \mathfrak{o}(f(-, q)) = 1}$$

J. Gutiérrez García and J. Picado
On the algebraic representation of semicontinuity
Journal of Pure and Applied Algebra, 210 (2007) 299–306.

Semicontinuity

semicontinuous real functions

But things become more complicated because $(\mathbb{R}, \mathcal{T}_l)$ fails to be sober. Indeed, the spectrum $\Sigma\mathfrak{L}_l(\mathbb{R})$ of $\mathfrak{L}_l(\mathbb{R})$ is homeomorphic to the space $(\mathbb{R}_{-\infty} = \mathbb{R} \cup \{-\infty\}, \mathcal{T}_l)$. Hence

$$\boxed{\mathrm{Top}(X, (\mathbb{R}, \mathcal{T}_l)) \subset \mathrm{Top}(X, (\mathbb{R}_{-\infty}, \mathcal{T}_l)) \simeq \mathrm{Frm}(\mathfrak{L}_l(\mathbb{R}), \mathcal{O}X).}$$

The frame homomorphisms $f \in \mathrm{Frm}(\mathfrak{L}_l(\mathbb{R}), \mathcal{O}X)$ corresponding to continue maps in $\mathrm{Top}(X, (\mathbb{R}, \mathcal{T}_l))$ are precisely those satisfying the additional condition:

$$\boxed{\bigvee_{q\in\mathbb{Q}} \mathfrak{o}(f(-,q)) = 1}$$

J. Gutiérrez García and J. Picado
On the algebraic representation of semicontinuity
Journal of Pure and Applied Algebra, 210 (2007) 299–306.

	Top	Frm
continuous	$f : X \to (\mathbb{R}, \mathcal{T}_e)$	$h : \mathfrak{L}(\mathbb{R}) \to L$
usc	$f : X \to (\mathbb{R}, \mathcal{T}_l)$	$h : \mathfrak{L}_l(\mathbb{R}) \to L$ satisfying$(\ldots)$
lsc	$f : X \to (\mathbb{R}, \mathcal{T}_u)$	$h : \mathfrak{L}_u(\mathbb{R}) \to L$ satisfying$(\ldots)$
$\boxed{C(X) = USC(X) \cap LSC(X)}$		???

$$\boxed{\mathrm{Top}(X, \mathcal{T}_e) \simeq \mathrm{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{O}X)}$$

B. Banaschewski,
The real numbers in pointfree topology
Textos de Matemática, Série B, 12, Univ. de Coimbra, 1997.

	Top	Frm
continuous	$f : X \longrightarrow (\mathbb{R}, \mathcal{T}_e)$	$h : \mathfrak{L}(\mathbb{R}) \to L$
usc	$f : X \longrightarrow (\mathbb{R}, \mathcal{T}_l)$	$h : \mathfrak{L}_l(\mathbb{R}) \to L$ *satisfying*$(\dots)$
lsc	$f : X \longrightarrow (\mathbb{R}, \mathcal{T}_u)$	$h : \mathfrak{L}_u(\mathbb{R}) \to L$ *satisfying*$(\dots)$
	$\boxed{C(X) = USC(X) \cap LSC(X)}$	???

Y.-M. Li and G.-J. Wang,
Localic Katětov-Tong insertion theorem and localic Tietze extension theorem,
Comment. Math. Univ. Carolinae, 38 (1997) 801–814.

	Top	**Frm**
continuous	$f : X \to (\mathbb{R}, \mathcal{T}_e)$	$h : \mathfrak{L}(\mathbb{R}) \to L$
usc	$f : X \to (\mathbb{R}, \mathcal{T}_l)$	$h : \mathfrak{L}_l(\mathbb{R}) \to L$ $\quad satisfying(\ldots)$
lsc	$f : X \to (\mathbb{R}, \mathcal{T}_u)$	$h : \mathfrak{L}_u(\mathbb{R}) \to L$ $\quad satisfying(\ldots)$
	$\boxed{C(X) = USC(X) \cap LSC(X)}$	???

$$\boxed{\mathrm{Top}(X, \mathcal{T}_l) \not\approx \mathrm{Frm}(\mathfrak{L}_l(\mathbb{R}), \mathcal{O}X)}\,!!!$$

J. Gutiérrez García and J. Picado
On the algebraic representation of semicontinuity
Journal of Pure and Applied Algebra, 210 (2007) 299–306.

	Top	**Frm**
continuous	$f : X \to (\mathbb{R}, \mathcal{T}_e)$	$h : \mathfrak{L}(\mathbb{R}) \to L$
usc	$f : X \to (\mathbb{R}, \mathcal{T}_l)$	$h : \mathfrak{L}_l(\mathbb{R}) \to L \quad satisfying(\dots)$
lsc	$f : X \to (\mathbb{R}, \mathcal{T}_u)$	$h : \mathfrak{L}_u(\mathbb{R}) \to L \quad satisfying(\dots)$
$\boxed{C(X) = USC(X) \cap LSC(X)}$		???

J. Gutiérrez García and J. Picado
On the algebraic representation of semicontinuity
Journal of Pure and Applied Algebra, 210 (2007) 299–306.

	Top	**Frm**
continuous	$f : X \to (\mathbb{R}, \mathcal{T}_e)$	$h : \mathfrak{L}(\mathbb{R}) \to L$
usc	$f : X \to (\mathbb{R}, \mathcal{T}_l)$	$h : \mathfrak{L}_l(\mathbb{R}) \to L$ *satisfying*$(\dots)$
lsc	$f : X \to (\mathbb{R}, \mathcal{T}_u)$	$h : \mathfrak{L}_u(\mathbb{R}) \to L$ *satisfying*$(\dots)$
	$\boxed{C(X) = USC(X) \cap LSC(X)}$	???

	Top	Frm
continuous	$f : X \to (\mathbb{R}, \mathcal{T}_e)$	$h : \mathfrak{L}(\mathbb{R}) \to L$
usc	$f : X \to (\mathbb{R}, \mathcal{T}_l)$	$h : \mathfrak{L}_l(\mathbb{R}) \to L \quad satisfying(\dots)$
lsc	$f : X \to (\mathbb{R}, \mathcal{T}_u)$	$h : \mathfrak{L}_u(\mathbb{R}) \to L \quad satisfying(\dots)$
$\boxed{C(X) = USC(X) \cap LSC(X)}$		???

	Top	Frm
continuous	$f : X \to (\mathbb{R}, \mathcal{T}_e)$	$h : \mathfrak{L}(\mathbb{R}) \to L$
usc	$f : X \to (\mathbb{R}, \mathcal{T}_l)$	$h : \mathfrak{L}_l(\mathbb{R}) \to L$ *satisfying*$(\dots)$
lsc	$f : X \to (\mathbb{R}, \mathcal{T}_u)$	$h : \mathfrak{L}_u(\mathbb{R}) \to L$ *satisfying*$(\dots)$
	$\boxed{C(X) = USC(X) \cap LSC(X)}$	???

(Q1) $\boxed{\text{How to remedy this?}}$

Top	Frm
Every $f : X \to \mathbb{R}$ admits lsc and usc regularizations	???

(Q2) How can we speak about general localic real functions?

Top	Frm
Every $f : X \to \mathbb{R}$ admits lsc and usc regularizations	???

(Q2) How can we speak about general localic real functions?

Top

Every $f : X \to \mathbb{R}$ admits lsc and usc regularizations

Frm

???

(Q2) How can we speak about general localic real functions?

Question 1

Is it possible to extend the treatment of continuous functions in the sense of Banaschewski to obtain nice algebraic descriptions of upper and lower semicontinuity?

Question 2

Which is the pointfree (localic) counterpart of the lattice-ordered ring $\mathbb{R}^X$?

Question 1

Is it possible to extend the treatment of continuous functions in the sense of Banaschewski to obtain nice algebraic descriptions of upper and lower semicontinuity?

Question 2

Which is the pointfree (localic) counterpart of the lattice-ordered ring $\mathbb{R}^X$?

J. Gutiérrez García, J. Picado,
On the algebraic representation of semicontinuity
Journal of Pure and Applied Algebra, 210 (2007) 299–306.

Question 1

Is it possible to extend the treatment of continuous functions in the sense of Banaschewski to obtain nice algebraic descriptions of upper and lower semicontinuity?

Question 2

Which is the pointfree (localic) counterpart of the lattice-ordered ring $\mathbb{R}^X$?

J. Gutiérrez García, J. Picado,
On the algebraic representation of semicontinuity
Journal of Pure and Applied Algebra, 210 (2007) 299–306.

Question 1

Is it possible to extend the treatment of continuous functions in the sense of Banaschewski to obtain nice algebraic descriptions of upper and lower semicontinuity?

Question 2

Which is the pointfree (localic) counterpart of the lattice-ordered ring $\mathbb{R}^X$?

J. Gutiérrez García, J. Picado,
On the algebraic representation of semicontinuity
Journal of Pure and Applied Algebra, 210 (2007) 299–306.

J. Gutiérrez García, T Kubiak, J. Picado,
Localic real-valued functions: a general setting
Journal of Pure and Applied Algebra, 213 (2009) 1064–1074.

Pointfree topology

sublocales (generalized subspaces)

The quotients in Frm (equivalently, the subobjects in the dual category Loc) have been described in several equivalent ways in the literature:

- as sublocale maps (i.e. onto frame homomorphisms),
- congruences,
- nuclei
- sublocale sets.

We follow the latter approach because, in our opinion, it has revealed to be the more intuitive and the easiest to work with:

A subset $S \subseteq L$ is a *sublocale* of L if it satisfies the following:

(S1) For every $A \subseteq S$, $\bigwedge A \in S$,

(S2) For every $a \in L$ and $s \in S$, $a \to s \in S$.

Pointfree topology sublocales (generalized subspaces)

The quotients in Frm (equivalently, the subobjects in the dual category Loc) have been described in several equivalent ways in the literature:

- as sublocale maps (i.e. onto frame homomorphisms),
- congruences,
- nuclei
- sublocale sets.

We follow the latter approach because, in our opinion, it has revealed to be the more intuitive and the easiest to work with:

A subset $S \subseteq L$ is a *sublocale* of L if it satisfies the following:

(S1) For every $A \subseteq S$, $\bigwedge A \in S$,

(S2) For every $a \in L$ and $s \in S$, $a \to s \in S$.

Pointfree topology

sublocales (generalized subspaces)

The quotients in Frm (equivalently, the subobjects in the dual category Loc) have been described in several equivalent ways in the literature:

- as sublocale maps (i.e. onto frame homomorphisms),
- congruences,
- nuclei
- sublocale sets.

We follow the latter approach because, in our opinion, it has revealed to be the more intuitive and the easiest to work with:

A subset $S \subseteq L$ is a *sublocale* of L if it satisfies the following:

(S1) For every $A \subseteq S$, $\bigwedge A \in S$,

(S2) For every $a \in L$ and $s \in S$, $a \rightarrow s \in S$.

Pointfree topology

sublocales (generalized subspaces)

The quotients in Frm (equivalently, the subobjects in the dual category Loc) have been described in several equivalent ways in the literature:

- as sublocale maps (i.e. onto frame homomorphisms),
- congruences,
- nuclei
- sublocale sets.

We follow the latter approach because, in our opinion, it has revealed to be the more intuitive and the easiest to work with:

A subset $S \subseteq L$ is a *sublocale* of L if it satisfies the following:

(S1) For every $A \subseteq S$, $\bigwedge A \in S$,

(S2) For every $a \in L$ and $s \in S$, $a \to s \in S$.

Pointfree topology

sublocales (generalized subspaces)

The quotients in Frm (equivalently, the subobjects in the dual category Loc) have been described in several equivalent ways in the literature:

- as sublocale maps (i.e. onto frame homomorphisms),
- congruences,
- nuclei
- sublocale sets.

We follow the latter approach because, in our opinion, it has revealed to be the more intuitive and the easiest to work with:

A subset $S \subseteq L$ is a *sublocale* of L if it satisfies the following:

(S1) For every $A \subseteq S$, $\bigwedge A \in S$,

(S2) For every $a \in L$ and $s \in S$, $a \to s \in S$.

Pointfree topology

sublocales (generalized subspaces)

Since the intersection of sublocales is again a sublocale, the set $\mathcal{S}L$ of all sublocales is a complete lattice under inclusion.

For convenience, we shall deal with the opposite order, i.e.:

$$S_1 \leq S_2 \quad \Longleftrightarrow \quad S_1 \supseteq S_2.$$

$(\mathcal{S}L, \leq)$ is a frame, in which $\{1\}$ is the top and L is the bottom.

Further, given $\{S_i \in \mathcal{S}L : i \in I\}$, we have

$$\bigvee_{i \in I} S_i = \bigcap_{i \in I} S_i \quad \text{and} \quad \bigwedge_{i \in I} S_i = \{\bigwedge A : A \subseteq \bigcup_{i \in I} S_i\}.$$

Pointfree topology

sublocales (generalized subspaces)

Since the intersection of sublocales is again a sublocale, the set $\mathcal{S}L$ of all sublocales is a complete lattice under inclusion.

For convenience, we shall deal with the opposite order, i.e.:

$$S_1 \leq S_2 \quad \Longleftrightarrow \quad S_1 \supseteq S_2.$$

$(\mathcal{S}L, \leq)$ is a frame, in which $\{1\}$ is the top and L is the bottom.

Further, given $\{S_i \in \mathcal{S}L : i \in I\}$, we have

$$\bigvee_{i\in I} S_i = \bigcap_{i\in I} S_i \quad \text{and} \quad \bigwedge_{i\in I} S_i = \{\bigwedge A : A \subseteq \bigcup_{i\in I} S_i\}.$$

Pointfree topology

sublocales (generalized subspaces)

Important examples of sublocales are the *open* and *closed* ones:

$$\mathfrak{o}(a) = \{a \to b : b \in L\} \quad \text{and} \quad \mathfrak{c}(a) = {\uparrow}a = \{b \in L : a \leq b\}.$$

Open and closed sublocales are complemented and

$$\neg\mathfrak{o}(a) = \mathfrak{c}(a) \quad \text{for each } a \in L.$$

Also, for each $a_i, a, b \in L$:

$$\bigvee_{i \in I} \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee_{i \in I} a_i), \qquad \mathfrak{c}(a) \wedge \mathfrak{c}(b) = \mathfrak{c}(a \wedge b),$$

$$\bigwedge_{i \in I} \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee_{i \in I} a_i) \quad \text{and} \quad \mathfrak{o}(a) \vee \mathfrak{o}(b) = \mathfrak{o}(a \wedge b)$$

Thus, $\mathfrak{c} : L \longrightarrow \mathcal{S}L$ is an embedding from L into $\mathfrak{c}(L) = \{\mathfrak{c}(a) : a \in L\}$ whereas $\mathfrak{o} : L \longrightarrow \mathcal{S}L$ is a dual lattice embedding taking finite meets to joins and arbitrary joins to meets.

Pointfree topology

sublocales (generalized subspaces)

Important examples of sublocales are the *open* and *closed* ones:

$$\mathfrak{o}(a) = \{a \to b : b \in L\} \quad \text{and} \quad \mathfrak{c}(a) = {\uparrow} a = \{b \in L : a \leq b\}.$$

Open and closed sublocales are complemented and

$$\neg\mathfrak{o}(a) = \mathfrak{c}(a) \quad \text{for each } a \in L.$$

Also, for each $a_i, a, b \in L$:

$$\bigvee_{i \in I} \mathfrak{c}(a_i) = \mathfrak{c}\Big(\bigvee_{i \in I} a_i\Big), \qquad \mathfrak{c}(a) \wedge \mathfrak{c}(b) = \mathfrak{c}(a \wedge b),$$

$$\bigwedge_{i \in I} \mathfrak{o}(a_i) = \mathfrak{o}\Big(\bigvee_{i \in I} a_i\Big) \quad \text{and} \quad \mathfrak{o}(a) \vee \mathfrak{o}(b) = \mathfrak{o}(a \wedge b)$$

Thus, $\mathfrak{c} : L \longrightarrow \mathcal{S}L$ is an embedding from L into $\mathfrak{c}(L) = \{\mathfrak{c}(a) : a \in L\}$ whereas $\mathfrak{o} : L \longrightarrow \mathcal{S}L$ is a dual lattice embedding taking finite meets to joins and arbitrary joins to meets.

Pointfree topology

sublocales (generalized subspaces)

Important examples of sublocales are the *open* and *closed* ones:

$$\mathfrak{o}(a) = \{a \to b : b \in L\} \quad \text{and} \quad \mathfrak{c}(a) = {\uparrow} a = \{b \in L : a \leq b\}.$$

Open and closed sublocales are complemented and

$$\neg\mathfrak{o}(a) = \mathfrak{c}(a) \quad \text{for each } a \in L.$$

Also, for each $a_i, a, b \in L$:

$$\bigvee_{i\in I} \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee_{i\in I} a_i), \qquad \mathfrak{c}(a) \wedge \mathfrak{c}(b) = \mathfrak{c}(a \wedge b),$$

$$\bigwedge_{i\in I} \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee_{i\in I} a_i) \quad \text{and} \quad \mathfrak{o}(a) \vee \mathfrak{o}(b) = \mathfrak{o}(a \wedge b)$$

Thus, $\mathfrak{c} : L \longrightarrow \mathcal{S}L$ is an embedding from L into $\mathfrak{c}(L) = \{\mathfrak{c}(a) : a \in L\}$ whereas $\mathfrak{o} : L \longrightarrow \mathcal{S}L$ is a dual lattice embedding taking finite meets to joins and arbitrary joins to meets.

Pointfree topology

sublocales (generalized subspaces)

Important examples of sublocales are the *open* and *closed* ones:

$$\mathfrak{o}(a) = \{a \to b : b \in L\} \quad \text{and} \quad \mathfrak{c}(a) = \uparrow a = \{b \in L : a \leq b\}.$$

Open and closed sublocales are complemented and

$$\neg\mathfrak{o}(a) = \mathfrak{c}(a) \quad \text{for each } a \in L.$$

Also, for each $a_i, a, b \in L$:

$$\begin{aligned} \bigvee_{i\in I} \mathfrak{c}(a_i) &= \mathfrak{c}(\bigvee_{i\in I} a_i), & \qquad & \mathfrak{c}(a) \wedge \mathfrak{c}(b) = \mathfrak{c}(a \wedge b), \\ \bigwedge_{i\in I} \mathfrak{o}(a_i) &= \mathfrak{o}(\bigvee_{i\in I} a_i) & \text{and} & \quad \mathfrak{o}(a) \vee \mathfrak{o}(b) = \mathfrak{o}(a \wedge b) \end{aligned}$$

Thus, $\mathfrak{c} : L \longrightarrow \mathcal{S}L$ is an embedding from L into $\mathfrak{c}(L) = \{\mathfrak{c}(a) : a \in L\}$ whereas $\mathfrak{o} : L \longrightarrow \mathcal{S}L$ is a dual lattice embedding taking finite meets to joins and arbitrary joins to meets.

Pointfree topology

sublocales (generalized subspaces)

For each sublocale S the *closure* and the *interior* of S are given by:

$$\overline{S} = \bigvee\{\mathfrak{c}(a) : \mathfrak{c}(a) \leq S\} \quad \text{and} \quad S^\circ = \bigwedge\{\mathfrak{o}(a) : S \leq \mathfrak{o}(a)\}.$$

In particular $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$ and $\overset{\circ}{\mathfrak{c}(a)} = \mathfrak{o}(a^*)$.

Also, for each $S, T \in \mathcal{S}L$:

(1) $\overline{\{1\}} = \{1\}$, $\overset{\circ}{L} = L$, $\overline{S} \leq S \leq \overset{\circ}{S}$, $\overline{\overline{S}} = \overline{S}$ and $\overset{\circ}{\overset{\circ}{S}} = \overset{\circ}{S}$.

(2) $\overline{S \wedge T} = \overline{S} \wedge \overline{T}$ and $\overset{\circ}{\overbrace{S \vee T}} = \overset{\circ}{S} \vee \overset{\circ}{T}$.

J. Picado and A. Pultr,
Sublocale sets and sublocale lattices,
Arch. Math. (Brno), 42 (2006) 409–418.

Pointfree topology

sublocales (generalized subspaces)

For each sublocale S the *closure* and the *interior* of S are given by:

$$\overline{S} = \bigvee\{\mathfrak{c}(a) : \mathfrak{c}(a) \leq S\} \quad \text{and} \quad S^\circ = \bigwedge\{\mathfrak{o}(a) : S \leq \mathfrak{o}(a)\}.$$

In particular $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$ and $\overset{\circ}{\mathfrak{c}(a)} = \mathfrak{o}(a^*)$.

Also, for each $S, T \in \mathcal{S}L$:

(1) $\overline{\{1\}} = \{1\}$, $\mathring{L} = L$, $\overline{S} \leq S \leq \mathring{S}$, $\overline{\overline{S}} = \overline{S}$ and $\mathring{\mathring{S}} = \mathring{S}$.

(2) $\overline{S \wedge T} = \overline{S} \wedge \overline{T}$ and $\overset{\circ}{\widehat{S \vee T}} = \mathring{S} \vee \mathring{T}$.

J. Picado and A. Pultr,
Sublocale sets and sublocale lattices,
Arch. Math. (Brno), 42 (2006) 409–418.

Pointfree topology

sublocales (generalized subspaces)

For each sublocale S the *closure* and the *interior* of S are given by:

$$\overline{S} = \bigvee\{\mathfrak{c}(a) : \mathfrak{c}(a) \leq S\} \quad \text{and} \quad S^\circ = \bigwedge\{\mathfrak{o}(a) : S \leq \mathfrak{o}(a)\}.$$

In particular $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$ and $\overset{\circ}{\mathfrak{c}(a)} = \mathfrak{o}(a^*)$.

Also, for each $S, T \in \mathcal{S}L$:

(1) $\overline{\{1\}} = \{1\}$, $\overset{\circ}{L} = L$, $\overline{S} \leq S \leq \overset{\circ}{S}$, $\overline{\overline{S}} = \overline{S}$ and $\overset{\circ}{\overset{\circ}{S}} = \overset{\circ}{S}$.

(2) $\overline{S \wedge T} = \overline{S} \wedge \overline{T}$ and $\overset{\circ}{\overbrace{S \vee T}} = \overset{\circ}{S} \vee \overset{\circ}{T}$.

J. Picado and A. Pultr,
Sublocale sets and sublocale lattices,
Arch. Math. (Brno), 42 (2006) 409–418.

Pointfree topology

sublocales (generalized subspaces)

For each sublocale S the *closure* and the *interior* of S are given by:

$$\overline{S} = \bigvee\{\mathfrak{c}(a) : \mathfrak{c}(a) \leq S\} \quad \text{and} \quad S^\circ = \bigwedge\{\mathfrak{o}(a) : S \leq \mathfrak{o}(a)\}.$$

In particular $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$ and $\overset{\circ}{\mathfrak{c}(a)} = \mathfrak{o}(a^*)$.

Also, for each $S, T \in \mathcal{S}L$:

(1) $\overline{\{1\}} = \{1\}$, $\overset{\circ}{L} = L$, $\overline{S} \leq S \leq \overset{\circ}{S}$, $\overline{\overline{S}} = \overline{S}$ and $\overset{\circ}{\overset{\circ}{S}} = \overset{\circ}{S}$.

(2) $\overline{S \wedge T} = \overline{S} \wedge \overline{T}$ and $\overset{\circ}{\overbrace{S \vee T}} = \overset{\circ}{S} \vee \overset{\circ}{T}$.

J. Picado and A. Pultr,
Sublocale sets and sublocale lattices,
Arch. Math. (Brno), 42 (2006) 409–418.

Localic real-valued functions

In order to motivate the idea, we first recall the isomorphism

$$\boxed{\mathsf{Top}(X, (\mathbb{R}, \mathcal{T}_e)) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{O}X)}$$

Now, if we observe that the set $\mathbb{R}^X$ is in an obvious bijection with $\mathsf{Top}((X, \mathcal{P}(X)), (\mathbb{R}, \mathcal{T})$ where $\mathcal{T}$ is *any* topology on $\mathbb{R}$, we would, in particular, have a bijection

$$\boxed{\mathbb{R}^X \simeq \mathsf{Top}((X, \mathcal{P}(X)), (\mathbb{R}, \mathcal{T}_e)) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{P}(X))}$$

Therefore,

$$\boxed{\mathbb{R}^X \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(\mathcal{O}X))}$$

where $\mathcal{S}(\mathcal{O}X)$ denotes the lattice of *all* subspaces of X.

Localic real-valued functions

In order to motivate the idea, we first recall the isomorphism

$$\boxed{\mathrm{Top}(X, (\mathbb{R}, \mathcal{T}_e)) \simeq \mathrm{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{O}X)}$$

Now, if we observe that the set $\mathbb{R}^X$ is in an obvious bijection with $\mathrm{Top}((X, \mathcal{P}(X)), (\mathbb{R}, \mathcal{T})$ where $\mathcal{T}$ is *any* topology on $\mathbb{R}$, we would, in particular, have a bijection

$$\boxed{\mathbb{R}^X \simeq \mathrm{Top}((X, \mathcal{P}(X)), (\mathbb{R}, \mathcal{T}_e))) \backsimeq \mathrm{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{P}(X))}$$

Therefore,

$$\boxed{\mathbb{R}^X \simeq \mathrm{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(\mathcal{O}X))}$$

where $\mathcal{S}(\mathcal{O}X)$ denotes the lattice of *all* subspaces of X.

Localic real-valued functions

In order to motivate the idea, we first recall the isomorphism

$$\boxed{\mathsf{Top}(X, (\mathbb{R}, \mathcal{T}_e)) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{O}X)}$$

Now, if we observe that the set $\mathbb{R}^X$ is in an obvious bijection with $\mathsf{Top}((X, \mathcal{P}(X)), (\mathbb{R}, \mathcal{T})$ where $\mathcal{T}$ is *any* topology on $\mathbb{R}$, we would, in particular, have a bijection

$$\boxed{\mathbb{R}^X \simeq \mathsf{Top}((X, \mathcal{P}(X)), (\mathbb{R}, \mathcal{T}_e))) \backsimeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{P}(X))}$$

Therefore,

$$\boxed{\mathbb{R}^X \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(\mathcal{O}X))}$$

where $\mathcal{S}(\mathcal{O}X)$ denotes the lattice of *all* subspaces of X.

Localic real-valued functions

$$\boxed{\mathbb{R}^X \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(\mathcal{O}X))}$$

If we finally recall one slogan of pointfree topology that elements of the frame $\mathcal{S}L$ are identified as *generalized subspaces*, we thus arrive at the conclusion that one can think of members of

$$\boxed{\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}L)}$$

as *arbitrary not necessarily continuous* real functions on L.

Thus the above bijection justifies to adopt the following:

Definition

A localic real function on L is a frame homomorphism $\mathfrak{L}(\mathbb{R}) \to \mathcal{S}L$.

Localic real-valued functions

$$\boxed{\mathbb{R}^X \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(\mathcal{O}X))}$$

If we finally recall one slogan of pointfree topology that elements of the frame $\mathcal{S}L$ are identified as *generalized subspaces*, we thus arrive at the conclusion that one can think of members of

$$\boxed{\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}L)}$$

as *arbitrary not necessarily continuous* real functions on L.

Thus the above bijection justifies to adopt the following:

Definition

A localic real function on L is a frame homomorphism $\mathfrak{L}(\mathbb{R}) \to \mathcal{S}L$.

Localic real-valued functions

$$\boxed{\mathbb{R}^X \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(\mathcal{O}X))}$$

If we finally recall one slogan of pointfree topology that elements of the frame $\mathcal{S}L$ are identified as *generalized subspaces*, we thus arrive at the conclusion that one can think of members of

$$\boxed{\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}L)}$$

as *arbitrary not necessarily continuous* real functions on L.

Thus the above bijection justifies to adopt the following:

Definition

A localic real function on L is a frame homomorphism $\mathfrak{L}(\mathbb{R}) \to \mathcal{S}L$.

Localic real-valued functions

We write: $\boxed{\mathrm{F}(L) = \mathrm{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}L)}$.

Recall now that the map $\mathfrak{c} : L \longrightarrow \mathcal{S}L$, associating to each $a \in L$ the closed sublocale $\mathfrak{c}(a)$, is an embedding.

Then for each frame M we have a further embedding

$$\begin{array}{cccc} \mathfrak{c} : & \mathrm{Frm}(M, L) & \longrightarrow & \mathrm{Frm}(M, \mathcal{S}L) \\ & \varphi & \longmapsto & \mathfrak{c} \circ \varphi \end{array}$$

Hence $\boxed{\mathrm{Frm}(M, L) \simeq \{ f \in \mathrm{Frm}(M, \mathcal{S}L) : f(M) \subseteq \mathfrak{c}(L) \}}$

In particular we have:

$$\boxed{\mathrm{c}(L) = \mathrm{Frm}(\mathfrak{L}(\mathbb{R}), L) \simeq \{ f \in \mathrm{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}L) : f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}(L) \}}$$

Localic real-valued functions

We write: $\boxed{\mathrm{F}(L) = \mathrm{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}L)}$.

Recall now that the map $\mathfrak{c} : L \longrightarrow \mathcal{S}L$, associating to each $a \in L$ the closed sublocale $\mathfrak{c}(a)$, is an embedding.

Then for each frame M we have a further embedding

$$\begin{array}{cccc} \mathfrak{c} : & \mathrm{Frm}(M, L) & \longrightarrow & \mathrm{Frm}(M, \mathcal{S}L) \\ & \varphi & \longmapsto & \mathfrak{c} \circ \varphi \end{array}$$

Hence $\boxed{\mathrm{Frm}(M, L) \simeq \{ f \in \mathrm{Frm}(M, \mathcal{S}L) : f(M) \subseteq \mathfrak{c}(L) \}}$

In particular we have:

$$\boxed{\mathrm{c}(L) = \mathrm{Frm}(\mathfrak{L}(\mathbb{R}), L) \simeq \{ f \in \mathrm{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}L) : f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}(L) \}}$$

Localic real-valued functions

We write: $\boxed{\mathrm{F}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}L)}$.

Recall now that the map $\mathfrak{c} : L \longrightarrow \mathcal{S}L$, associating to each $a \in L$ the closed sublocale $\mathfrak{c}(a)$, is an embedding.

Then for each frame M we have a further embedding

$$\begin{array}{cccccc} \mathfrak{c} & : & \mathsf{Frm}(M, L) & \longrightarrow & \mathsf{Frm}(M, \mathcal{S}L) \\ & & \varphi & \longmapsto & \mathfrak{c} \circ \varphi \end{array}$$

Hence $\boxed{\mathsf{Frm}(M, L) \simeq \{f \in \mathsf{Frm}(M, \mathcal{S}L) : f(M) \subseteq \mathfrak{c}(L)\}}$

In particular we have:

$$\boxed{\mathrm{c}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L) \simeq \{f \in \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}L) : f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}(L)\}}$$

Localic real-valued functions

We write: $\boxed{\mathrm{F}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}L)}$.

Recall now that the map $\mathfrak{c} : L \longrightarrow \mathcal{S}L$, associating to each $a \in L$ the closed sublocale $\mathfrak{c}(a)$, is an embedding.

Then for each frame M we have a further embedding

$$\begin{array}{cccc} \mathfrak{c} & : & \mathsf{Frm}(M, L) & \longrightarrow & \mathsf{Frm}(M, \mathcal{S}L) \\ & & \varphi & \longmapsto & \mathfrak{c} \circ \varphi \end{array}$$

Hence $\boxed{\mathsf{Frm}(M, L) \simeq \{f \in \mathsf{Frm}(M, \mathcal{S}L) : f(M) \subseteq \mathfrak{c}(L)\}}$

In particular we have:

$$\boxed{\mathrm{C}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L) \simeq \{f \in \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}L) : f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}(L)\}}$$

Localic real-valued functions

We write: $\boxed{\mathrm{F}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}L)}$.

Recall now that the map $\mathfrak{c} : L \longrightarrow \mathcal{S}L$, associating to each $a \in L$ the closed sublocale $\mathfrak{c}(a)$, is an embedding.

Then for each frame M we have a further embedding

$$\begin{array}{rcccc} \mathfrak{c} & : & \mathsf{Frm}(M, L) & \longrightarrow & \mathsf{Frm}(M, \mathcal{S}L) \\ & & \varphi & \longmapsto & \mathfrak{c} \circ \varphi \end{array}$$

Hence $\boxed{\mathsf{Frm}(M, L) \simeq \{f \in \mathsf{Frm}(M, \mathcal{S}L) : f(M) \subseteq \mathfrak{c}(L)\}}$

In particular we have:

$$\boxed{\mathrm{C}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L) \simeq \{f \in \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}L) : f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}(L)\}}$$

Localic real-valued functions

semicontinuity

Definition

We shall say that a localic real function $f \in \mathrm{F}(L)$ is:

(1) *continuous* if $f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

(2) *upper semicontinuous* if $f(\mathfrak{L}_l(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

(3) *lower semicontinuous* if $f(\mathfrak{L}_u(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

We denote by $\mathrm{C}(L)$, $\mathrm{USC}(L)$, and $\mathrm{LSC}(L)$ the corresponding collections of members of $\mathrm{F}(L)$.

Of course, one has

$$\boxed{\mathrm{C}(L) = \mathrm{LSC}(L) \cap \mathrm{USC}(L)}$$

J. Gutiérrez García, T Kubiak, J. Picado,
Localic real-valued functions: a general setting
Journal of Pure and Applied Algebra, 213 (2009) 1064–1074.

Localic real-valued functions

semicontinuity

Definition

We shall say that a localic real function $f \in \mathrm{F}(L)$ is:

(1) *continuous* if $f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

(2) *upper semicontinuous* if $f(\mathfrak{L}_l(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

(3) *lower semicontinuous* if $f(\mathfrak{L}_u(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

We denote by $\mathrm{C}(L)$, $\mathrm{USC}(L)$, and $\mathrm{LSC}(L)$ the corresponding collections of members of $\mathrm{F}(L)$.

Of course, one has

$$\boxed{\mathrm{C}(L) = \mathrm{LSC}(L) \cap \mathrm{USC}(L)}$$

J. Gutiérrez García, T Kubiak, J. Picado,
Localic real-valued functions: a general setting
Journal of Pure and Applied Algebra, 213 (2009) 1064–1074.

Localic real-valued functions

semicontinuity

Definition

We shall say that a localic real function $f \in \mathrm{F}(L)$ is:

(1) *continuous* if $f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

(2) *upper semicontinuous* if $f(\mathfrak{L}_l(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

(3) *lower semicontinuous* if $f(\mathfrak{L}_u(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

We denote by $\mathrm{C}(L)$, $\mathrm{USC}(L)$, and $\mathrm{LSC}(L)$ the corresponding collections of members of $\mathrm{F}(L)$.

Of course, one has

$$\boxed{\mathrm{C}(L) = \mathrm{LSC}(L) \cap \mathrm{USC}(L)}$$

J. Gutiérrez García, T Kubiak, J. Picado,
Localic real-valued functions: a general setting
Journal of Pure and Applied Algebra, 213 (2009) 1064–1074.

Localic real-valued functions

semicontinuity

Definition

We shall say that a localic real function $f \in \mathrm{F}(L)$ is:

(1) *continuous* if $f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

(2) *upper semicontinuous* if $f(\mathfrak{L}_l(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

(3) *lower semicontinuous* if $f(\mathfrak{L}_u(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

We denote by $\mathrm{C}(L)$, $\mathrm{USC}(L)$, and $\mathrm{LSC}(L)$ the corresponding collections of members of $\mathrm{F}(L)$.

Of course, one has

$$\boxed{\mathrm{C}(L) = \mathrm{LSC}(L) \cap \mathrm{USC}(L)}$$

J. Gutiérrez García, T Kubiak, J. Picado,
Localic real-valued functions: a general setting
Journal of Pure and Applied Algebra, 213 (2009) 1064–1074.

Localic real-valued functions

semicontinuity

Definition

We shall say that a localic real function $f \in \mathrm{F}(L)$ is:

(1) *continuous* if $f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

(2) *upper semicontinuous* if $f(\mathfrak{L}_l(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

(3) *lower semicontinuous* if $f(\mathfrak{L}_u(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

We denote by $\mathrm{C}(L)$, $\mathrm{USC}(L)$, and $\mathrm{LSC}(L)$ the corresponding collections of members of $\mathrm{F}(L)$.

Of course, one has

$$\boxed{\mathrm{C}(L) = \mathrm{LSC}(L) \cap \mathrm{USC}(L)}$$

J. Gutiérrez García, T Kubiak, J. Picado,
Localic real-valued functions: a general setting
Journal of Pure and Applied Algebra, 213 (2009) 1064–1074.

Localic real-valued functions

semicontinuity

Definition

We shall say that a localic real function $f \in \mathrm{F}(L)$ is:

(1) *continuous* if $f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

(2) *upper semicontinuous* if $f(\mathfrak{L}_l(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

(3) *lower semicontinuous* if $f(\mathfrak{L}_u(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

We denote by $\mathrm{C}(L)$, $\mathrm{USC}(L)$, and $\mathrm{LSC}(L)$ the corresponding collections of members of $\mathrm{F}(L)$.

Of course, one has

$$\boxed{\mathrm{C}(L) = \mathrm{LSC}(L) \cap \mathrm{USC}(L)}$$

J. Gutiérrez García, T Kubiak, J. Picado,
Localic real-valued functions: a general setting
Journal of Pure and Applied Algebra, 213 (2009) 1064–1074.

Localic real-valued functions

the isomorphism

For each $\varphi \in \mathrm{usc}(L)$ we define $f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}L$ by:

$$f(-,q) := \mathfrak{c}(f(-,q)) \quad \text{and} \quad \varphi(p,-) := \bigvee_{r>p} \mathfrak{o}(\varphi(-,r)).$$

Then

$$\begin{aligned} f \in \mathrm{USC}(L) \quad &\iff \quad \bigvee_{q\in\mathbb{Q}} f(-,q) = 1 = \bigvee_{p\in\mathbb{Q}} f(p,-) \\ &\iff \quad \bigvee_{q\in\mathbb{Q}} f(-,q) = 1 \quad \text{and} \quad \boxed{\bigvee_{p\in\mathbb{Q}} \mathfrak{o}(\varphi(-,p)) = 1} \\ &\iff \quad f \in \mathrm{usc}(L). \end{aligned}$$

We conclude that the restriction to $\mathrm{usc}(L)$ is also an order-isomorphism between $\mathrm{usc}(L)$ and $\mathrm{USC}(L)$.

Localic real-valued functions

the isomorphism

For each $\varphi \in \mathrm{usc}(L)$ we define $f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}L$ by:

$$f(-,q) := \mathfrak{c}(f(-,q)) \quad \text{and} \quad \varphi(p,-) := \bigvee_{r>p} \mathfrak{o}(\varphi(-,r)).$$

Then

$$f \in \mathrm{USC}(L) \iff \bigvee_{q\in\mathbb{Q}} f(-,q) = 1 = \bigvee_{p\in\mathbb{Q}} f(p,-)$$

$$\iff \bigvee_{q\in\mathbb{Q}} f(-,q) = 1 \quad \text{and} \quad \boxed{\textstyle\bigvee_{p\in\mathbb{Q}} \mathfrak{o}(\varphi(-,p)) = 1}$$

$$\iff f \in \mathrm{usc}(L).$$

We conclude that the restriction to $\mathrm{usc}(L)$ is also an order-isomorphism between $\mathrm{usc}(L)$ and $\mathrm{USC}(L)$.

Localic real-valued functions

the isomorphism

For each $\varphi \in \mathrm{usc}(L)$ we define $f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}L$ by:

$$f(-,q) := \mathfrak{c}(f(-,q)) \quad \text{and} \quad \varphi(p,-) := \bigvee_{r>p} \mathfrak{o}(\varphi(-,r)).$$

Then

$$\begin{aligned} f \in \mathrm{USC}(L) \quad &\iff \quad \bigvee_{q\in\mathbb{Q}} f(-,q) = 1 = \bigvee_{p\in\mathbb{Q}} f(p,-) \\ &\iff \quad \bigvee_{q\in\mathbb{Q}} f(-,q) = 1 \quad \text{and} \quad \boxed{\textstyle\bigvee_{p\in\mathbb{Q}} \mathfrak{o}(\varphi(-,p)) = 1} \\ &\iff \quad f \in \mathrm{usc}(L). \end{aligned}$$

We conclude that the restriction to $\mathrm{usc}(L)$ is also an order-isomorphism between $\mathrm{usc}(L)$ and $\mathrm{USC}(L)$.

Localic real-valued functions

the isomorphism

For each $\varphi \in \mathrm{usc}(L)$ we define $f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}L$ by:

$$f(-,q) := \mathfrak{c}(f(-,q)) \quad \text{and} \quad \varphi(p,-) := \bigvee_{r>p} \mathfrak{o}(\varphi(-,r)).$$

Then

$$\begin{aligned} f \in \mathrm{USC}(L) \quad &\Longleftrightarrow \quad \bigvee_{q\in\mathbb{Q}} f(-,q) = 1 = \bigvee_{p\in\mathbb{Q}} f(p,-) \\ &\Longleftrightarrow \quad \bigvee_{q\in\mathbb{Q}} f(-,q) = 1 \quad \text{and} \quad \boxed{\textstyle\bigvee_{p\in\mathbb{Q}} \mathfrak{o}(\varphi(-,p)) = 1} \\ &\Longleftrightarrow \quad f \in \mathrm{usc}(L). \end{aligned}$$

We conclude that the restriction to $\mathrm{usc}(L)$ is also an order-isomorphism between $\mathrm{usc}(L)$ and $\mathrm{USC}(L)$.

Localic real-valued functions

the isomorphism

For each $\varphi \in \mathrm{usc}(L)$ we define $f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}L$ by:

$$f(-,q) := \mathfrak{c}(f(-,q)) \quad \text{and} \quad \varphi(p,-) := \bigvee_{r>p} \mathfrak{o}(\varphi(-,r)).$$

Then

$$\begin{aligned} f \in \mathrm{USC}(L) &\iff \bigvee_{q\in\mathbb{Q}} f(-,q) = 1 = \bigvee_{p\in\mathbb{Q}} f(p,-) \\ &\iff \bigvee_{q\in\mathbb{Q}} f(-,q) = 1 \quad \text{and} \quad \boxed{\textstyle\bigvee_{p\in\mathbb{Q}} \mathfrak{o}(\varphi(-,p)) = 1} \\ &\iff f \in \mathrm{usc}(L). \end{aligned}$$

We conclude that the restriction to $\mathrm{usc}(L)$ is also an order-isomorphism between $\mathrm{usc}(L)$ and $\mathrm{USC}(L)$.

Localic real-valued functions

the isomorphism

$$\begin{tikzcd}
 & & \mathrm{F}(L) & & \\
\mathrm{usc}(L) \arrow[r, <->, squiggly] & \mathrm{USC}(L) \arrow[ur] & & \mathrm{LSC}(L) \arrow[ul] \arrow[r, <->, squiggly] & \mathrm{lsc}(L) \\
 & & \mathrm{C}(L) \arrow[ul] \arrow[ur] \arrow[d, <->, squiggly] & & \\
 & & \mathrm{c}(L) & &
\end{tikzcd}$$

Localic real-valued functions

characteristic functions

Given a complemented sublocale $S \in \mathcal{S}L$ the characteristic function $\chi_S : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}L$ is defined by

$$\chi_S(-,q) = \begin{cases} 0 & \text{if } q \leq 0 \\ S & \text{if } 0 < q \leq 1, \\ 1 & \text{if } q > 1 \end{cases} \quad \chi_S(p,-) = \begin{cases} 1 & \text{if } p < 0 \\ \neg S & \text{if } 0 \leq p < 1 \\ 0 & \text{if } p \geq 1. \end{cases}$$

Note that,

- $\chi_S \in \mathrm{USC}(L)$ if and only if S is closed.
- $\chi_S \in \mathrm{LSC}(L)$ if and only if S is open.
- $\chi_S \in \mathrm{C}(L)$ if and only if S is clopen.

Localic real-valued functions

characteristic functions

Given a complemented sublocale $S \in \mathcal{S}L$ the characteristic function $\chi_S : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}L$ is defined by

$$\chi_S(-,q) = \begin{cases} 0 & \text{if } q \leq 0 \\ S & \text{if } 0 < q \leq 1, \\ 1 & \text{if } q > 1 \end{cases} \quad \chi_S(p,-) = \begin{cases} 1 & \text{if } p < 0 \\ \neg S & \text{if } 0 \leq p < 1 \\ 0 & \text{if } p \geq 1. \end{cases}$$

Note that,

- $\chi_S \in \mathrm{USC}(L)$ if and only if S is closed.
- $\chi_S \in \mathrm{LSC}(L)$ if and only if S is open.
- $\chi_S \in \mathrm{C}(L)$ if and only if S is clopen.

Localic real-valued functions

regularization

For $f \in \mathrm{F}(L)$ we define the *lower regularization f°*:

$$f^\circ(-,q) = \bigvee_{s<q} \neg \overline{f(s,-)} \quad \text{and} \quad f^\circ(p,-) = \bigvee_{r>p} \overline{f(r,-)}.$$

$f^\circ \leq f$

$f^{\circ\circ} = f^\circ$

$f^\circ \in \mathrm{LSC}(L)$

$g \in \mathrm{LSC}(L)$ and $g \leq f \quad \Rightarrow \quad g \leq f^\circ$

$(\chi_S)^\circ = \chi_{\overset{\circ}{S}}$

Localic real-valued functions

regularization

For $f \in \mathrm{F}(L)$ we define the *lower regularization f°*:

$$\boxed{f^\circ(-,q) = \bigvee_{s<q} \neg\, \overline{f(s,-)}} \quad \text{and} \quad \boxed{f^\circ(p,-) = \bigvee_{r>p} \overline{f(r,-)}.}$$

$f^\circ \leq f$

$f^{\circ\circ} = f^\circ$

$f^\circ \in \mathrm{LSC}(L)$

$g \in \mathrm{LSC}(L)$ and $g \leq f \quad \Rightarrow \quad g \leq f^\circ$

$(\chi_S)^\circ = \chi_{\overset{\circ}{S}}$

Localic real-valued functions

regularization

For $f \in \mathrm{F}(L)$ we define the *lower regularization f°*:

$$\boxed{f^\circ(-,q) = \bigvee_{s<q} \neg\, \overline{f(s,-)}} \quad \text{and} \quad \boxed{f^\circ(p,-) = \bigvee_{r>p} \overline{f(r,-)}.}$$

$f^\circ \leq f$

$f^{\circ\circ} = f^\circ$

$f^\circ \in \mathrm{LSC}(L)$

$g \in \mathrm{LSC}(L)$ and $g \leq f \quad \Rightarrow \quad g \leq f^\circ$

$(\chi_S)^\circ = \chi_{S^\circ}$

Localic real-valued functions regularization

For $f \in \mathrm{F}(L)$ we define the *lower regularization f°*:

$$\boxed{f^\circ(-,q) = \bigvee_{s<q} \neg\, \overline{f(s,-)}} \quad \text{and} \quad \boxed{f^\circ(p,-) = \bigvee_{r>p} \overline{f(r,-)}.}$$

$f^\circ \le f$

$f^{\circ\circ} = f^\circ$

$f^\circ \in \mathrm{LSC}(L)$

$g \in \mathrm{LSC}(L)$ and $g \le f \quad \Rightarrow \quad g \le f^\circ$

$(\chi_S)^\circ = \chi_{S^\circ}$

Localic real-valued functions

regularization

For $f \in \mathrm{F}(L)$ we define the *lower regularization f°*:

$$\boxed{f^\circ(-,q) = \bigvee_{s<q} \neg\,\overline{f(s,-)}} \quad \text{and} \quad \boxed{f^\circ(p,-) = \bigvee_{r>p} \overline{f(r,-)}.}$$

$f^\circ \leq f$

$f^{\circ\circ} = f^\circ$

$f^\circ \in \mathrm{LSC}(L)$

$g \in \mathrm{LSC}(L)$ and $g \leq f \quad \Rightarrow \quad g \leq f^\circ$

$(\chi_S)^\circ = \chi_{\overset{\circ}{S}}$

Localic real-valued functions

regularization

For $f \in \mathrm{F}(L)$ we define the *lower regularization f°*:

$$\boxed{f^\circ(-,q) = \bigvee_{s<q} \neg\,\overline{f(s,-)}} \quad \text{and} \quad \boxed{f^\circ(p,-) = \bigvee_{r>p} \overline{f(r,-)}.}$$

$$f^\circ \leq f$$

$$f^{\circ\circ} = f^\circ$$

$$f^\circ \in \mathrm{LSC}(L)$$

$g \in \mathrm{LSC}(L)$ and $g \leq f \quad \Rightarrow \quad g \leq f^\circ$

$$(\chi_S)^\circ = \chi_{\overset{\circ}{S}}$$

Localic real-valued functions

regularization

For $f \in \mathrm{F}(L)$ we define the *lower regularization f°*:

$$\boxed{f^\circ(-,q) = \bigvee_{s<q} \neg\,\overline{f(s,-)}} \quad \text{and} \quad \boxed{f^\circ(p,-) = \bigvee_{r>p} \overline{f(r,-)}.}$$

$$f^\circ \leq f$$

$$f^{\circ\circ} = f^\circ$$

$$f^\circ \in \mathrm{LSC}(L)$$

$$g \in \mathrm{LSC}(L) \text{ and } g \leq f \quad \Rightarrow \quad g \leq f^\circ$$

$$(\chi_S)^\circ = \chi_{\mathring{S}}$$

Localic real-valued functions

regularization

For $f \in \overline{\mathrm{F}}(L)$ we define the *upper regularization f^-*:

$$\boxed{f^-(-,q) = \bigvee_{s<q} \overline{f(-,s)}} \quad \text{and} \quad \boxed{f^-(p,-) = \bigvee_{r>p} \neg\, \overline{f(-,r)}.}$$

$f \leq f^-$

$f^{--} = f^-$

$f^- \in \mathrm{USC}(L)$

$g \in \mathrm{USC}(L)$ and $f \leq g \quad \Rightarrow \quad f^- \leq g$

$(\chi_S)^- = \chi_{\overline{S}}$

Localic real-valued functions

regularization

For $f \in \overline{\mathrm{F}}(L)$ we define the *upper regularization f^{-}*:

$$\boxed{f^{-}(-,q) = \bigvee_{s<q} \overline{f(-,s)}} \quad \text{and} \quad \boxed{f^{-}(p,-) = \bigvee_{r>p} \neg\, \overline{f(-,r)}.}$$

$$f \le f^{-}$$

$$f^{--} = f^{-}$$

$$f^{-} \in \mathrm{USC}(L)$$

$$g \in \mathrm{USC}(L) \text{ and } f \le g \quad \Rightarrow \quad f^{-} \le g$$

$$(\chi_S)^{-} = \chi_{\overline{S}}$$

Localic real-valued functions

Achievements

- One can see semicontinuous functions as a particular kind of real-valued functions on the frame of congruences, with the same domain, namely $\mathfrak{L}(\mathbb{R})$.
- Being all upper and lower semicontinuous functions particular kinds of real-valued functions on the frame of congruences, we can compare them.
- By considering the algebraic operations of the ring $\mathrm{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}L)$, we obtain, in particular, a way of defining the sum of upper and lower semicontinuous functions.
- The class of continuous functions is precisely the intersection of the classes of lower and upper ones.
- The situation with respect to regularization is precisely the same as in the topological setting.

Localic real-valued functions

Achievements

- One can see semicontinuous functions as a particular kind of real-valued functions on the frame of congruences, with the same domain, namely $\mathfrak{L}(\mathbb{R})$.
- Being all upper and lower semicontinuous functions particular kinds of real-valued functions on the frame of congruences, we can compare them.
- By considering the algebraic operations of the ring $\mathrm{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}L)$, we obtain, in particular, a way of defining the sum of upper and lower semicontinuous functions.
- The class of continuous functions is precisely the intersection of the classes of lower and upper ones.
- The situation with respect to regularization is precisely the same as in the topological setting.

Localic real-valued functions

Achievements

- One can see semicontinuous functions as a particular kind of real-valued functions on the frame of congruences, with the same domain, namely $\mathfrak{L}(\mathbb{R})$.
- Being all upper and lower semicontinuous functions particular kinds of real-valued functions on the frame of congruences, we can compare them.
- By considering the algebraic operations of the ring Frm$(\mathfrak{L}(\mathbb{R}), \mathcal{S}L)$, we obtain, in particular, a way of defining the sum of upper and lower semicontinuous functions.
- The class of continuous functions is precisely the intersection of the classes of lower and upper ones.
- The situation with respect to regularization is precisely the same as in the topological setting.

Localic real-valued functions

Achievements

- One can see semicontinuous functions as a particular kind of real-valued functions on the frame of congruences, with the same domain, namely $\mathfrak{L}(\mathbb{R})$.
- Being all upper and lower semicontinuous functions particular kinds of real-valued functions on the frame of congruences, we can compare them.
- By considering the algebraic operations of the ring $\mathrm{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}L)$, we obtain, in particular, a way of defining the sum of upper and lower semicontinuous functions.
- The class of continuous functions is precisely the intersection of the classes of lower and upper ones.
- The situation with respect to regularization is precisely the same as in the topological setting.

Localic real-valued functions

Achievements

- One can see semicontinuous functions as a particular kind of real-valued functions on the frame of congruences, with the same domain, namely $\mathfrak{L}(\mathbb{R})$.
- Being all upper and lower semicontinuous functions particular kinds of real-valued functions on the frame of congruences, we can compare them.
- By considering the algebraic operations of the ring $\mathrm{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}L)$, we obtain, in particular, a way of defining the sum of upper and lower semicontinuous functions.
- The class of continuous functions is precisely the intersection of the classes of lower and upper ones.
- The situation with respect to regularization is precisely the same as in the topological setting.

Insertion theorems

Theorem (Katětov-Tong)

The following conditions on a frame L are equivalent:

(1) *L is normal.*

(2) *For every $f \in \mathrm{USC}(L)$ and every $g \in \mathrm{LSC}(L)$ with $f \leq g$, there exists $h \in \mathrm{C}(L)$ such that $f \leq h \leq g$.*

📄 Y.-M. Li and G.-J. Wang,
Localic Katětov-Tong insertion theorem and localic Tietze extension theorem,
Comment. Math. Univ. Carolinae, 38 (1997) 801–814.

📄 J. Gutiérrez García and J. Picado,
On the algebraic representation of semicontinuity,
Journal of Pure and Applied Algebra, 210 (2007) 299–306.

Insertion theorems

Theorem (Stone)

The following conditions on a frame L are equivalent:

1. *L is extremally disconnected.*
2. $\mathrm{C}(L) = \{f^- : f \in \mathrm{LSC}(L)\}$.
3. $\mathrm{C}(L) = \{g^\circ : g \in \mathrm{USC}(L)\}$.
4. *For every $f \in \mathrm{USC}(L)$ and every $g \in \mathrm{LSC}(L)$ with $g \leq f$, there exists $h \in \mathrm{C}(L)$ such that $g \leq h \leq f$.*

Y.-M. Li and Z.-H. Li,
Constructive insertion theorems and extension theorems on extremally disconnected frames,
Algebra Universalis, 44 (2000), 271Ð281.

J. Gutiérrez García and J. Picado,
Lower and upper regularizations of frame semicontinuous real functions,
Algebra Universalis, 60 (2009) 169–184.

Insertion theorems

Let $\mathrm{UL}(L) = \{(f,g) \in \mathrm{USC}(L) \times \mathrm{LSC}(L) : f \leq g\}$ with the order $(f_1, g_1) \leq (f_2, g_2) \iff f_2 \leq f_1$ and $g_1 \leq g_2$.

Theorem (Monotone Katětov-Tong)

For a frame L, the following are equivalent:

1. *L is monotonically normal.*
2. *There exists a monotone function $\Lambda : \mathrm{UL}(L) \to \mathrm{C}(L)$ such that $f \leq \Lambda(f,g) \leq g$ for all $(f,g) \in \mathrm{UL}(L)$.*

J. Gutiérrez García, T. Kubiak and J. Picado,
Monotone insertion and monotone extension of frame homomorphisms,
Journal of Pure and Applied Algebra, 212 (2008) 955–968.

Insertion theorems

Let $\mathrm{UL}(L) = \{(f, g) \in \mathrm{USC}(L) \times \mathrm{LSC}(L) : f \leq g\}$ with the order $(f_1, g_1) \leq (f_2, g_2) \iff f_2 \leq f_1$ and $g_1 \leq g_2$.

Theorem (Monotone Katětov-Tong)

For a frame L, the following are equivalent:

(1) *L is monotonically normal.*

(2) *There exists a monotone function* $\Lambda : \mathrm{UL}(L) \to \mathrm{C}(L)$ *such that* $f \leq \Lambda(f, g) \leq g$ *for all* $(f, g) \in \mathrm{UL}(L)$.

J. Gutiérrez García, T. Kubiak and J. Picado,
Monotone insertion and monotone extension of frame homomorphisms,
Journal of Pure and Applied Algebra, 212 (2008) 955–968.

Insertion theorems

Strict insertion

Michael insertion theorem for perfectly normal frames. . .

Dowker insertion theorem for normal and countably paracompact frames. . .

J. Gutiérrez García, T. Kubiak and J. Picado,
Pointfree forms of Dowker and Michael insertion theorems,
Journal of Pure and Applied Algebra, 213 (2009) 98–108.

Insertion theorems

Strict insertion

Michael insertion theorem for perfectly normal frames. . .

Dowker insertion theorem for normal and countably paracompact frames. . .

J. Gutiérrez García, T. Kubiak and J. Picado,
Pointfree forms of Dowker and Michael insertion theorems,
Journal of Pure and Applied Algebra, 213 (2009) 98–108.

Insertion theorems

Strict insertion

Michael insertion theorem for perfectly normal frames. . .

Dowker insertion theorem for normal and countably paracompact frames. . .

J. Gutiérrez García, T. Kubiak and J. Picado,
Pointfree forms of Dowker and Michael insertion theorems,
Journal of Pure and Applied Algebra, 213 (2009) 98–108.

Insertion theorems

Theorem

The following conditions on a frame L are equivalent:

(1) *L is completely normal.*

(2) *L is hereditarily normal (each sublocale of L is normal.*

(3) *Each open sublocale of L is normal.*

(4) *For every $f, g \in \mathrm{F}(L)$, if $f^{-} \leq g$ and $f \leq g^{\circ}$, then there exists an $h \in \mathrm{LSC}(L)$ such that $f \leq h \leq h^{-} \leq g$.*

M.J. Ferreira, J. Gutiérrez García and J. Picado
Completely normal frames and real-valued functions,
Topology and its Applications, in press.

Insertion theorems

Theorem

The following conditions on a frame L are equivalent:

1. *L is completely normal.*
2. *L is hereditarily normal (each sublocale of L is normal.*
3. *Each open sublocale of L is normal.*
4. *For every* $f, g \in \mathrm{F}(L)$*, if* $f^- \leq g$ *and* $f \leq g^\circ$*, then there exists an* $h \in \mathrm{LSC}(L)$ *such that* $f \leq h \leq h^- \leq g$*.*

M.J. Ferreira, J. Gutiérrez García and J. Picado
Completely normal frames and real-valued functions,
Topology and its Applications, in press.

Insertion theorems

Theorem

The following conditions on a frame L are equivalent:

1. *L is completely normal.*
2. *L is hereditarily normal (each sublocale of L is normal.*
3. *Each open sublocale of L is normal.*
4. *For every* $f, g \in \mathrm{F}(L)$, *if* $f^- \leq g$ *and* $f \leq g^\circ$, *then there exists an* $h \in \mathrm{LSC}(L)$ *such that* $f \leq h \leq h^- \leq g$.

M.J. Ferreira, J. Gutiérrez García and J. Picado
Completely normal frames and real-valued functions,
Topology and its Applications, in press.

Insertion theorems

Theorem

The following conditions on a frame L are equivalent:

(1) *L is completely normal.*

(2) *L is hereditarily normal (each sublocale of L is normal.*

(3) *Each open sublocale of L is normal.*

(4) *For every* $f, g \in \mathrm{F}(L)$*, if* $f^- \leq g$ *and* $f \leq g^\circ$*, then there exists an* $h \in \mathrm{LSC}(L)$ *such that* $f \leq h \leq h^- \leq g$*.*

M.J. Ferreira, J. Gutiérrez García and J. Picado
Completely normal frames and real-valued functions,
Topology and its Applications, in press.

Insertion theorems

Theorem

The following conditions on a frame L are equivalent:

(1) *L is completely normal.*

(2) *L is hereditarily normal (each sublocale of L is normal.*

(3) *Each open sublocale of L is normal.*

(4) *For every* $f, g \in \mathrm{F}(L)$*, if* $f^- \leq g$ *and* $f \leq g^\circ$*, then there exists an* $h \in \mathrm{LSC}(L)$ *such that* $f \leq h \leq h^- \leq g$*.*

M.J. Ferreira, J. Gutiérrez García and J. Picado
Completely normal frames and real-valued functions,
Topology and its Applications, in press.

Extension theorems

Each $\theta \in \mathfrak{C}L$ determines a unique sublocale $S_\theta \subseteq L$ and a unique frame quotient $c_\theta \in \mathrm{Frm}(L, S_\theta)$.

$\widetilde{H} \in \mathrm{C}(L)$ is said to be a *continuous extension* of $H \in \mathrm{C}(S_\theta)$ if and only if the following diagram commutes

$$\begin{tikzcd}
 & \nabla L \arrow[r, leftrightarrow, "\nabla"] & L \arrow[d, "c_\theta"] \\
\mathfrak{L}(\mathbb{R}) \arrow[ur, "\widetilde{H}"] \arrow[r, "H"] & \nabla S_\theta \arrow[r, leftrightarrow, "\nabla"] & S_\theta
\end{tikzcd}$$

i.e. $c_\theta \circ \nabla \circ \widetilde{H} = \nabla \circ H$.

Extension theorems

Each $\theta \in \mathfrak{C}L$ determines a unique sublocale $S_\theta \subseteq L$ and a unique frame quotient $c_\theta \in \mathrm{Frm}(L, S_\theta)$.

Given $H \in \mathrm{C}(S_\theta)$, $\widetilde{H} \in \mathrm{C}(L)$ is said to be a *continuous extension* of $H \in \mathrm{C}(S_\theta)$ if and only if the following diagram commutes

$$\begin{tikzcd}
 & \nabla L \arrow[r, leftrightarrow, "\nabla"] & L \arrow[d, "c_\theta"] \\
\mathfrak{L}(\mathbb{R}) \arrow[ur, "\widetilde{H}"] \arrow[r, "H"] & \nabla S_\theta \arrow[r, leftrightarrow, "\nabla"] & S_\theta
\end{tikzcd}$$

i.e. $c_\theta \circ \nabla \circ \widetilde{H} = \nabla \circ H$.

Extension theorems

Each $\theta \in \mathfrak{C}L$ determines a unique sublocale $S_\theta \subseteq L$ and a unique frame quotient $c_\theta \in \mathsf{Frm}(L, S_\theta)$.

Given $H \in \mathrm{C}(S_\theta)$, $\widetilde{H} \in \mathrm{C}(L)$ is said to be a *continuous extension* of $H \in \mathrm{C}(S_\theta)$ if and only if the following diagram commutes

$$\begin{tikzcd}
 & \nabla L \arrow[r, leftrightarrow, "\nabla"] & L \arrow[d, "c_\theta"] \\
\mathfrak{L}(\mathbb{R}) \arrow[ur, "\widetilde{H}"] \arrow[r, "H"] & \nabla S_\theta \arrow[r, leftrightarrow, "\nabla"] & S_\theta
\end{tikzcd}$$

i.e. $c_\theta \circ \nabla \circ \widetilde{H} = \nabla \circ H$.

Extension theorems

Each $\theta \in \mathfrak{C}L$ determines a unique sublocale $S_\theta \subseteq L$ and a unique frame quotient $c_\theta \in \mathrm{Frm}(L, S_\theta)$.

Given $H \in \mathrm{C}(S_\theta)$, $\widetilde{H} \in \mathrm{C}(L)$ is said to be a *continuous extension* of $H \in \mathrm{C}(S_\theta)$ if and only if the following diagram commutes

$$\begin{tikzcd}
 & \nabla L \arrow[r, leftrightarrow, "\nabla"] & L \arrow[d, "c_\theta"] \\
\mathfrak{L}(\mathbb{R}) \arrow[r, "H"] \arrow[ru, "\widetilde{H}"] & \nabla S_\theta \arrow[r, leftrightarrow, "\nabla"] & S_\theta
\end{tikzcd}$$

i.e. $c_\theta \circ \nabla \circ \widetilde{H} = \nabla \circ H$.

Extension theorems

Each $\theta \in \mathfrak{C}L$ determines a unique sublocale $S_\theta \subseteq L$ and a unique frame quotient $c_\theta \in \mathrm{Frm}(L, S_\theta)$.

Given $H \in \mathrm{C}(S_\theta)$, $\widetilde{H} \in \mathrm{C}(L)$ is said to be a *continuous extension* of $H \in \mathrm{C}(S_\theta)$ if and only if the following diagram commutes

$$\begin{tikzcd}
 & \nabla L \arrow[r, leftrightarrow, "\nabla"] & L \arrow[d, "c_\theta"] \\
\mathfrak{L}(\mathbb{R}) \arrow[ur, "\widetilde{H}"] \arrow[r, "H"] & \nabla S_\theta \arrow[r, leftrightarrow, "\nabla"] & S_\theta
\end{tikzcd}$$

i.e. $c_\theta \circ \nabla \circ \widetilde{H} = \nabla \circ H$.

Extension theorems

Theorem (Tietze)

The following conditions on a frame L are equivalent:

1. *L is normal.*
2. *For each closed sublocale S of L and each $H \in \mathrm{C}(S)$, there exists a continuous extension $\widetilde{H} \in \mathrm{C}(L)$ of H.*

Theorem

The following conditions on a frame L are equivalent:

1. *L is extremally disconnected.*
2. *For each open sublocale S of L and each $H \in \mathrm{C}(S)$, there exists a continuous extension $\overline{H} \in \mathrm{C}(L)$ of H.*

Extension theorems

Theorem (Tietze)

The following conditions on a frame L are equivalent:

(1) *L is normal.*

(2) *For each closed sublocale S of L and each $H \in \mathrm{C}(S)$, there exists a continuous extension $\widetilde{H} \in \mathrm{C}(L)$ of H.*

Theorem

The following conditions on a frame L are equivalent:

(1) *L is extremally disconnected.*

(2) *For each open sublocale S of L and each $H \in \mathrm{C}(S)$, there exists a continuous extension $\overline{H} \in \mathrm{C}(L)$ of H.*

Extension theorems

Also versions for monotone normality, perfect normality, ...

Theorem

For a frame L, the following are equivalent:

1. *L is monotonically normal.*
2. *For every closed sublocale S there exists an extender $\Phi_S : \overline{\mathrm{C}}(S) \to \overline{\mathrm{C}}(L)$ such that for each S_1, S_2 and $H_i \in \overline{\mathrm{C}}(S_i)$ $(i = 1, 2)$ with $\widehat{H}_1 \leq \widehat{H}_2$ one has $\Phi_{S_1}(H_1) \leq \Phi_{S_2}(H_2)$.*

Theorem

For a frame L, the following are equivalent:

1. *L is perfectly normal.*
2. *For every closed sublocale S and $H \in \overline{\mathrm{C}}(S)$, there exists a continuous extension $\widetilde{H} \in \overline{\mathrm{C}}(L)$ of H such that $\widetilde{H}(\bigvee_{p,q\in\mathbb{Q}} (p, q)) \in S$.*

Extension theorems

Also versions for monotone normality, perfect normality, ...

Theorem

For a frame L, the following are equivalent:

1. *L is monotonically normal.*
2. *For every closed sublocale S there exists an extender* $\Phi_S : \overline{\mathrm{C}}(S) \to \overline{\mathrm{C}}(L)$ *such that for each* S_1, S_2 *and* $H_i \in \overline{\mathrm{C}}(S_i)$ $(i = 1, 2)$ *with* $\widehat{H}_1 \leq \widehat{H}_2$ *one has* $\Phi_{S_1}(H_1) \leq \Phi_{S_2}(H_2)$.

Theorem

For a frame L, the following are equivalent:

1. *L is perfectly normal.*
2. *For every closed sublocale S and* $H \in \overline{\mathrm{C}}(S)$, *there exists a continuous extension* $\widetilde{H} \in \overline{\mathrm{C}}(L)$ *of H such that* $\widetilde{H}(\bigvee_{p,q\in\mathbb{Q}} (p,q)) \in S$.

Extension theorems

Also versions for monotone normality, perfect normality, ...

Theorem

For a frame L, the following are equivalent:

(1) *L is monotonically normal.*

(2) *For every closed sublocale S there exists an extender $\Phi_S : \overline{\mathrm{C}}(S) \to \overline{\mathrm{C}}(L)$ such that for each S_1, S_2 and $H_i \in \overline{\mathrm{C}}(S_i)$ $(i = 1, 2)$ with $\widehat{H}_1 \le \widehat{H}_2$ one has $\Phi_{S_1}(H_1) \le \Phi_{S_2}(H_2)$.*

Theorem

For a frame L, the following are equivalent:

(1) *L is perfectly normal.*

(2) *For every closed sublocale S and $H \in \overline{\mathrm{C}}(S)$, there exists a continuous extension $\widetilde{H} \in \overline{\mathrm{C}}(L)$ of H such that $\widetilde{H}(\bigvee_{p,q\in\mathbb{Q}} (p, q)) \in S$.*

with J. Picado

- On the algebraic representation of semicontinuity
  Journal of Pure and Applied Algebra, 210 (2007) 299–306.

with T. Kubiak and J. Picado

- Monotone insertion and monotone extension of frame homomorphisms
  Journal of Pure and Applied Algebra, 212 (2008) 955–968.
- Lower and upper regularizations of frame semicontinuous real functions
  Algebra Universalis, 60 (2009) 169–184.
- Pointfree forms of Dowker and Michael insertion theorems
  Journal of Pure and Applied Algebra, 213 (2009) 98–108.
- Localic real-valued functions: a general setting
  Journal of Pure and Applied Algebra, 213 (2009) 1064–1074.

with M.J. Ferreira and J. Picado

- Completely normal frames and real-valued functions
  To appear in: *Topology and its Applications*.