*Acta Math. Hungar.*, **122 (1–2)** (2009), 71–80.
DOI: 10.1007/s10474-008-7234-3
First published online July 13, 2008

# MONOTONE NORMALITY FREE OF $T_1$ AXIOM*

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*(Received December 5, 2007; revised March 6, 2008; accepted March 10, 2008)*

**Abstract.** Monotone normality is usually defined in the class of $T_1$ spaces. In this paper new characterizations of monotone normality, free of $T_1$ axiom, are provided and it is shown that in this context it is not a hereditary property. Also, a Tietze-type extension theorem for lattice-valued functions for this class of spaces is given.

## 1. Introduction

There has been an extensive literature devoted to monotonically normal spaces (see the surveys [3, 5] and the references therein) since the notion was introduced in [1, 7, 13]. With the exception of [8], monotone normality has always been studied in the restricted class of $T_1$ spaces.

The influence of computer science not only has given relevance to those spaces not satisfying $T_1$ axiom, but also has focused attention on functions with values in ordered sets rather than in the reals. Continuous lattices or domains with their Scott topology are an important class among the spaces

*This research was supported by the MEyC and FEDER under grant MTM2006-14925-C02-02/ and by the University of the Basque Country under grant UPV05/101.

*Key words and phrases:* monotone normality, completely distributive lattice, Raney relation, $\lhd$-separable, lower semicontinuous function, upper semicontinuous function, extension theorem.

*2000 Mathematics Subject Classification:* 54D15, 54C20, 54C99, 06D10.

which do not satisfy the $T_1$ axiom. In concordance with these ideas, we explore monotone normality in a $T_1$ free context. Also lattice-valued functions rather than real-valued functions are considered throughout.

In this paper, after some lattice theoretic preliminaries collected in Section 2, the notion of monotone normality, free of $T_1$ axiom, is studied. In Section 3 several characterizations of monotone normality in this context are provided and some deviation from $T_1$-monotonically normal spaces is exhibited. It is well known that in the class of normal spaces (either $T_1$ or not), complete normality and hereditary normality are equivalent concepts as well as the fact that open subsets inherit the property [12]. As to the class of $T_1$-monotonically normal spaces is concerned, it has been proved [2, 7, 10] that monotone normality is equivalent to any one of the following notions: complete monotone normality, hereditary monotone normality, open subsets inherit the property. The proof of these equivalences depends strongly on the axiom $T_1$. It relies upon a new property, also equivalent to monotone normality, which can be properly called monotone regularity and implies the Hausdorff axiom. The question as to whether the above equivalences hold in spaces not satisfying the axiom $T_1$, is answered in the negative. The answer is based on a construction of a non $T_1$ monotonically normal compactification associated to any topological space. It is important to notice that, when characterizing monotone normality, the role of points will now be played by the closure of singletons (the minimal closed sets in a non $T_1$-space). This idea is as simple as effective. It is also used in Section 4, to provide an extension property of lattice-valued functions for monotonically normal spaces, extending the result given in [9] to a $T_1$-free context (see also [11] for the case of real valued functions).

## 2. Semicontinuous lattice-valued functions

In this paper $L$ denotes a (complete) completely distributive lattice (with bounds 0 and 1). For general concepts regarding lattices and complete distributivity we refer the reader to [4]. We shall use the Raney's characterization of complete distributivity in terms of an extra order $\lhd$ with the following approximation property: Given a complete lattice $L$ and $a, b \in L$, we write $a \lhd b$ if and only if, whenever $A \subset L$ and $b \leqq \bigvee A$, there is $c \in A$ with $a \leqq c$. The lattice $L$ is then *completely distributive* if and only if $\lhd$ has the approximation property, i.e., $a = \bigvee\{b \in L : b \lhd a\}$ for each $a \in L$. We shall use the following properties of the extra order: (1) $a \lhd b$ implies $a \leqq b$; (2) $c \leqq a \lhd b \leqq d$ implies $c \lhd d$; (3) $a \lhd b$ implies $a \lhd c \lhd b$ for some $c \in L$ (Interpolation Property).

A subset $D \subset L$ is called *join-dense* (or a *base*) if $a = \bigvee\{d \in D : d \leqq a\}$ for each $a \in L$. An element $a \in L$ is called *supercompact* if $a \lhd a$ holds. As in [6], any completely distributive lattice which has a countable join-dense subset free of supercompact elements will be $\lhd$-*separable*. Examples of this kind of lattices are the unit interval $[0,1]$ or the Tychonoff cube $[0,1]^{\mathbb{N}}$.

Given a set $X$, $L^X$ denotes the complete lattice of all maps from $X$ into $L$ ordered pointwisely, i.e., $f \leqq g$ in $L^X$ if and only if $f(x) \leqq g(x)$ in $L$ for each $x \in X$. Given $f \in L^X$ and $a \in L$, we write $[f \geqq a] = \{x \in X : a \leqq f(x)\}$ and similarly for $[f \rhd a]$.

We now recall the general procedure for generating lattice-valued functions developed in [6]. Let $D \subset L$. A family $\mathcal{F} = \{F_d \subset X : d \in D\}$ is called a *scale* in $X$ if $\mathcal{F}$ is $\lhd$-antitone (i.e., $F_{d_1} \supset F_{d_2}$ whenever $d_1 \lhd d_2$). The function $f \in L^X$ defined by $f(x) = \bigvee\{d \in D : x \in F_d\}$ is said to be *generated* by the scale $\mathcal{F}$. Given $f \in L^X$, both $\{[f \geqq a] : a \in L\}$ and $\{[f \rhd a] : a \in L\}$ are scales that generate the function $f$.

Among the different possibilities to define semicontinuity for lattice-valued functions, in this paper, as in [6] and [9], we will work with the definition below:

For any topological space $X$ and any $f \in L^X$ let

$$f_*(x) = \bigvee_{U \in \mathcal{N}_x} \bigwedge_{y \in U} f(y) \quad \text{and} \quad f^*(x) = \bigwedge_{U \in \mathcal{N}_x} \bigvee_{y \in U} f(y)$$

where $\mathcal{N}_x$ is the family of all open neighborhoods of $x$. It is said that $f$ is *lower* [*upper*] *semicontinuous* if and only if $f = f_*$ $[f = f^*]$. The collections of all lower and upper semicontinuous functions of $L^X$ will be denoted by $LSC(X,L)$ and $USC(X,L)$, respectively. Elements of $C(X,L) = LSC(X,L) \cap USC(X,L)$ are called *continuous*. One easily checks that $f_* \in LSC(X,L)$ and $f^* \in USC(X,L)$. The operations $(\cdot)_*$ and $(\cdot)^*$ are monotone and $f_* \leqq f \leqq f^*$ for each $f \in L^X$. Besides, $(1_A)_* = 1_{\operatorname{Int} A}$ and $(1_A)^* = 1_{\overline{A}}$, where $1_A$ denotes the characteristic function of $A \subset X$. Therefore, $A$ is open (closed) in $X$ iff $1_A \in LSC(X,L)$ $(1_A \in USC(X,L))$. The following properties, proved in [6], will also be needed afterwards:

(P1) $f \in USC(X,L)$ iff $[f \geqq a]$ is closed in $X$ for each $a \in L$,
(P2) $f \in LSC(X,L)$ iff $[f \rhd a]$ is open for each $a \in L$,
(P3) $f \in USC(X,L)$ iff $\overline{F_{a_1}} \subset F_{a_2}$ whenever $a_2 \lhd a_1$,
(P4) $f \in LSC(X,L)$ iff $F_{a_1} \subset \operatorname{Int} F_{a_2}$ whenever $a_2 \lhd a_1$,

where $\{F_a \subset X : a \in L\}$ is any scale generating the function $f \in L^X$.

## 3. Monotone normality

In this section some characterizations of monotone normality, as well as some examples, are given. Of particular interest is the construction of a monotonically normal compactification of an arbitrary topological space, which will have interesting consequences.

Before we proceed some notation should be fixed. Let $X$ be a topological space with topology $o(X)$ and let us denote by $\kappa(X)$ the family of closed subsets of $X$. We shall need the following sets:

$$\mathcal{D}_X = \big\{ (F,U) \in \kappa(X) \times o(X) :\ F \subset U \big\},$$

$$\mathcal{S}_X = \big\{ (A,B) \in 2^X \times 2^X :\ \overline{A} \subset B \text{ and } A \subset \operatorname{Int} B \big\},$$

$$\widehat{\mathcal{S}_X} = \Big\{ (A,B) \in 2^X \times 2^X :\ \overline{A} \subset \bigcap_{y \in X \setminus B} \operatorname{Int}\big(X \setminus \{y\}\big) \text{ and } \bigcup_{x \in A} \overline{\{x\}} \subset \operatorname{Int} B \Big\}.$$

All these sets are partially ordered considering the componentwise order. Note that $\mathcal{D}_X \subset \widehat{\mathcal{S}_X} \subset \mathcal{S}_X$, $\mathcal{S}_X = \widehat{\mathcal{S}_X}$ if $X$ is $T_1$ and $(A,B) \in \widehat{\mathcal{S}_X}$ if and only if $(X \setminus B, X \setminus A) \in \widehat{\mathcal{S}_X}$.

We now recall the definition of monotonically normal spaces of [7] (see also [8] for the formulation presented below). Note that in the definition $T_1$ axiom is not assumed.

Definition 3.1. A topological space $X$ is called *monotonically normal* if there exists and order-preserving function $\Delta : \mathcal{D}_X \to o(X)$ such that

$$K \subset \Delta(K,U) \subset \overline{\Delta(K,U)} \subset U$$

for any $(K,U) \in \mathcal{D}_X$. The function $\Delta$ is called a *monotone normality operator.*

A trivial example of a monotonically normal space, not satisfying $T_1$ axiom, is provided by the reals endowed with the right-order topology (Kolmogorov line).

The proposition below extends Lemma 2.2 in [7] (also Proposition 3 in [10]) and part of Theorem 2.4 in [2]. The proof follows the lines of Proposition 3 in [10].

Proposition 3.2. *Let $X$ be a topological space. The following are equivalent:*

(1) *$X$ is monotonically normal.*

(2) *There exists an order preserving function $\widehat{\Sigma} : \widehat{\mathcal{S}_X} \to o(X)$ such that*

$$A \subset \widehat{\Sigma}(A,B) \subset \overline{\widehat{\Sigma}(A,B)} \subset B$$

*for any $(A,B) \in \widehat{\mathcal{S}_X}$.*

(3) *For each point $x$ and open set $U$ containing $\overline{\{x\}}$ we can assign an open set $H(x,U)$ such that*

(i) $\overline{\{x\}} \subset H(x,U) \subset U$*;*

(ii) *if* $V$ *is open and* $\overline{\{x\}} \subset U \subset V$*, then* $H(x,U) \subset H(x,V)$*;*

(iii) *if* $\overline{\{x\}} \cap \overline{\{y\}} = \emptyset$*, then* $H\big(x, X \setminus \overline{\{y\}}\big) \cap H\big(y, X \setminus \overline{\{x\}}\big) = \emptyset$*.*

(4) *For each point* $x$ *and open set* $U$ *containing* $\overline{\{x\}}$ *we can assign an open set* $G(x,U)$ *containing* $\overline{\{x\}}$ *such that if* $G(x,U) \cap G(y,V) \neq \emptyset$ *then either* $x \in V$ *or* $y \in U$*.*

Proof. (1) $\Rightarrow$ (4). First assume that the monotone normality operator $\Delta$ satisfies $\Delta(F,U) \cap \Delta(X \setminus U, X \setminus F) = \emptyset$ for all $(F,U) \in \mathcal{D}_X$ (see Lemma 2.2 in [7]). For each $x \in X$ and open set $U$ with $\overline{\{x\}} \subset U$ let

$$G(x,U) = \Delta\big(\overline{\{x\}}, U\big)$$

and suppose that $U$ and $V$ are open sets with $\overline{\{x\}} \subset U$ and $\overline{\{y\}} \subset V$ such that neither $x \in V$ nor $y \in U$. The latter means that $V \subset X \setminus \overline{\{x\}}$ and $U \subset X \setminus \overline{\{y\}}$ which implies that

$$G(x,U) \cap G(y,V) \subset \Delta\big(\overline{\{x\}}, X \setminus \overline{\{y\}}\big) \cap \Delta\big(\overline{\{y\}}, X \setminus \overline{\{x\}}\big) = \emptyset.$$

(4) $\Rightarrow$ (3). Take, for each $x \in X$ and open set $U$ containing $\overline{\{x\}}$,

$$H(x,U) = \bigcup_{\overline{\{x\}} \subset W \subset U} G(x,W) \cap U.$$

Conditions (i) and (ii) of (3) are straightforward. Now, if $\overline{\{x\}} \cap \overline{\{y\}} = \emptyset$ assume $H\big(x, X \setminus \overline{\{y\}}\big) \cap H\big(y, X \setminus \overline{\{x\}}\big) \neq \emptyset$. This would mean that there exist open sets $V$ and $W$ such that $\overline{\{x\}} \subset V \subset X \setminus \overline{\{y\}}$ and $\overline{\{y\}} \subset W \subset X \setminus \overline{\{x\}}$ with $G(x,V) \cap G(y,W) \neq \emptyset$. By statement (4) we would get that $x \in W$ or $y \in V$ which is a contradiction.

(3) $\Rightarrow$ (2). For $(A,B) \in \widehat{\mathcal{S}_X}$ let

$$\widehat{\Sigma}(A,B) = \bigcup_{x \in A} H(x, \operatorname{Int} B).$$

Clearly $A \subset \widehat{\Sigma}(A,B)$. Besides, for any $x \in A$ and each $y \in X \setminus B$, we have $\overline{\{x\}} \cap \overline{\{y\}} = \emptyset$ and hence, by (ii) and (iii), $H(x, \operatorname{Int} B) \cap H\big(y, \operatorname{Int}(X \setminus A)\big) = \emptyset$. Since $(X \setminus B, X \setminus A) \in \widehat{\mathcal{S}_X}$, it follows that $\widehat{\Sigma}(A,B) \cap \widehat{\Sigma}(X \setminus B, X \setminus A) = \emptyset$. We conclude that $\overline{\widehat{\Sigma}(A,B)} \subset B$.

Finally, the fact that $\widehat{\Sigma}$ is order-preserving follows immediately from (ii).

(2) $\Rightarrow$ (1). Take the restriction $\Delta = \widehat{\Sigma}|_{\mathcal{D}_X}$. $\square$

Notice that opposite to what happened in Proposition 3 in [10] (or Lemma 2.2 in [7] or Theorem 2.4 in [2]), the equivalence between monotone normality and hereditary monotone normality cannot be derived directly from the previous proposition. Even more, for spaces not satisfying the axiom $T_1$, this equivalence does not hold, as the following construction shows.

REMARK 3.3. Any topological space has a monotonically normal non $T_1$ compactification. Indeed, for a topological space $(X,\tau)$, let $Y$ be a set such that $X \subset Y$ and $Y \setminus X \neq \emptyset$. Define on $Y$ the topology $\tau^\star = \tau \cup \{Y\}$. Then, $X$ is an open, dense subspace of the monotonically normal non $T_1$ compact space $Y$.

Some other examples of monotonically normal non $T_1$ spaces, come from the field of quasi-pseudo-metrics (where by a quasi-pseudo-metric we mean a map $d\colon X \times X \to [0,\infty)$ such that $d(x,y) = d(y,x) = 0$ iff $x = y$ and $d(x,z) \leqq d(x,y) + d(y,z)$ for any $x,y,z \in X$).

EXAMPLE 3.4. Let $K > 0$ and $X = (-\infty,0] \cup [K,+\infty)$. Define the map $d\colon X \times X \to [0,\infty)$ as follows:

$$d(x,y) = \begin{cases} |x-y| & \text{if } \ x,y \leqq 0 \text{ or } x,y \geqq K, \\ y-x-K & \text{if } \ x \leqq 0 \text{ and } y \geqq K, \\ x-y & \text{if } \ y \leqq 0 \text{ and } y \geqq K. \end{cases}$$

The map $d$ defined above is a quasi-pseudo-metric and the collection $\{B_d(x,\varepsilon) : x \in X,\ \varepsilon > 0\}$ (where $B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$) forms a base for a topology $\tau_d$ on $X$.

Clearly the space $(X,\tau_d)$ is not $T_1$ (notice that $\overline{\{0^*\}} = \{0,0^*\}$). Even if monotone normality is not a property easy to manage with, condition (3) of Proposition 3.2 turns out to be very effective to prove that the previous space is monotonically normal.

One could think of the class of quasi-pseudo-metric spaces as a source of examples of monotonically normal spaces. Nevertheless, this is not the case as the following examples show:

EXAMPLES 3.5. (1) Let $X$ be a nonempty set with at least three points, and let $x_0 \in X$. Define $d\colon X \times X \to [0,\infty)$ as $d(x,y) = 0$ if $x = y$ or $y = x_0$ and $d(x,y) = 1$ otherwise. It is obvious that $d$ is a quasi-pseudo-metric, which generates the included point topology, a non normal topology.

(2) The Sorgenfrey plane $\mathbb{R}_s \times \mathbb{R}_s$ is a well known example of a $T_1$ non normal space which is quasi-pseudo-metrizable.

The previous discussion suggests the following open question:

QUESTION. Characterize those quasi-pseudo-metric spaces that are monotonically normal.

## 4. Monotone normality and lattice-valued functions

Monotonically normal spaces will now be characterized in terms of insertion and extension of some kind of lattice-valued functions. Before doing so, we shall need some more notation. Let us consider the following families:

$$UL(X,L) = \left\{ (f,g) \in USC(X,L) \times LSC(X,L) : f \leqq g \right\},$$

$$SF(X,L) = \left\{ (f,g) \in L^X \times L^X : f^* \leqq g \text{ and } f \leqq g_* \right\},$$

$$\widehat{SF}(X,L) = \left\{ (f,g) \in L^X \times L^X : \bigvee_{y \in \overline{\{x\}}} f^*(y) \leqq g(x) \text{ and } f(x) \leqq \bigwedge_{y \in \overline{\{x\}}} g_*(y) \text{ for each } x \in X \right\},$$

which are partially ordered considering the componentwise order. Note that $SF(X,L) = \widehat{SF}(X,L)$ if $X$ is $T_1$.

The following facts will be of interest later on.

Remarks 4.1. (a) $UL(X,L) \subset \widehat{SF}(X,L) \subset SF(X,L)$.

(b) $(A,B) \in \widehat{\mathcal{S}_X}$ if and only if $(1_A, 1_B) \in \widehat{SF}(X,L)$.

The proposition below is a characterization of monotonically normal spaces in terms of insertion of semicontinuous lattice-valued functions. Note that it extends Proposition 3.3 and improves Proposition 3.7 in [9], since $L$ is now only assumed to be completely distributive, instead of being also required to have a countable base. For the case of real-valued functions, the equivalence (1) $\Leftrightarrow$ (3) in the $T_1$-free context was obtained in [8].

Proposition 4.2. *Let $X$ be a topological space and $L$ be a completely distributive lattice. The following are equivalent:*

(1) *$X$ is monotonically normal.*

(2) *There exists an order preserving function $\widehat{\Theta} : \widehat{SF}(X,L) \to LSC(X,L)$ such that $f \leqq \widehat{\Theta}(f,g) \leqq \widehat{\Theta}(f,g)^* \leqq g$ for any $(f,g) \in \widehat{SF}(X,L)$.*

(3) *There exists an order preserving function $\Gamma : UL(X,L) \to LSC(X,L)$ such that $f \leqq \Gamma(f,g) \leqq \Gamma(f,g)^* \leqq g$ for any $(f,g) \in UL(X,L)$.*

Proof. (1) $\Rightarrow$ (2). Let $(f,g) \in \widehat{SF}(X,L)$ and $a \in L$. Then, for each $x \in [f \rhd a]$ we have $a \lhd f(x) \leqq \bigwedge_{y \in \overline{\{x\}}} g_*(y)$ so $\overline{\{x\}} \subset [g_* \rhd a] \subset [g \geqq a]$ (with $[g_* \rhd a]$ open in $X$). Moreover, for any $y \in X \setminus [g \geqq a]$, since $\bigvee_{z \in \overline{\{y\}}} f^*(z) \leqq g(y)$, it follows that $\overline{\{y\}} \subset X \setminus [f^* \geqq a]$ and hence, $[f \rhd a] \subset$

$[f^* \geqq a] \subset \operatorname{Int}\bigl(X \setminus \{y\}\bigr)$ (with $[f^* \geqq a]$ closed in $X$). Therefore, $\bigl([f \rhd a], [g \geqq a]\bigr) \in \widehat{\mathcal{S}_X}$.

Let us define $U_a = \widehat{\Sigma}\bigl([f \rhd a], [g \geqq a]\bigr)$ and $F_a = \overline{\widehat{\Sigma}\bigl([f \rhd a], [g \geqq a]\bigr)}$ (with $\widehat{\Sigma}$ the order-preserving function of Proposition 3.2 (2)). The families $\{U_a\}_{a\in L}$ and $\{F_a\}_{a\in L}$ are scales of open and closed sets, respectively, which generate a pair of maps $h \in LSC(X,L)$ and $k \in USC(X,L)$ such that $f \leqq h \leqq k \leqq g$. Consequently, $f \leqq h \leqq h^* \leqq g$. Now, define $\widehat{\Theta} : \widehat{SF}(X,L) \to LSC(X,L)$ as $\widehat{\Theta}(f,g) = h$ and the proof is done.

(2) $\Rightarrow$ (3). Take $\Gamma = \widehat{\Theta}|_{UL(X,L)}$.

(3) $\Rightarrow$ (1). Let $a \in L \setminus \{0\}$ be such that $0 \lhd a \lhd 1$. Let us define $\Delta : \mathcal{D}_X \to o(X)$ by $\Delta(K,U) = \bigl[\Gamma(1_K, 1_U) \rhd a\bigr]$. Then, since $1_K \leqq \Gamma(1_K, 1_U) \leqq \Gamma(1_K, 1_U)^* \leqq 1_U$, it follows that

$$K = [1_K \rhd a] \subset \bigl[\Gamma(1_K, 1_U) \rhd a\bigr] \subset \bigl[\Gamma(1_K, 1_U)^* \geqq a\bigr] \subset [1_U \geqq a] = U$$

i.e., $K \subset \Delta(K,U) \subset \overline{\Delta(K,U)} \subset U$. Moreover, since $\Gamma$ is order-preserving then so is $\Delta$. $\square$

Let us now recall a result from [9], which will be needed to prove the characterization of monotone normality in terms of extension of lattice-valued functions in the $T_1$-free context. Note that for the case of real-valued functions, the result below was given by Kubiak in [8].

THEOREM 4.3. *Let $X$ be a topological space and $L$ be a completely distributive $\lhd$-separable lattice. The following statements are equivalent:*

(1) *$X$ is monotonically normal.*

(2) [Monotone Katětov–Tong theorem] *There exists an order-preserving function $\Lambda : UL(X,L) \to C(X,L)$ such that $f \leqq \Lambda(f,g) \leqq g$ for any $(f,g) \in UL(X,L)$ ($\Lambda$ is called a monotone inserter).*

Our final result extends Theorem 2.3 in [9] to the $T_1$-free context. That theorem was a generalization to lattice-valued functions of the extension theorem for monotonically normal spaces given by Stares in [11, Theorem 2.3].

THEOREM 4.4. *Let $X$ be a topological space and $L$ be a completely distributive $\lhd$-separable lattice. The following are equivalent:*

(1) *$X$ is monotonically normal.*

(2) *For every closed subspace $A \subset X$ there exists an order-preserving function $\Phi_A : C(A,L) \to C(X,L)$ such that $\Phi_A(f)\big|_A = f$ for all $f \in C(A,L)$ and which satisfies the following two conditions:*

(a) *If $A_1 \subset A_2$ are closed subspaces and $f_1 \in C(A_1,L)$, $f_2 \in C(A_2,L)$ are such that $f_2|_{A_1} \geqq f_1$ and $f_2(x) = 1$ for any $x \in A_2 \setminus A_1$, then $\Phi_{A_2}(f_2) \geqq \Phi_{A_1}(f_1)$.*

(b) *If $A_1 \subset A_2$ are closed subspaces and $f_1 \in C(A_1, L)$, $f_2 \in C(A_2, L)$ are such that $f_2|_{A_1} \leqq f_1$ and $f_2(x) = 0$ for any $x \in A_2 \setminus A_1$, then $\Phi_{A_2}(f_2) \leqq \Phi_{A_1}(f_1)$.*

PROOF. (1) $\Rightarrow$ (2). For any closed $A \subset X$ let $\Phi_A : C(A, L) \to C(X, L)$ be defined by $\Phi_A(f) = \Lambda(h_f, g_f)$, where $h_f, g_f : X \to L$ are such that $h_f = f = g_f$ on $A$, $h_f = 0$ and $g_f = 1$ on $X \setminus A$ and $\Lambda$ the monotone inserter of Theorem 4.3. If $A_1 \subset A_2$ are closed subspaces and $f_1 \in C(A_1, L)$, $f_2 \in C(A_2, L)$ are such that $f_2|_{A_1} \geqq f_1$ and $f_2(x) = 1$ for any $x \in A_2 \setminus A_1$, then $h_{f_1} \leqq h_{f_2}$ and $g_{f_1} \leqq g_{f_2}$ so

$$\Phi_{A_1}(f_1) = \Lambda(h_{f_1}, g_{f_1}) \leqq \Lambda(h_{f_2}, g_{f_2}) = \Phi_{A_2}(f_2)$$

and hence condition (a) is satisfied. Condition (b) is proved similarly.

(2) $\Rightarrow$ (1). In order to prove monotone normality we will use (3) of Proposition 3.2. Let $U$ be open and $x \in X$ such that $\overline{\{x\}} \subset U$. We take the closed subspace $A_U^x = \overline{\{x\}} \cup (X \setminus U)$ and define the maps

$$f_{A_U^x}, g_{A_U^x} : \overline{\{x\}} \cup (X \setminus U) \to L$$

as $f_{A_U^x} = 1_{X \setminus U}$ and $g_{A_U^x} = 1_{\overline{\{x\}}}$. Then $f_{A_U^x}, g_{A_U^x} \in C(A_U^x, L)$ and hence the extensions $\Phi_{A_U^x}(f_{A_U^x}), \Phi_{A_U^x}(g_{A_U^x})$ belong to $C(X, L)$. Let $a \in L \setminus \{0\}$ be such that $0 \lhd a \lhd 1$ and define

$$H(x, U) = \big(X \setminus \big[\Phi_{A_U^x}(f_{A_U^x}) \geqq a\big]\big) \cap \big[\Phi_{A_U^x}(g_{A_U^x}) \rhd a\big].$$

Then, clearly $H(x, U)$ is open and $\overline{\{x\}} \subset H(x, U) \subset U$. Now, if $V$ is open and $\overline{\{x\}} \subset U \subset V$, by property (a) it easy to prove that $\big[\Phi_{A_V^x}(f_{A_V^x}) \geqq a\big] \subset \big[\Phi_{A_U^x}(f_{A_U^x}) \geqq a\big]$ and property (b) yields the inclusion $\big[\Phi_{A_U^x}(g_{A_U^x}) \rhd a\big] \subset \big[\Phi_{A_V^x}(g_{A_V^x}) \rhd a\big]$ so

$$H(x, U) \subset H(x, V).$$

Moreover, if $x, y \in X$ are such that $\overline{\{x\}} \cap \overline{\{y\}} = \emptyset$, one easily checks that

$$H\big(x, X \setminus \overline{\{y\}}\big) \cap H\big(y, X \setminus \overline{\{x\}}\big) = \emptyset.$$

By Proposition 3.2 the space is monotonically normal. $\square$

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