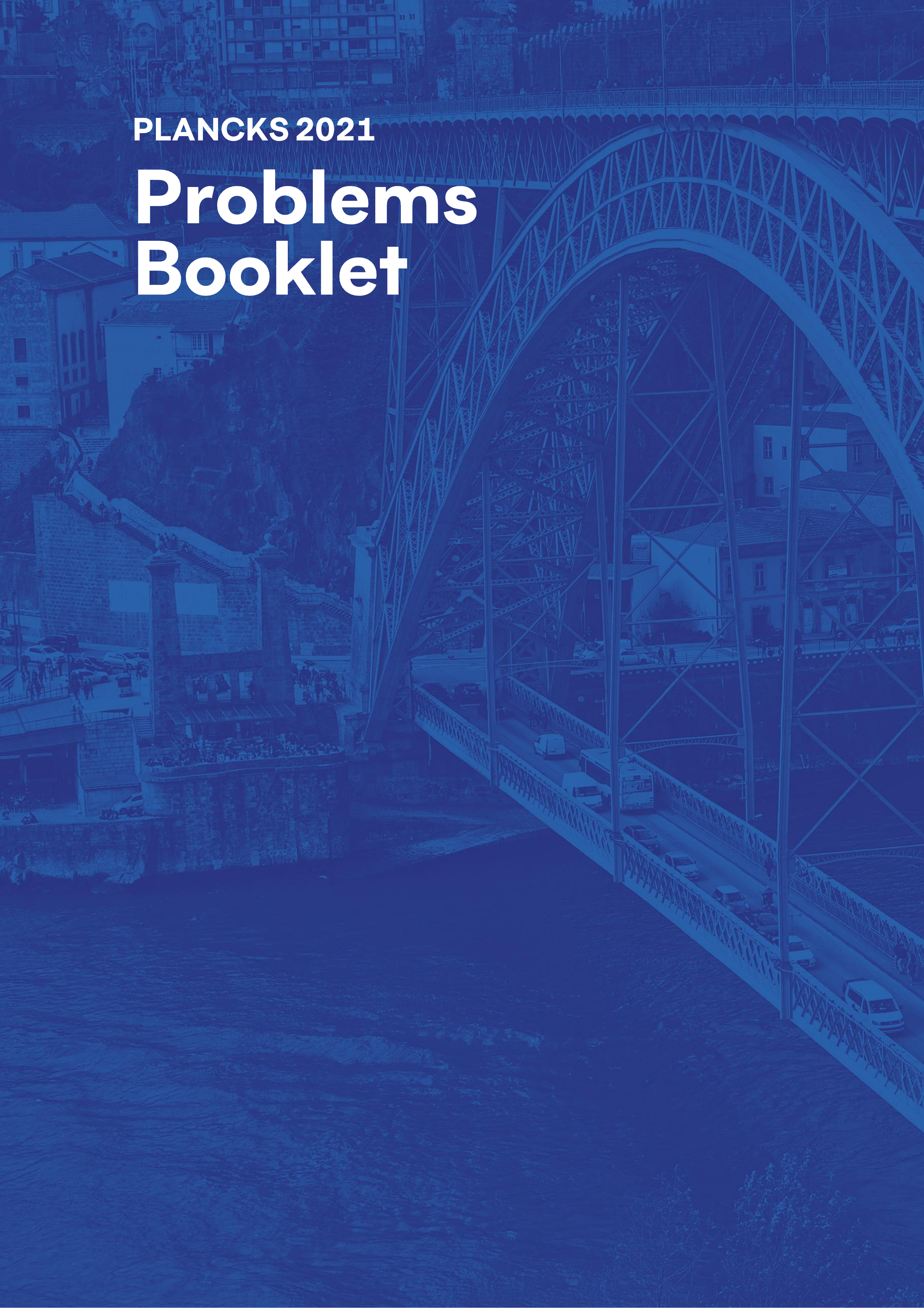
PLANCKS 2021

Problems Booklet



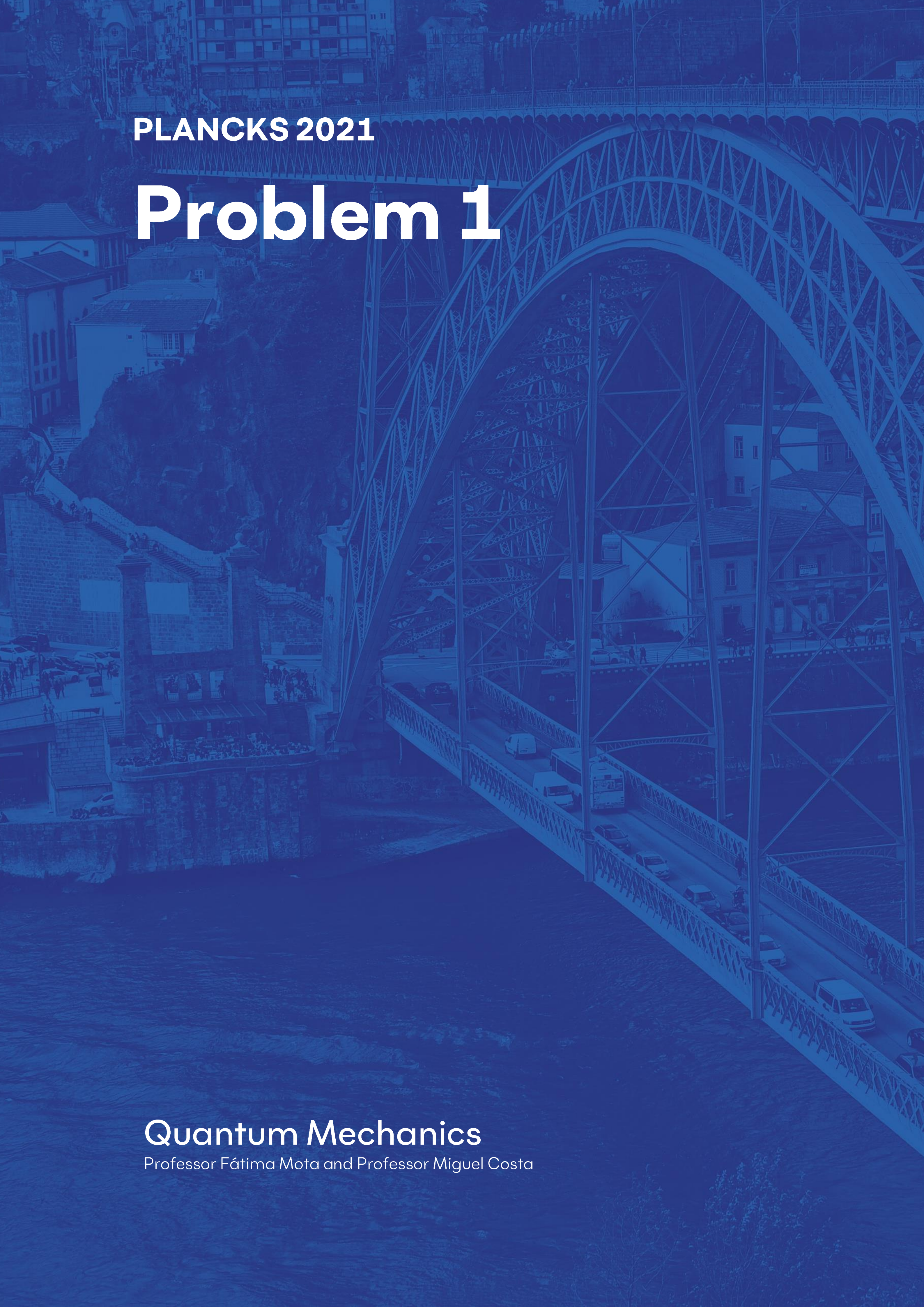
Rules

Dear contestants, Welcome to PLANCKS 2021!

Here are some rules and information you must read before starting to solve each problem:

1. The language used in the competition is English. If you use another language, it will not be considered.
2. The contest consists of 12 problems, each worth 100/1200 points. Subdivisions of points are indicated in the exercises.
3. All the exercises must be handed in separately and only one PDF per problem. For this purpose, you have different problem spaces in Moodle.
4. Please, when scanning your solution to a problem (you can use for instance CamScanner - Phone PDF Creator), write your team number on each page.
5. Make sure that your resolution is readable. Otherwise, the marking team has the right to not consider and reject the submission.
6. If you identify your team through other means that are not your team number, the submission will be rejected.
7. When a problem is unclear, a participant can ask, by tagging a @Supervisor - Problem X on Discord (where X is the problem number) for a clarification. If the response is relevant to all teams, the OC will provide this information to all.
8. You have 16 hours to submit this problem.
9. All the resolutions must be submitted before Saturday, May 8th, 12:00 (GMT+1).
10. Books and other sources can be consulted during the competition.
11. The organisation has the right to disqualify teams for misbehavior or breaking the rules. So please play fair, because we will know if you have not.
12. In situations to which no rule applies, the OC decides.

Best of luck!
May the best physics team win!



PLANCKS 2021

Problem 1

Quantum Mechanics

Professor Fátima Mota and Professor Miguel Costa

Consider a charged particle of mass m and charge q placed in a 3-dimensional isotropic harmonic potential of frequency ω_0.

Question 1 [30 points]

Assume the particle is placed in a time dependent and spatially uniform magnetic field $\boldsymbol{B} = \boldsymbol{B}_0 \sin(\omega t)$. Using first order perturbation theory indicate the allowed transitions from the ground state, calculate the correspondent amplitude for transition probability and show how you could calculate the transition probability (you don't have to carry out this final calculation).

Question 2 [70 points]

Now consider a system of twenty non-interacting identical particles of mass m and spin 1/2, placed in such an harmonic potential.

a. [25 points] Determine the ground state energy of the system.
b. [45 points] Assume that a constant magnetic field $\boldsymbol{B}$ is applied. Show that the diamagnetic susceptibility is constant.

Useful information

Remember:

- minimal coupling procedure $\boldsymbol{p} \to \boldsymbol{p} - q\boldsymbol{A}$
- the vector potential for an uniform magnetic field may be written as: $\boldsymbol{A} = \frac{1}{2}\boldsymbol{B} \times \boldsymbol{R}$

You may use the following information for the eigenfunctions of the 3-dimensional isotropic harmonic potential, in the $\{|n_x, n_y, n_z\rangle\}$ basis:

$$\begin{aligned}
|n_x = 0, n_y = 0, n_z = 0\rangle &= |k = 0, l = 0, m = 0\rangle \\
\hline
|n_x = 0, n_y = 0, n_z = 1\rangle &= |k = 0, l = 1, m = 0\rangle \\
|n_x = 1, n_y = 0, n_z = 0\rangle &= \tfrac{1}{\sqrt{2}}[|k = 0, l = 1, m = -1\rangle - |k = 0, l = 1, m = 1\rangle] \\
|n_x = 0, n_y = 1, n_z = 0\rangle &= \tfrac{i}{\sqrt{2}}[|k = 0, l = 1, m = -1\rangle + |k = 0, l = 1, m = 1\rangle] \\
\hline
|n_x = 1, n_y = 1, n_z = 0\rangle &= \tfrac{-i}{\sqrt{2}}[|k = 0, l = 2, m = 2\rangle + |k = 0, l = 2, m = -2\rangle] \\
|n_x = 1, n_y = 0, n_z = 1\rangle &= \tfrac{-1}{\sqrt{2}}[|k = 0, l = 2, m = 1\rangle - |k = 0, l = 2, m = -1\rangle] \\
|n_x = 0, n_y = 1, n_z = 1\rangle &= \tfrac{i}{\sqrt{2}}[|k = 0, l = 2, m = 1\rangle + |k = 0, l = 2, m = -1\rangle] \\
|n_x = 2, n_y = 0, n_z = 0\rangle &= -\tfrac{1}{\sqrt{3}}|k = 2, l = 0, m = 0\rangle - \tfrac{1}{\sqrt{6}}|k = 0, l = 2, m = 0\rangle \\
&\quad + \tfrac{1}{2}|k = 0, l = 2, m = 2\rangle + \tfrac{1}{2}|k = 0, l = 2, m = -2\rangle \\
|n_x = 0, n_y = 2, n_z = 0\rangle &= -\tfrac{1}{\sqrt{3}}|k = 2, l = 0, m = 0\rangle - \tfrac{1}{\sqrt{6}}|k = 0, l = 2, m = 0\rangle \\
&\quad - \tfrac{1}{2}|k = 0, l = 2, m = 2\rangle - \tfrac{1}{2}|k = 0, l = 2, m = -2\rangle \\
|n_x = 0, n_y = 0, n_z = 2\rangle &= -\tfrac{2}{\sqrt{3}}|k = 2, l = 0, m = 0\rangle + \sqrt{\tfrac{2}{3}}|k = 2, l = 2, m = 0\rangle
\end{aligned}$$

Solutions

Question 1 [30 points]

The Hamiltonian of the particle in the magnetic field is:

$$\hat{H} = \frac{1}{2m}\left[\hat{P} - qA\right]^2 + V(\hat{R}) + q\phi$$

We can choose to work in the Coulomb gauge: $\nabla . A(r,t) = 0$. In this gauge, the scalar potential is zero. So we obtain:

$$\hat{H} = \frac{\widehat{P^2}}{2m} + V(\hat{R}) - \frac{q}{m}\hat{P}.A + \frac{q^2}{2m}A^2$$

The vector potential for the uniform magnetic field may be written as:

$$A = \frac{1}{2}B \times R$$

And we can choose a frame where:

$$B(t) = B_0 \sin(\omega t)\,\hat{e_z}$$

So we obtain (neglecting the coupling between orbital and spin magnetic moments):

$$\hat{H} = \hat{H}_0 - \frac{q}{2m}B_0 \sin(\omega t)\hat{L}_z + \frac{q^2 B_0^2}{8m}\sin^2(\omega t)\left(\hat{X}^2 + \hat{Y}^2\right).$$

The ground state of the 3-dimensional isotropic harmonic potential has $l = 0$ and so the term

$$\frac{q}{2m}B_0 \sin(\omega t)\hat{L}_z$$

does not induce any transition from the ground state. Next we consider the perturbation quadratic in the magnetic field. In the number representation we have:

$$\hat{X} = C\left(\hat{a}_x^\dagger + \hat{a}_x\right) \qquad \hat{Y} = C\left(\hat{a}_y^\dagger + \hat{a}_y\right)$$

where C is a constant, and so, in first order perturbation theory we just have to analyze when the matrix element

$$\langle 000|\hat{a}_x^\dagger\hat{a}_x^\dagger + \hat{a}_x^\dagger\hat{a}_x + \hat{a}_x\hat{a}_x^\dagger + \hat{a}_x\hat{a}_x + \hat{a}_y^\dagger\hat{a}_y^\dagger + \hat{a}_y^\dagger\hat{a}_y + \hat{a}_y\hat{a}_y^\dagger + \hat{a}_y\hat{a}_y)|n_x n_y n_z\rangle$$

is different from zero. We see that the allowed transitions are:

$$\boxed{(000) \leftrightarrow (200) \qquad (000) \leftrightarrow (020)}$$

Now we evaluate the transition probability using time dependent perturbation theory. The amplitude of probability, at first order, for a transition between an initial state $|i\rangle$ and a final state $|f\rangle$ is given by:

$$A_{if}^{(1)} = \frac{-i}{\hbar}\int_0^\infty d\,t' V_{fi}(t') e^{i\omega_{fi}t'}$$

where

$$V_{fi}(t') = \langle f|\hat{V}(t')|i\rangle$$

and

$$\omega_{fi} = \frac{E_f - E_i}{\hbar}$$

The amplitudes of transition are equals in both cases, so we just consider the transition $(000) \to (200)$. In terms of creation and destrution operators:

$$\hat{X}^2 = \frac{\hbar}{2m\omega_0}(\hat{a}_x^\dagger + \hat{a}_x)(\hat{a}_x^\dagger + \hat{a}_x); \qquad \hat{Y}^2 = \frac{\hbar}{2m\omega_0}(\hat{a}_y^\dagger + \hat{a}_y)(\hat{a}_y^\dagger + \hat{a}_y)$$

so

$$\begin{aligned} V_{fi}(t') &= \left\langle 000 \left| \frac{q^2 B_0^2}{8m} \sin^2(\omega t')(\hat{X}^2 + \hat{Y}^2) \right| 200 \right\rangle \\ &= \frac{q^2 B_0^2}{8m} \sin^2(\omega t') \frac{\hbar}{\sqrt{2} m\omega_0} = \frac{q^2 B_0^2 \hbar}{8\sqrt{2} m^2 \omega_0} \sin^2(\omega t') \end{aligned}$$

Consequently we can write:

$$A_{if}^{(1)}(t) = \frac{-i}{\hbar} \frac{q^2 B_0^2 \hbar}{8\sqrt{2} m^2 \omega_0} e^{-E_i t/\hbar} \int_0^t d\,t' sin^2(\omega t) e^{i\omega_{fi} t'}$$

We now do some calculations:

$$\begin{aligned} A_{if}^{(1)} &= \frac{-i}{\hbar} \frac{q^2 B_0^2 \hbar}{8\sqrt{2} m^2 \omega_0} \frac{1}{2} e^{-E_i t/\hbar} \int_0^t d\,t' \left(1 - \cos(2\omega t')\right) e^{i\omega_{fi} t'} \\ &= \frac{-i}{\hbar} \frac{q^2 B_0^2 \hbar}{8\sqrt{2} m^2 \omega_0} \left(\frac{1}{2}\right) e^{-E_i t/\hbar} \int_0^t d\,t' \left[e^{i\omega_{fi} t'} - e^{i(\omega_{fi} + 2\omega) t'} - e^{i(\omega_{fi} - 2\omega) t'}\right] \\ &= \frac{q^2 B_0^2 \hbar}{16\sqrt{2} m^2 \omega_0} e^{-E_i t/\hbar} \left\{ e^{i\omega_{fi} t} \left[\frac{-i}{\omega_{if}} + \frac{i e^{2i\omega t}}{\omega_{fi} + 2\omega} + \frac{i e^{-2i\omega t}}{\omega_{fi} - 2\omega}\right] - i \frac{\omega_{if}^2 + 4\omega^2}{\omega_{if}(\omega_{if}^2 + 4\omega^2)} \right\} \end{aligned}$$

The transition probability is given by:

$$P_{if} = \left|A_{if}^{(1)}\right|^2$$

Question 2 [70 points]

a. The energy of a particle in a 3-dimensional isotropic harmonic potential of frequency ω is given by:

$$E_n = \left(n + \frac{3}{2}\right)\hbar\omega$$

and the degeneracy of each level is, accounting for a factor of 2 for fermions $s = 1/2$ each state can accommodate 2 particles. So we have:

$$g_n = (n+1)(n+2)$$

So:

(-) 2 particles in state $n = 1$; so a contribution $2 \times 3/2(\hbar\omega)$;

(-) 6 particles in state $n = 2$; so a contribution $6 \times 5/2(\hbar\omega)$;

(-) 12 particles in state $n = 2$; so a contribution $12 \times 7/2(\hbar\omega)$.

So the ground state energy is $60\hbar\omega$

b. As the 20 particles occupy completely 3 shells, we have $L = S = J = 0$ for each shell. Since we have to calculate the correction to ground state energy of the 3-dimensional isotropic harmonic potential, we have to construct the wave functions for each states (a total of 1 + 3+ 6 functions). Now since we are dealing with fermions, the total function must be antisymmetric with respect to permutation of particles. Since the space functions are symmetric, spins states must be antisymmetric and so only the state $|S = 0, M_s = 0\rangle$ need to be considered. For the pair of particles i and j:

$$\left[\left|S = 0, M_S = 0\right\rangle_{ij} = \frac{1}{\sqrt{2}}\left|\underset{i\,j}{\uparrow\downarrow}\right\rangle - \left|\underset{j\,i}{\uparrow\downarrow}\right\rangle$$

Using the information we write for the space functions in the basis $\{|k, l, m\rangle\}$. For example, for the ground state and the first excited state of the unperturbed Hamiltonian, we should obtain:

• for the ground level

$$\left|\psi_0\right\rangle = \left[\frac{1}{\sqrt{2}}(|k=0, l=0, m=0\rangle_1 + |k=0, l=0, m=0\rangle_2)\right]\left|S = 0, M_S = 0\right\rangle_{12}$$

• for the first excited level (using the same notation)

$$|\psi_{11}\rangle = \frac{1}{2}[(|k=0, l=1, m=-1\rangle_3 - |k=0, l=1, m=1\rangle_3)|k=0, l=1, m=-1\rangle_4 - |k=0, l=1, m=1\rangle_4)]$$
$$|S = 0, M_S = 0\rangle_{34}$$

$$|\psi_{12}\rangle = \frac{-1}{2}[(|k=0, l=1, m=-1\rangle_5 + |k=0, l=1, m=1\rangle_5)(|k=0, l=1, m=-1\rangle_6 + |k=0, l=1, m=1\rangle_6)]$$
$$|S = 0, M_S = 0\rangle|S = 0, M_S = 0\rangle_{56}$$

$$\left|\psi_{13}\right\rangle = \frac{1}{\sqrt{2}}(|k=0, l=1, m=0\rangle_7 + |k=0, l=1, m=0\rangle_8)\left|S = 0, M_S = 0\right\rangle_{78}$$

And analogously for the second excited state.

The Hamiltonian of the particle in the magnetic field is (ignoring the couplings between orbital and spin moments, which would in fact contribute to zero):

$$\hat{H} = \sum_{i=1}^{8}\left[\frac{\widehat{P_\iota^2}}{2m} + \frac{1}{2}m\omega^2\widehat{R_\iota^2} - \sum_i^{20}\widehat{\mu_{s_\iota}}.\boldsymbol{B} - \sum_i^{20}\frac{q}{2m}\widehat{\boldsymbol{L}_\iota}.\boldsymbol{B} + \frac{q}{2m}A^2\right]$$

Now the correction due to term $\widehat{\mu_s}.B$ gives zero as we show in the following.

Remember that, using the frame we chosen:

$$\sum_i^{20}\widehat{\mu_{s_\iota}}B = \gamma\left(\widehat{S_{1z}} + \widehat{S_{2z}} + \cdots\right)B$$

Since the spins functions are antisymmetric in respect to permutations of particles:

$$\langle\psi_{\ldots}\,\Sigma_i^{20}\,\widehat{\mu_{s_\iota}}B\,|\psi_{\ldots}\rangle = 0$$

In same way we see that since the orbitals functions have the same coefficient for symmetric values of m:

$$\sum_i^{20}\frac{q}{2m}\widehat{L_\iota}.\boldsymbol{B} = \frac{q}{2m}\left(\widehat{\boldsymbol{L}_{1z}} + \widehat{\boldsymbol{L}_{2z}} + \cdots\right)B$$

And as subshells are complete:

$$\left\langle\psi_{\ldots}\,\frac{q}{2m}\left(\widehat{L_{1z}} + \widehat{L_{2z}} + \cdots\right)B\middle|\psi_{\ldots}\right\rangle = 0$$

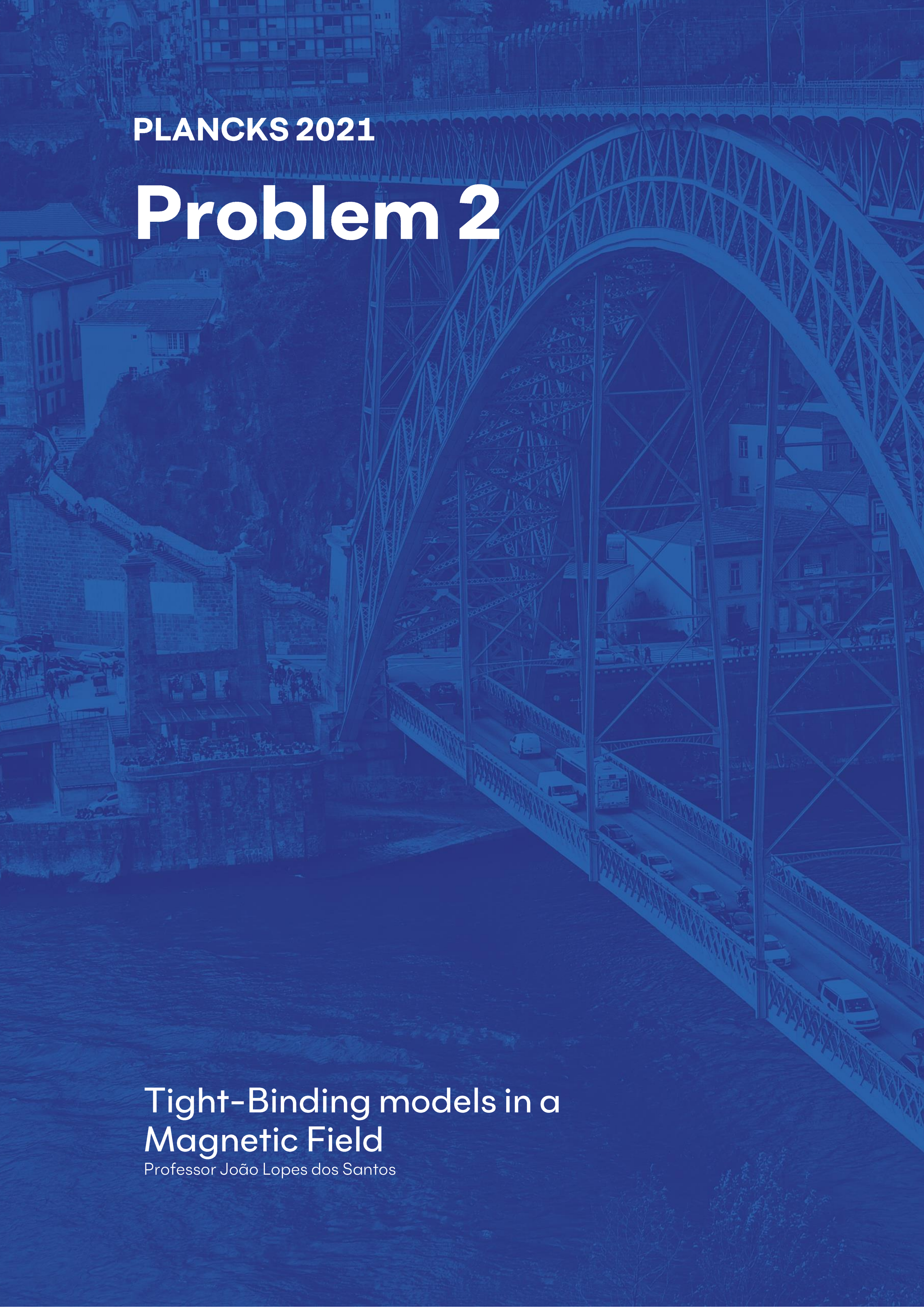
So the correction to energy is proportional to B^2:

$$E = 87\hbar\omega + \frac{q^2}{8m}\sum_i^{20}\left(\hat{B}\times\widehat{R_\iota}\right)^2$$

and so the magnetization M is:

$$M = -\frac{dE}{dB} = \chi B$$

and so the diamagnetic susceptibility is constant.



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Problem 2

Tight-Binding models in a Magnetic Field

Professor João Lopes dos Santos

Introduction

The problem of the spectrum of an electron gas in a magnetic field is addressed by the minimal coupling prescription

$$\frac{\nabla}{i} \to \frac{\nabla}{i} - \frac{q}{\hbar}\mathbf{A}(\mathbf{r}) \quad (1)$$

where $\mathbf{A}$ is the vector potential and the magnetic field is $\mathbf{B} = \nabla \times \mathbf{A}$. It is not difficult to show that in that an electron gas in two dimensions, for uniform $\mathbf{B}$ normal to the electron gas, one has a discrete spectrum of *Landau Levels*,

$$\epsilon_n = \left(n + \frac{1}{2}\right)\hbar\omega_c, \qquad \omega_c = \frac{eB}{m} \quad (2)$$

where ω_c is the classical angular frequency of electron orbits. Furthermore, each level has a degeneracy (not counting spin) equal to the number of flux quanta in the sample, i.e. equal to BA/ϕ_0 where A is the area and $\phi_0 = h/e$ the flux quantum.

In the presence of a lattice potential this problem becomes considerably more complicated.

A tight binding (TB) model assumes a local basis, with one or more states $|\phi_i\rangle$ in each lattice unit cell and is defined by two sets of parameters:

- local orbital energies: $|\phi_i\rangle\epsilon_i\langle\phi_i|$;
- hopping amplitudes between local orbitals: $|\phi_i\rangle t_{ij}\langle\phi_j|$;

The general state is defined by its amplitudes in this basis

$$|\psi\rangle = \sum_i c_i\,|\phi_i\rangle \quad (3)$$

and for a crystalline lattice, Bloch's theorem can be used to solve for the Hamiltonian eigenstates.

The magnetic field is introduced by adding complex phases to the hopping amplitudes (Peierls substitution)

$$t_{ij} \to e^{i\phi_{ij}}t_{ij} \quad (4)$$

with the only requirement that the sum of phases along a loop is proportional to the magnetic flux in the area enclosed by the loop. The choice in Fig. 1 corresponds to a flux in the square of $Ba^2 = \phi$.

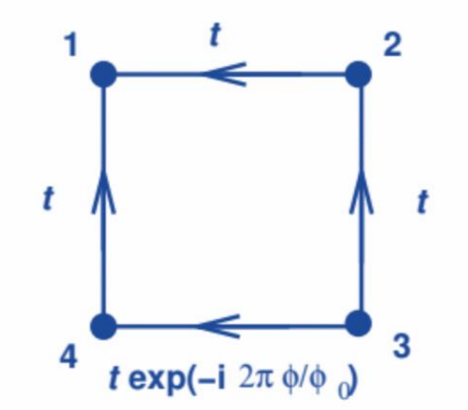


Figure 1 – With t real this choice of phase describes motion in a magnetic field perpendicular to the plane with flux in the square of $Ba^2 = \phi$

1. The square lattice

Assume a square lattice with sites $\mathbf{R}_{m.n} := a\left(m\hat{\mathbf{e}}_x + n\hat{\mathbf{e}}_y\right)$ with non-zero hopping amplitudes $-t$ ($t > 0$) only between nearest neighbors. Take $0 \leq m \leq N_L - 1$ and $0 \leq n \leq N_W - 1$ ($N_W N_L$ unit cells). You may take the site energy to be zero (band center).

1.1. [20 points] You can make a choice of phases to represent a uniform magnetic field affecting only bonds along x, $\langle\psi_{m,n}|\mathcal{H}|\psi_{m\pm1,n}\rangle$. Go ahead and do so. Notice that you break translational invariance only along y direction. Use Bloch's theorem along x to reduce the Hamiltonian eigenvalue problem, for each Bloch wave vector k_x, to a 1D tight-binding chain with a on-site energy $V_n(k_x)$ that varies along the chain and is periodic. Determine $V_n(k_x)$.

1.2. [15 points] In realistic situation, the flux per unit cell, Ba^2, is much smaller that the flux quantum. Confirm this by estimating for $a \sim 1\text{Å}$ the value of B such that $Ba^2 = \phi_0$. Show that the potential $V_n(k)$, for sensible values of B, has a wavelength much greater than the lattice spacing, a. Obtain a continuum limit for your TB equation by assuming that the tight binding amplitudes vary slowly with the index n and can be represented by a continuous function of y computed at the site coordinate $y_n = na$, which can be expanded in a power series. Show that for each k_x, you obtain a Schroedinger equation in 1D (along y) for a particle in a potential $V(y, k_x)$.

1.3. [15points] Look for solution close to the minimum of the potential. Reduce the problem to that of an harmonic oscillator and try to prove the following:

- The low energy spectrum takes the form

$$\epsilon_r = -4t + \left(r + \frac{1}{2}\right)\hbar\omega, \qquad r = 0,1,2 \ldots (5)$$

- if the potential period Ma is larger than the width $N_W a$ of the sample, each level has a degeneracy

$$p = N_W N_L / M;$$

- The form of the spectrum of Eq. 5 only holds for $r \ll M$.

2. The Graphene Lattice

The case of the graphene lattice brings further complications. The honeycomb lattice is not a Bravais lattice and has two carbon atoms per unit cell (see Fig. 2). The general state is

$$|\psi\rangle = \sum_{\mathbf{R}_{m,n}} a_{m,n} \left|\phi^{A}_{m,n}\right\rangle + b_{m,n}\left|\phi^{B}_{m,n}\right\rangle$$

where $\mathbf{R}_{m,n} = m\mathbf{a}_1 + n\mathbf{a}_2$ is a Bravais lattice site, and $\left|\phi^{A}_{m,n}\right\rangle$ and $\left|\phi^{B}_{m,n}\right\rangle$ are the two local orbitals in the unit cell at $\mathbf{R}_{m,n}$. In a minimal model the site energies of all local orbitals is taken to be zero and a non-zero hopping amplitude, $-t$, ($t \approx 3$ eV) exists only between nearest neighbors. The magnetic field can be introduced with phases affecting only the bonds connecting orbitals with the same n.

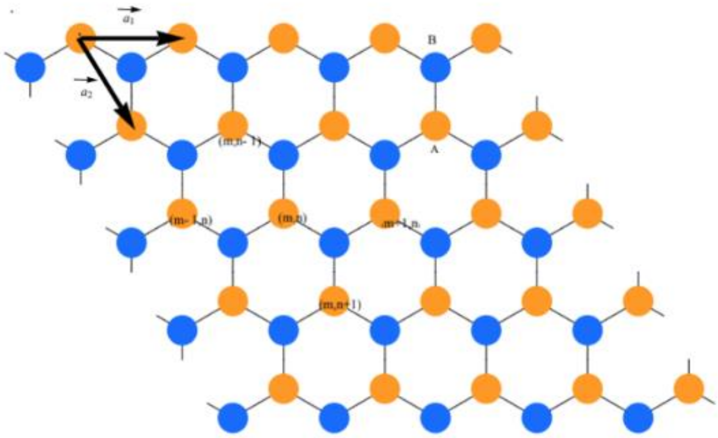


Figure 2- Graphene lattice with basic lattice translations

2.1. [20 points] Use Bloch's Theorem to reduce the eigenvalue problem to a 1D chain with two types of atoms with a space dependent hopping $t_n(k_1)$.

In graphene, the interesting energies are close zero. To obtain these states consider the following suggestions.

2.2. [20 points] Use a phase change of the local orbitals to reduce the TB equation of the AB chain to the form

$$\begin{aligned} \epsilon\tilde{a}_n(k_1) &= -t\left[2\cos\left(\frac{k_1 a}{2} - n\pi\frac{\varphi}{\phi_0}\right)\tilde{b}_n(k_1) - \tilde{b}_{n-1}(k_1)\right] \\ \epsilon\tilde{b}_n(k_1) &= -t\left[2\cos\left(\frac{k_1 a}{2} - n\pi\frac{\varphi}{\phi_0}\right)\tilde{a}_n(k_1) - \tilde{a}_{n+1}(k_1)\right] \end{aligned}$$

2.3. [10 points] You can obtain low energy states $|\epsilon| \ll t$, with slowly varying wavefunctions in an atomic scale, near values of n such that

$$2\cos\left(\frac{k_1 a}{2} - \bar{n}\pi\frac{\varphi}{\phi_0}\right) = 1.$$

Expand the cosine term about $\bar{n}$, use a continuum aproximation for the amplitudes as you did for the square lattice, and obtain equations in the form

$$\begin{bmatrix} 0 & \hat{A}^\dagger(\partial_y, y) \\ \hat{A}(\partial_y, y) & 0 \end{bmatrix}\begin{bmatrix} \psi_a(y) \\ \psi_b(y) \end{bmatrix} = \epsilon \begin{bmatrix} \psi_a(y) \\ \psi_b(y) \end{bmatrix}$$

were the commutator $\left[\hat{A}(\partial_y, y), \hat{A}^\dagger(\partial_y, y)\right]$ is a c-number. Recall the commutation relation of harmonic oscillator operators, $[a, a^\dagger] = 1$, and try to figure out the low energy spectrum from this.

Solutions

1. The square lattice

Model

A square lattice, with lattice parameter a, of dimensions $N_W a$ along x with open boundary conditions (BC) and $N_L a$ along y with periodic BC. Sites

$$\mathbf{R}_{m.n} := a\left(m\hat{\mathbf{e}}_x + n\hat{\mathbf{e}}_y\right)$$

$m = ,0, \dots, N_L - 1$ $n = 0, N_W - 1$. Notation $c_{m,n}$ is amplitude of state in local orbital at $\mathbf{R}_{m.n}$.

Site energies $\epsilon_{m,n} = 0$;hopping $-t$ between nearest neighbors.

Tight Binding Equations

$$\epsilon c_{m,n} = -t\left[c_{m+1,n} + c_{m-1,n} + c_{m,n+1} + c_{m,n-1}\right]$$

With magnetic field with Peierls phases along horizontal bonds:

$$\epsilon c_{m,n} = -t\left[e^{-i2\pi n\varphi/\phi_0} c_{m+1,n} + e^{i2\pi n\varphi/\phi_0} c_{m-1,n} + c_{m,n+1} + c_{m,n-1}\right]$$

Over an elementary square run anti-clockwise

$$\begin{aligned}\prod_{<ij>} t_{ij} &= t^4 \exp\left[i2\pi n\frac{\varphi}{\phi_0} - i2\pi(n+1)\frac{\varphi}{\phi_0}\right] t^4 \\ &= t^4 \exp\left[-i2\pi\frac{\varphi}{\phi_0}\right]\end{aligned}$$

Thus describing uniform field along Oz $Ba^2 = \varphi$.

Bloch's theorem along Ox direction

$$c_{m,n} = c_n(k)^{ikma}$$

$$\begin{aligned}\epsilon c_n(k) &= -t\left[c_{n+1}(k) + c_{n-1}(k) + \left(e^{i(ka-2\pi n\varphi/\phi_0)} + e^{-i(ka-2\pi n\varphi/\phi_0)}\right)c_n(k)\right] \\ &= -t[c_{n+1}(k) + c_{n-1}(k) + 2\cos(ka - 2\pi n\varphi/\phi_0)c_n(k)] \\ \epsilon \quad & c_n(k) = V_n(k)c_n(k) - t\left(c_{n+1}(k) + c_{n-1}(k)\right)\end{aligned}$$

For each k this is a TB hamiltonian for a 1D chain with onsite energy $V_n(k) = -2t\cos(ka + 2\pi n\varphi/\phi_0)$.

The continuum limit

Period of $V_n(k)$ in n

$$2\pi N\varphi/\phi_0 = 2\pi$$

or

$$N = \frac{\phi_0}{\varphi}$$

so if $\varphi \ll \phi_0$ or

$$Ba^2 \ll \frac{h}{e} = 4.2 \times 10^{-15}\,\mathrm{T\,m^2}$$

For $a \sim 10^{-10}, B \ll 4 \times 10^5$ T the onsite potencial is slowly varying on an atomic scale: wavelenght of potential $\gg$ lattice parameter.

Look for solutions which are slowly varying of an atomic scale $c_n(k) = \psi(y,k)$ where $\psi(y,k)$ is continuous in x

$$\begin{aligned} c_{n\pm1}(k) &= \psi(y_n,k) \pm a\partial_y\psi(y,k)\big|_{y=y_n} + \frac{a^2}{2}\partial_y^2\psi(y,k)\big|_{y=y_n} \\ c_{n+1}(k) + c_{n-1}(k) &= 2\psi(y_n,k) + a^2\partial_y^2\psi(y,k)\big|_{y=y_n} \end{aligned}$$

$$\epsilon\psi(y,k) = V(y,k) - 2t\psi(y,k) - ta^2\,\partial_y^2\psi(y,k)$$

or

$$-ta^2\,\partial_y^2\psi(y,k) - 2t\left[1 + \cos\left(ka - 2\pi\frac{y}{a}\frac{\varphi}{\phi_0}\right)\right]\psi(y,k) = \epsilon\psi(y,k)$$

Furthermore if $N > N_w$ there is only one minimum inside ribbon for

$$\begin{aligned} n &= \frac{ka}{2\pi}\frac{\phi_0}{\varphi} = \frac{ka}{2\pi}N \\ 0 < k &< \frac{2\pi}{a}\frac{N_W}{N} \end{aligned}$$

For states near minimum

$$\begin{aligned} ka - 2\pi\frac{y}{a}\frac{\varphi}{\phi_0} &= 0 \\ \overline{y}(k) &= ka^2\frac{\phi_0}{2\varphi} \end{aligned}$$

$$1 + \cos\left(k - 2\pi\frac{\overline{y}}{a}\frac{\varphi}{\phi_0} - 2\pi\frac{y-\overline{y}}{a}\frac{\varphi}{\phi_0}\right) \approx 2 - \frac{1}{2}4\pi^2\left(\frac{\varphi}{\phi_0}\right)^2\frac{(y-\overline{y})^2}{a^2}$$

so

$$\left[-ta^2\,\partial_y^2 - 4t + t\frac{4\pi^2}{a^2}\left(\frac{\varphi}{\phi_0}\right)^2(y-\overline{y}(k))^2\right]\psi(y,k) = \epsilon\psi(y,k)$$

1 D harmonic oscillator

$$\hbar^2/2m^* = ta^2$$

$$\begin{aligned} \frac{1}{2}m^*\omega^2 &= 4\pi^2\frac{t}{a^2}\left(\frac{\varphi}{\phi_0}\right)^2 \\ \omega^2 &= 8\pi^2\frac{t}{m^*a^2}\left(\frac{\varphi}{\phi_0}\right)^2 = \frac{16\pi^2}{\hbar^2}t^2\left(\frac{\varphi}{\phi_0}\right)^2 \end{aligned}$$

or

$$\hbar\omega = 4\pi t\left(\frac{\varphi}{\phi_0}\right) = 4\pi ta^2\frac{B}{\phi_0} = \frac{2\pi\hbar^2 eB}{m^*h} = \hbar\frac{eB}{m^*}$$

The characteristic length scale of harmonic oscillator wave functions

$$\ell_\omega^2 = \frac{\hbar}{m^*\omega} = \left(\frac{2ta^2}{\hbar}\right)\left(\frac{\hbar}{4\pi t}\frac{\phi_0}{\varphi}\right) = \frac{1}{2\pi}a^2\frac{\phi_0}{\varphi}$$

The expansion of the cosine, only works if

$$2\pi\frac{\ell_\omega^2}{a^2}\left(\frac{\varphi}{\phi_0}\right)^2 \ll 1$$

or

$$\frac{\varphi}{\phi_0} \ll 1.$$

In that limit we find evenly spaced states of energy and only for states with energy smaller than t

$$r\hbar\omega \ll t$$

or

$$r \ll \frac{\phi_0}{\varphi} \ll N$$

In this limit the spectrum is a series of evenly spaced states of energies

$$-4t + \left(n + \frac{1}{2}\right)\hbar\omega$$

with a degeneracy correponding to the k values

$$\frac{2\pi/a}{2\pi/N_l a}\frac{N_w}{N} = \frac{N_w N_L}{N}$$

2. The Graphene Lattice

Phases

We start by setting up the TB equations with Field

$$\begin{aligned} \epsilon a_{m,n} &= -t\left(b_{m,n} + b_{m,n-1} + b_{m-1n}\right) \\ \epsilon b_{m,n} &= -t\left(a_{m,n} + a_{m,n+1} + a_{m+1,n}\right) \end{aligned}$$

Next we choose gauge by adding phase along bonds between sites of same n

$$\begin{aligned} \epsilon a_{m,n} &= -t\left(e^{-i\pi n\varphi/\phi_o} b_{m,n} + b_{m,n-1} + e^{i\pi n\varphi/\phi_o} b_{m-1n}\right) \\ \epsilon b_{m,n} &= -t\left(e^{i\pi n\varphi/\phi_o} a_{m,n} + a_{m,n+1} + e^{-i\pi n\varphi/\phi_o} a_{m+1,n}\right) \end{aligned}$$

Notice that

$$\varphi = \frac{\sqrt{3}}{2}Ba^2$$

Bloch theorem along $\mathbf{a}_1$:

$$\begin{aligned} a_{m,n} &= a_n(k)e^{ikma} \\ b_{m,n} &= b_n(k)e^{ikma} \end{aligned}$$

and

$$\begin{aligned} \epsilon a_n(k) &= -t\left(e^{-i\pi n\varphi/\phi_o}b_n(k) + b_{n-1}(k) + e^{-ika}e^{i\pi n\varphi/\phi_o}b_n(k)\right) \\ \epsilon b_{m,n} &= -t\left(e^{i\pi n\varphi/\phi_o}a_n(k) + a_{n+1}(k) + e^{ika}e^{-i\pi n\varphi/\phi_o}a_n(k)\right) \end{aligned}$$

$$\begin{aligned} \epsilon a_n(k) &= -t\big[\big(e^{-i\pi n\varphi/\phi_o} + e^{-ika}e^{i\pi n\varphi/\phi_o}\big)b_n(k) + b_{n-1}(k)\big] \\ \epsilon b_n(k) &= -t\big[\big(e^{i\pi n\varphi/\phi_o} + e^{ika}e^{-i\pi n\varphi/\phi_o}\big)a_n(k) + a_{n+1}(k)\big] \end{aligned}$$

This has the form

$$\begin{aligned} \epsilon a_n(k) &= [t_n^*(k)b_n(k) - tb_{n-1}(k)] \\ \epsilon b_n(k) &= [t_n(k)a_n(k) - ta_{n+1}(k)] \end{aligned}$$

$$t_n(k) = -t\big(e^{i\pi n\varphi/\phi_o} + e^{ika}e^{-i\pi n\varphi/\phi_o}\big) = -2te^{ika/2}\cos\left(\frac{ka}{2} - n\pi\frac{\varphi}{\phi_0}\right)$$

So

$$\begin{aligned} \epsilon a_n(k) &= -t\left[2e^{-ika/2}\cos\left(\frac{ka}{2} - n\pi\frac{\varphi}{\phi_0}\right)b_n(k) + b_{n-1}(k)\right] \\ \epsilon b_n(k) &= -t\left[2e^{ika/2}\cos\left(\frac{ka}{2} - n\pi\frac{\varphi}{\phi_0}\right)a_n(k) + a_{n+1}(k)\right]. \end{aligned}$$

If we define

$$\psi_b(k, y_n) \quad := (-1)^n e^{-ikna/2}b_n(k)$$

$$\begin{aligned} \epsilon a_n(k) &= -t\left[2\cos\left(\frac{ka}{2} - n\pi\frac{\varphi}{\phi_0}\right)(-1)^n e^{ik(n-1)a/2}\psi_n(k) + (-1)^{n-1}e^{ik(n-1)a/2}\psi_{n-1}(k)\right] \\ &= -t(-1)^n e^{ik(n-1)a/2}\left[2\cos\left(\frac{ka}{2} - n\pi\frac{\varphi}{\phi_0}\right)\psi_n(k) - \psi_{n-1}(k)\right] \end{aligned}$$

and

$$\psi_a(k, y_n) = (-1)^n e^{-ik(n-1)a/2}a_n(k)$$

$$\epsilon b_n(k) = -t(-1)^n e^{ikna/2}a_n(k)\left[2\cos\left(\frac{ka}{2} - n\pi\frac{\varphi}{\phi_0}\right)\psi_a(k,y) - \psi_a(k, y+a)\right]$$

and the equations reduce to

$$\begin{aligned} \epsilon\psi_a(k,y) &= -t\left[2\cos\left(\frac{ka}{2} - n\pi\frac{\varphi}{\phi_0}\right)\psi_b(k,y) - \psi_b(k, y-a)\right] \\ \epsilon\psi_b(k,y) &= -t\left[2\cos\left(\frac{ka}{2} - n\pi\frac{\varphi}{\phi_0}\right)\psi_a(k,y) - \psi_a(k, y+a)\right] \end{aligned}$$

These equations have slowly varying solutions of low energy near

$$2\cos\left(\frac{ka}{2} - n\pi\frac{\varphi}{\phi_0}\right) = 1$$

Expanding about that point

$$\begin{aligned} 2\cos\left(\frac{ka}{2}-\frac{y}{a}\pi\frac{\varphi}{\phi_0}\right) &= 2\cos\left(\frac{ka}{2}-\frac{\overline{y}}{a}\pi\frac{\varphi}{\phi_0}-\frac{y-\overline{y}}{a}\pi\frac{\varphi}{\phi_0}\right) \\ &= 1+\sin(\pi/3)\frac{y-\overline{y}(k)}{a}2\pi\frac{\varphi}{\phi_0} \\ & 1+\frac{\sqrt{3}}{2}\frac{(y-\overline{y})}{a}2\pi\frac{\varphi}{\phi_0} \end{aligned}$$

We can write the eigenvalue equations as

$$\begin{aligned} \epsilon\psi_a(k,y) &= -ta\left[\left(\partial_y\psi_b(k,y)\right)+\pi\sqrt{3}\frac{\varphi}{\phi_0}\frac{y-\overline{y}}{a^2}\psi_b(k,y)\right] \\ \epsilon\psi_b(k,y) &= -ta\left[-\partial_y\psi_a(k,y)+\pi\sqrt{3}\frac{\varphi}{\phi_0}\frac{y-\overline{y}}{a^2}\psi_a(k,y)\right] \end{aligned}$$

Or

$$\begin{bmatrix} 0 & \hat{A}\left(y,\partial_y\right) \\ \hat{A}^{\dagger}\left(y,\partial_y\right) & 0 \end{bmatrix}\begin{bmatrix}\psi_a \\ \psi_b\end{bmatrix}=\epsilon\begin{bmatrix}\psi_a \\ \psi_{\ b}\end{bmatrix}$$

$$\begin{aligned} A^{\dagger}\left(y,\partial_y\right) &= -ta\left[-i\frac{\partial_y}{i}+\pi\sqrt{3}\frac{\varphi}{\phi_0}\frac{y-\overline{y}}{a^2}\right] \\ A\left(y,\partial_y\right) &= -ta\left[i\frac{\partial_y}{i}+\pi\sqrt{3}\frac{\varphi}{\phi_0}\frac{y-\overline{y}}{a^2}\right] \end{aligned}$$

$$[A,A^{\dagger}]=2\pi t^2\sqrt{3}\frac{\varphi}{\phi_0}$$

$$\hat{a}=\frac{1}{t}\sqrt{\frac{1}{2\pi\sqrt{3}}\frac{\phi_0}{\varphi}}A$$

$$[\hat{a},\hat{a}^{\dagger}]=1$$

and

$$-t\sqrt{2\pi\sqrt{3}\frac{\varphi}{\phi_0}}\begin{bmatrix} 0 & a\left(y,\partial_y\right) \\ a^{\dagger}\left(y,\partial_y\right) & 0 \end{bmatrix}\begin{bmatrix}\psi_a \\ \psi_b\end{bmatrix}=\epsilon\begin{bmatrix}\psi_a \\ \psi_{\ b}\end{bmatrix}$$

This prompts the solution

$$\begin{bmatrix} 0 & a\left(y,\partial_y\right) \\ a^{\dagger}\left(y,\partial_y\right) & 0 \end{bmatrix}\begin{bmatrix}\varphi_{n-1} \\ \pm\varphi_n\end{bmatrix}=\pm\sqrt{n}\begin{bmatrix}\varphi_{n-1} \\ \varphi_n\end{bmatrix}$$

And the spectrum is

$$\epsilon_n=\mp ta\sqrt{2\pi\sqrt{3}\frac{\varphi}{\phi_0}n}=\mp\hbar v_F\frac{1}{l_B}\sqrt{2n}$$

To connect with results as presented in the literature, we note that the area of the elementary hexagon is $\sqrt{3}a^2/2$, and

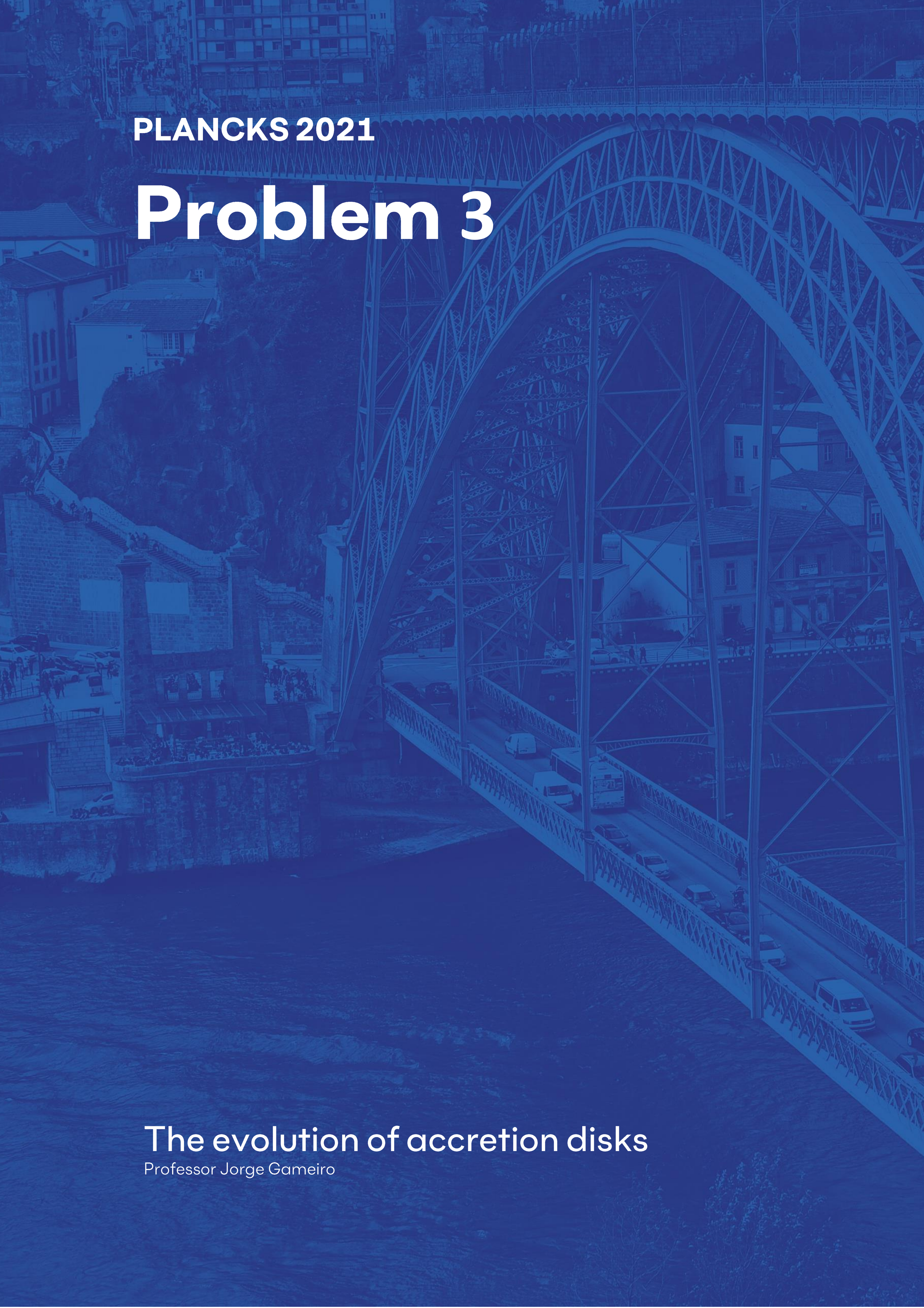
$$2\pi\frac{\varphi}{\phi_0} = \frac{\sqrt{3}}{2}2\pi\frac{Ba^2}{\phi_0} := \frac{\sqrt{3}}{2}\frac{a^2}{l_B^2}$$

with

$$\begin{aligned} t\sqrt{2\pi\sqrt{3}\frac{\varphi}{\phi_0}n} &= t\sqrt{\sqrt{3}\frac{\sqrt{3}}{2}\frac{a^2}{l_B^2}n} \\ &= ta\frac{\sqrt{3}}{2}\frac{1}{l_B}\sqrt{2n} \end{aligned}$$

or , since $v_F = \sqrt{3}ta/2\hbar$

$$\epsilon_n = \mp\hbar v_F\frac{1}{l_B}\sqrt{2n}$$



PLANCKS 2021

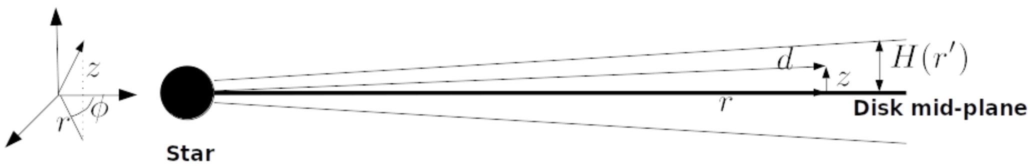
Problem 3

The evolution of accretion disks

Professor Jorge Gameiro

Introduction

The stars form through the gravitational collapse of dense molecular clouds. The specific angular momentum of gas in the molecular cloud typically matches the specific angular momentum of the gas in a circumstellar disk. In many cases the disk is confined so closely to the disk mid-plane that to a first approximation, one can assume the disk as a two-dimensional gas flow, the so-called thin disk approximation, where the thickness of the disk, $H(r) \ll r$.



Question 1 [20 points]

Assume for simplicity that the disk is optically thick and vertically isothermal, with a constant sound speed, c_s, and pressure given by $P(r,z) = \rho(r,z)c_s^2(r)$. If the gas is in hydrostatic equilibrium in the vertical z-direction (no mass motion in this direction), show that the vertical density profile, $\rho(z)$, is a gaussian profile. In this exercise one can consider the mass of the disk negligible when compared with the mass of the star, so the gravitational force is mainly due to the star.

Question 2 [30 points]

The evolution of a flat and geometrically thin disk follows from the equations of mass and angular momentum conservations. Consider a thin disk characterized by a surface density $\Sigma(r,t)$ (the mass per unit surface area of the disk, $\Sigma(r,t) \approx \rho(r,t) \times H(r)$), radial velocity $V_r(r,t)$ and angular velocity $\Omega(r)$.

The angular momentum conservation is given by the equation

$$r\frac{\partial}{\partial t}(r^2\Omega\Sigma) + \frac{\partial}{\partial r}(r^2\Omega . r\Sigma V_r) = \frac{1}{2\pi}\frac{\partial G}{\partial r} (1)$$

ν is the kinematic viscosity, G is the viscous torque exerted by the outer ring on the inner ring and has the form $G = 2\pi r . \nu\Sigma r\frac{d\Omega}{dr} . r$.

Get the mass conservation equation in cylindrical co-ordinates and together with equation 1 obtain the equation that represents the disk evolution (assume the angular velocity is keplerian),

$$\frac{\partial\Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(\nu\Sigma r^{1/2}\right)\right] (2)$$

Question 3 [30 points]

In general, equation 2 is a nonlinear diffusion equation, because the kinematic viscosity ν may be a function of Σ, r and time. If we assume a constant viscosity ν in the full disk, the equation is linear.

Show that for a constant viscosity disk, the equation is a pure diffusion equation,

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial R^2}$$

and determine the typical timescale of the disk evolution (also called the viscous timescale). [Hint: Consider $\mathrm{R} = 2\mathrm{r}^{1/2}$ and $\mathrm{f} = (3/2)\Sigma\mathrm{R}$]

In your opinion, how could the disk evolution timescale be obtained from observations?

Question 4 [20 points]

In the previous questions we have assumed that the mass of the disk is negligible when compared with the mass of the star. Here, we want to discuss the validity of this assumption. As the disk is very large, for simplicity we approximate the disk by an infinite sheet with constant surface density Σ and thickness H. Show that the mass of the disk is not negligible when

$$\frac{M_{disk}}{M_*} > 0.5\left(\frac{H}{R_{disk}}\right)$$

[Hint: Determine the gravitational acceleration above the sheet due to the mass of the disk and compare this acceleration with the vertical component of the stellar gravity at $z = H$]

Solutions

Question 1 [20 points]

Consider the vertical hydrostatic equilibrium

$$\frac{dP}{dz} = -\rho g_z \quad (1)$$

We ignore any contribution to the gravitational force from the disk. In this case, the vertical component of gravity is given by $g_z = \frac{GM_*}{d^3} z$.

For a thin disk $z \ll r$, the previous equation becomes

$$\frac{d\rho}{\rho} = -\frac{1}{c_s^2}\frac{GM_*}{r^3} zdz \quad (2)$$

which integrates to give

$$\rho(r,z) = \rho(r,0)e^{-z^2/(2H^2)} \quad (3)$$

with $\mathrm{H} = c_s/\Omega_\mathrm{K}$ and $\Omega_K^2 = \frac{GM_*}{r^3}$

Question 2 [30 points]

Angular momentum conservation equation

$$r\frac{\partial}{\partial t}(r^2\Omega\Sigma) + \frac{\partial}{\partial r}(r^2\Omega . r\Sigma V_r) = \frac{1}{2\pi}\frac{\partial G}{\partial r} \quad (4)$$

and $G = 2\pi r . \nu\Sigma r \frac{d\Omega}{dr} . r$

The mass conservation equation in cylindrical coordinates is

$$r\frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial r}(r\Sigma V_r) = 0 \quad (5)$$

The angular momentum equation can be written as

$$r\frac{\partial \Sigma}{\partial t}r^2\Omega + r\Sigma\frac{\partial}{\partial t}(r^2\Omega) + r^2\Omega\frac{\partial}{\partial r}(r\Sigma V_r) + r\Sigma V_r\frac{\partial}{\partial r}(r^2\Omega) = \frac{1}{2\pi}\frac{\partial G}{\partial r}$$

$$\Rightarrow \cancel{-r^2\Omega\frac{\partial}{\partial r}(rV_r\Sigma)} + \cancelto{0}{r\Sigma\frac{\partial}{\partial t}(r^2\Omega)} + \cancel{r^2\Omega\frac{\partial}{\partial r}(r\Sigma V_r)} + \mathrm{r}\Sigma V_r\frac{\partial}{\partial r}(r^2\Omega) = \frac{1}{2\pi}\frac{\partial G}{\partial r}$$

(the second term is zero because $r^2\Omega$ does not depend on time) and so

$$r\Sigma V_r\frac{\partial}{\partial r}(r^2\Omega) = \frac{1}{2\pi}\frac{\partial G}{\partial r} \quad (6)$$

We can rewrite the mass conservation equation 5 using the equation 6 and obtain

$$r\frac{\partial \Sigma}{\partial t} = -\frac{\partial}{\partial r}\left[\frac{1}{2\pi(r^2\Omega)'}\frac{\partial G}{\partial r}\right]$$

By developing this equation we obtain

$$r\frac{\partial\Sigma}{\partial t} = \frac{\partial}{\partial r}\left[\frac{2}{r\Omega}\frac{\partial}{\partial r}(3/2\nu\Sigma\Omega r^2)\right]$$

Taking the angular velocity as keplerian, $\Omega = (GM_*/r^3)^{1/2}$, we finally obtain the result

$$\frac{\partial\Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}(\nu\Sigma r^{1/2})\right] \quad (7)$$

Question 3 [30 points]

With the change of variables $R = 2r^{1/2}$ and $f = 3/2\Sigma R$, the partial derivative in equation 7 can be written as

$$\frac{\partial}{\partial r} = \frac{\partial R}{\partial r}\frac{\partial}{\partial R} \quad \Rightarrow \quad \frac{\partial}{\partial r} = \frac{2}{R}\frac{\partial}{\partial R}$$

and we get

$$\frac{\partial}{\partial t}(3\Sigma)\frac{3\times 4}{R^2}\frac{2}{R}\frac{\partial}{\partial R} = \left[\frac{R}{2}\frac{2}{R}\frac{\partial}{\partial R}\left(\nu 3\Sigma\frac{R}{2}\right)\right]$$

and so the evolution equation takes the form of a diffusion equation

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial R^2}$$

with the diffusion coefficient $D = 12\nu/R^2$. The diffusion time scale across a scale ΔR is $\Delta R^2/D$. If we convert this scale on the surface density spread on a radial scale Δr we get $t_{dif} \sim \Delta r^2/\nu$.

Question 4 [20 points]

We have calculated in question 1 the vertical component of the gravity due to the star

$$g_z(star) = \frac{GM_*}{r^3}z$$

The gravitational acceleration above the sheet due to the disk is given by the expression

$$g_z(disk) = 2\pi G\Sigma$$

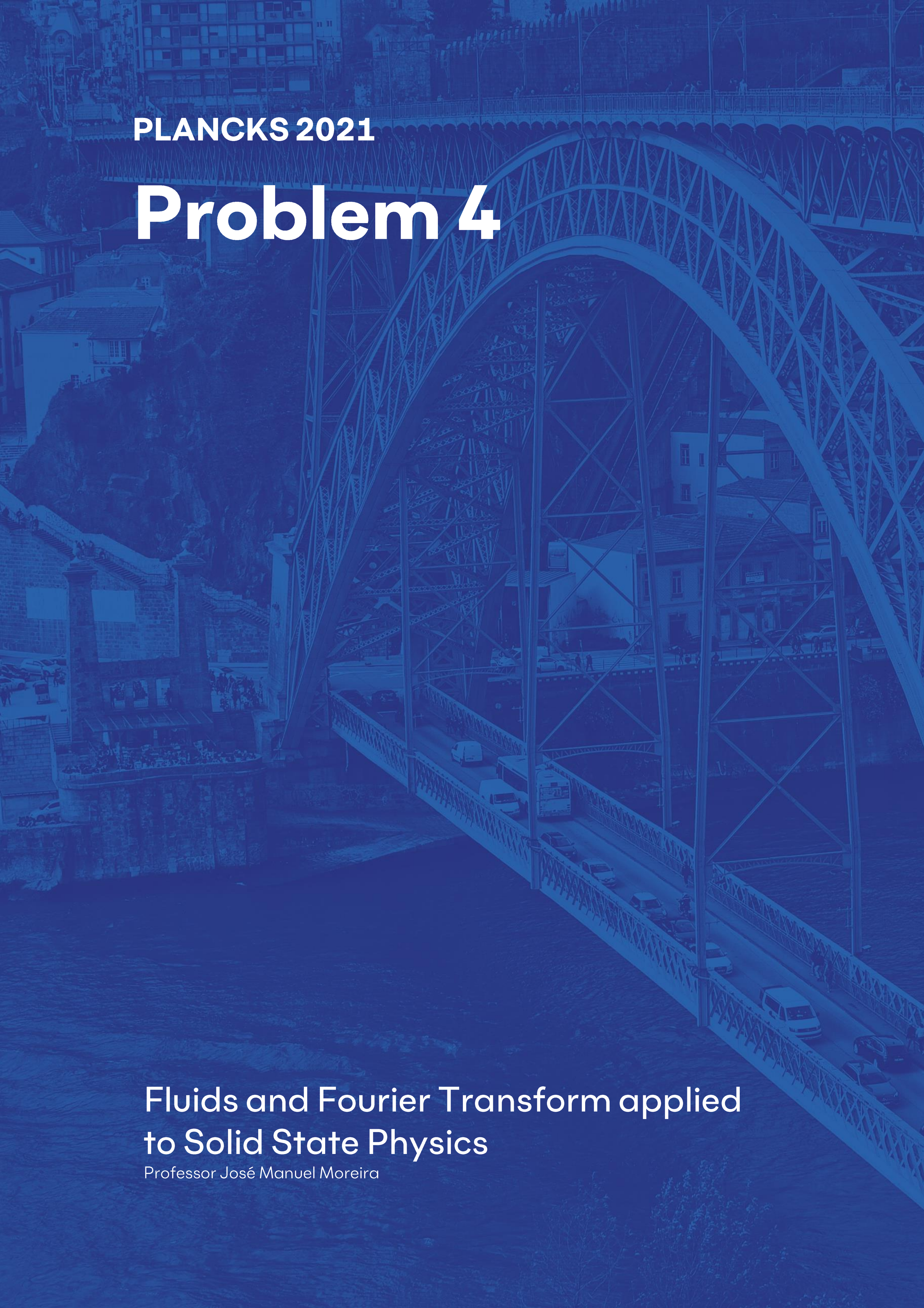
which is independent of height (we assume that the disk is large enough that we can take it as infinite). There are several form to determine the above result, for example using Gauss's theorem.

By calculating the ratio of gravitational acceleration caused by star and disk one gets

$$\frac{g_z(disk)}{g_z(star)} = \frac{2\pi G\Sigma}{GM_* z}r^3$$

We are looking for the case when $g_z(disk) > g_z(star)$.The disk mass at a distance r is roughly $M_{disk}(r) = \pi r^2\Sigma$ and $z\sim H$, so the condition is

$$\frac{M_{disk}}{M_*} > 0.5\left(\frac{H}{r}\right)$$



PLANCKS 2021

Problem 4

Fluids and Fourier Transform applied to Solid State Physics

Professor José Manuel Moreira

1. An empty sphere inside a fluid

A spherical hole of radius R_0 (see Fig. 1) suddenly forms in a perfect incompressible fluid (specific mass ρ; weight negligible). The radius of the hole is small compared to the dimension of surrounding fluid.

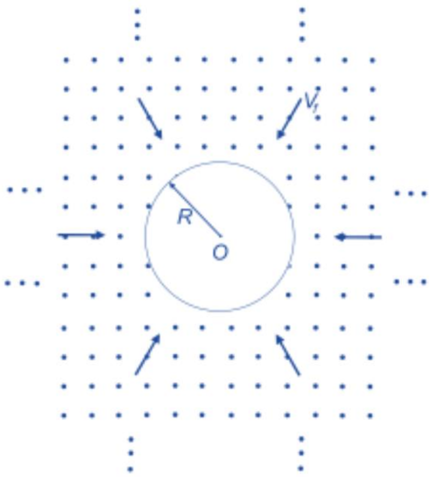


Figure 1 - Spherical hole forming in a perfect incompressible fluid

1.1. [32.5 points] Prove that the time it takes for the fluid to completely fill the hole is

$$T = \sqrt{\frac{3\rho}{2p_0}} \int_0^{R_0} \frac{dr}{\sqrt{(R_0/r)^3 - 1}} \quad (1)$$

with p_0 corresponding to the pressure at "infinity".

1.2. [17.5 points] In order to calculate this time, we need to evaluate the integral:

$$\int_0^{R_0} \frac{dr}{\sqrt{(R_0/r)^3 - 1}} \quad (2)$$

Your task is to find the value of this integral in terms of beta or gamma functions. After that, choose your favorite numerical integration method and find the numerical value for this same integral and compare both. Hint: use the normalization $x = r/R_0$.

You can do the numerical iterations by hand or you can put your computer to do them for you, using a programming language like Python or a similar one.

2. Polarization field

Polarization is the vector field that expresses the density (C/m^2) of permanent or induced electric dipole moments in a dielectric material. When we apply an external electric field, the molecules of the material will acquire an additional electric dipole moment and the medium is said to be polarized. In a first approximation, the second order differential equation that relates the polarization of a material to the applied electric field can be written, in the time domain:

$$\frac{d^2P}{dt^2} + \gamma\frac{dP}{dt} + \omega_0^2 P - \nu\omega_p^2 P = \epsilon_0\omega_p^2 E(t)(3)$$

where $\omega_0^2 = \frac{e^2}{4\pi\epsilon_0 m R_0^3}$, $\gamma = \frac{1}{\tau}$, $\omega_\mathrm{p}^2 = \frac{\mathrm{Ne}^2}{\epsilon_0 \mathrm{m}}$ and N the number of charges per unit volume. In metals $\nu = 0$ and for an isotropic non-polar dielectric ν is theoretically $1/3$.

As we know, Fourier transforms, direct and inverse, and convolution integrals are one of the most valuable theoretical tools of physicists and engineers. They give us a clear correspondence between time and frequency domains. One can choose a domain to solve a problem (time or frequency domain) allowing us to do the math in the easier one. The equation 3 can be written in the frequency domain as:

$$\widetilde{P}\,(\omega) = \widetilde{R}\,(\omega)\,\widetilde{E}\,(\omega)$$

where $\widetilde{R}\,(\omega)$ is the transfer function (susceptibility) of the medium. In the general case, susceptibility is a tensor but in this exercise we consider it a scalar quantity.

2.1. [20 points] In the limit of $\gamma \to 0$, find expressions for the real and imaginary parts of $\widetilde{R}\,(\omega) = \widetilde{\chi}\,(\omega) = \chi_r(\omega) + i\chi_i(\omega)$.

2.2. [7.5 points] In this limit, find the response of the system to an applied electric field that is zero for $t < 0$ and E_0 for $t > 0$ (Heaviside function). Plot $P(t)$ and comment physically the result.

2.3.[22.5 points] If $\gamma \neq 0$, with $\frac{\gamma}{\omega_0} \ll 1$, comment on the amplitude of the resultant oscillation, its energy and corresponding oscillation frequency. Sketch the graphs for different values of increasing γ (always obeying the condition $\gamma \ll \omega_0$). What is the value of P when $t \to \infty$? Neglect irradiated electromagnetic energy and all other decay modes due to interactions with other particles.

Solutions

1.1 [32.5 points]

$$v_r = v(r), v_\theta = v_\phi = 0$$

The Euler equation in spherical coordinates:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

From $\nabla \cdot \vec{v} = 0$ (continuity equation) in spherical coordinates:

$$r^2 v = f(t)$$

From the above we get

$$\frac{1}{r^2}\frac{df}{dt} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

Integrating this equation between $R(t)$ $[R(t) \le R_0]$ and infinity leads to

$$\frac{df}{dt}\int_{R(t)}^{\infty}\frac{1}{r^2}dr + \int_{V(R)}^{0} v\,dv = -\frac{1}{\rho}\int_0^{p_0} d\,p$$

$$-\frac{1}{R}\frac{df}{dt} + \frac{1}{2}V^2 = \frac{p_0}{\rho}$$

where $V = \frac{dR}{dt}$ is the temporal rate of the radius of the hole; the velocity $v(\infty)$ is zero and the pressure on the surface of the hole is obviously zero as well. For points on the surface of the hole, we have:

$$f(t) = R^2(t)V(t)$$

Plugging this equation into the former we get:

$$-\frac{3}{2}\mathrm{V}^2 - \frac{1}{2}\mathrm{R}\frac{\mathrm{d}\,\mathrm{V}^2}{\mathrm{d}\,\mathrm{R}} = \frac{\mathrm{p}_0}{\rho}$$

Integrating this equation with the initial values $V = 0$ for $R = R_0$, one obtains:

$$V = \frac{dR}{dt} = -\sqrt{\frac{2p_0}{3\rho}\left(\frac{R_0^3}{R^3} - 1\right)}$$

where the minus signal ensures the filling of the hole.

Therefore, the total time spent will be:

$$T = \sqrt{\frac{3\rho}{2p_0}}\int_0^{R_0}\frac{1}{\sqrt{(R_0/R)^3 - 1}}dR$$

1.2 [17.5 points] The calculation of the definite integral $I = \int_0^{R_0}\frac{1}{\sqrt{\left(\frac{R_0}{R}\right)^3 - 1}}dR$ can be performed numerically. Performing the variable change $x = \frac{R}{R_0}$ results in:

$$I = R_0 \int_0^1 \frac{x^{\frac{3}{2}}}{\sqrt{1-x^3}} dx = R_0 I_0$$

One of the most frequently used numerical methods of definite integrals is Simpson's method. However, in this case the calculation is inappropriate due to the discontinuity in the upper boundary. To overcome this difficulty we decomposed the integral into two additive intervals $[0; 0.99]$ and $[0.99; 1]$. In the first one, Simpson's rule was applied and in the second one, where the discontinuity exists, the extended midpoint rule was applied (the computational code was described in python language). The results found are:

$$\int_0^{0.99} \frac{x^{3/2}}{\sqrt{1-x^3}} dx = 0.632 \quad \text{and} \quad \int_{0.99}^{1} \frac{x^{3/2}}{\sqrt{1-x^3}} dx = 0.116$$

so $I_0 = 0.748$ i.e.

$$T = \sqrt{\frac{3}{2}} \times 0.748 \times R_0 \sqrt{\frac{\rho}{p_0}} = 0.916 R_0 \sqrt{\frac{\rho}{p_0}}$$

Note, however, that the integral I_0 has its own mathematical expression expressed in special functions (beta functions or gamma functions). These functions appear in the theoretical solution of many physics problems:

$$I_0 = \int_0^1 \frac{x^{3/2}}{\sqrt{1-x^3}} dx$$

$$= \frac{1}{3} B\left(\frac{1}{2}, \frac{5}{6}\right) = \frac{1}{3} \frac{\Gamma\left(-\frac{1}{2}\right) \cdot \Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{3}\right)} =$$

$$= \frac{\sqrt{3}}{2\sqrt{\pi}} \Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{5}{6}\right) = 0.747$$

2.1. **[20 points]** Using the differential equation given and the relation between Fourier transforms and its derivatives we get:

$$(-i\omega)^2 P(\omega) - i\gamma\omega P(\omega) + (\omega_o^2 - \nu\omega_P^2) P(\omega) = \varepsilon_0 \omega_p^2 E(\omega)$$

and with $R(\omega) = \frac{P(\omega)}{E(\omega)}$ it follows that

$$\tilde{\chi}(\omega) = \tilde{R}(\omega) = \frac{\epsilon_0 \omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega - \nu\omega_p^2}$$

where $\tilde{\chi}(\omega)$ is the medium susceptibility.

Using elementary algebra, we get:

$$\frac{\tilde{\chi}(\omega)}{1 + \frac{\nu\tilde{\chi}(\omega)}{\varepsilon_0}} = \frac{\varepsilon_0 \omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

and using the dielectric constant definition $\hat{K} = 1 + \tilde{\chi}/\varepsilon_0$

$$\frac{\hat{K} - 1}{1 + \nu(\hat{K} - 1)} = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

The effective resonance frequency can be written as $\sqrt{\omega_0^2 - \nu\omega_p^2}$. The imaginary and real parts of $\widehat{K}$ for $\nu = 0$ are:

$$K_r = 1 + \frac{\omega_p^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}$$

$$K_i = \frac{\omega_p^2\gamma\omega}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}$$

If $\gamma \to 0$, we get

$$K_r - 1 = \frac{\omega_p^2}{\omega_0^2 - \omega^2}$$

and

$$K_i = \lim_{\gamma\to 0}\left[\frac{\omega_p^2\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}\right]$$

$$= \frac{\omega_p^2}{\omega}\lim_{\gamma\to 0}\left[\frac{\gamma}{\left(\frac{\omega_0^2 - \omega^2}{\omega}\right)^2 + \gamma^2}\right] = \frac{\omega_p^2}{\omega}\pi\delta\left(\frac{\omega_0^2 - \omega^2}{\omega}\right) = \frac{\omega_p^2}{\omega}\frac{\pi}{2}\delta(\omega_0 - \omega)$$

when $\omega > 0$. $K_i \neq 0$ describes the energy dissipation of the system.

Using the last two equations and χ_0 for $\chi|_{\omega=0} = \frac{\varepsilon_0\omega_p^2}{\omega_0^2}$:

$$\tilde{\chi}(\omega) = \chi_r(\omega) + i\chi_i(\omega) = \frac{\chi_0}{1 - \omega^2/\omega_0^2} + i\frac{\pi}{2}\chi_0\frac{\omega_0^2}{\omega}\delta(\omega - \omega_0)$$

2.2. [7.5 points] The response function $R(t)$ is the inverse Fourier transform of $\tilde{\chi}(\omega)$. Since $R(t)$ is real, the transform may be expressed as an integral over positive frequencies only:

$$R(t'') = \sqrt{\frac{2}{\pi}}\int_0^\infty [\chi_r(\omega)\cos(\omega t'') + \chi_i(\omega)\sin(\omega t'')]d\omega$$

Both terms integrate to give the same result:

$$\sqrt{\tfrac{2}{\pi}}\int_0^\infty \chi_i(\omega)\sin\omega\, t''d\omega = \sqrt{\tfrac{\pi}{2}}\chi_0\omega_0\sin\omega_0\, t''$$

And finally:

$$P(t) = \chi_0\omega_0\int_0^\infty E(t - t'')\sin\omega_0\, t''dt''$$

to get

$$P(t) = \chi_0 E_0\big(1 - cos(\omega_0 t)\big)$$

for $t > 0$, because the field is zero when $t < 0$.

The plot of the previous equation can be seen in figure 2 .

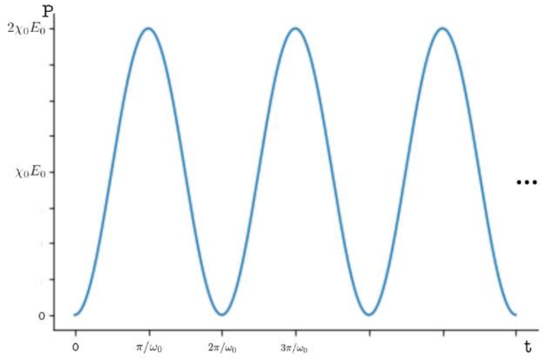


Figure 2-Polarization, a time function

2.3.[22.5 points] If $\gamma \neq 0$, with $\frac{\gamma}{\omega_0} \ll 1$, the general solution would be

$$P(t) = \chi_0 E_0 e^{-\frac{\gamma t}{2}}(1 - \cos(\omega_0' t))$$

and it can be shown by different methods (eigenvalues, or time domain) that

$$\omega_0' = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)}$$

In figure 3 one can see the plot for different values of γ for a $\omega_0 = 1$.

It is necessary to note:

- the different time stretches due to the factor ω'_0
- the different damping due to the $e^{-\gamma t/2}$ factor
- the first peak amplitude
- the asymptotic value for $t \to \infty$

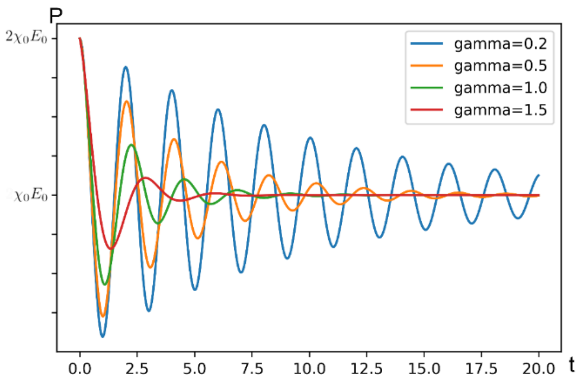
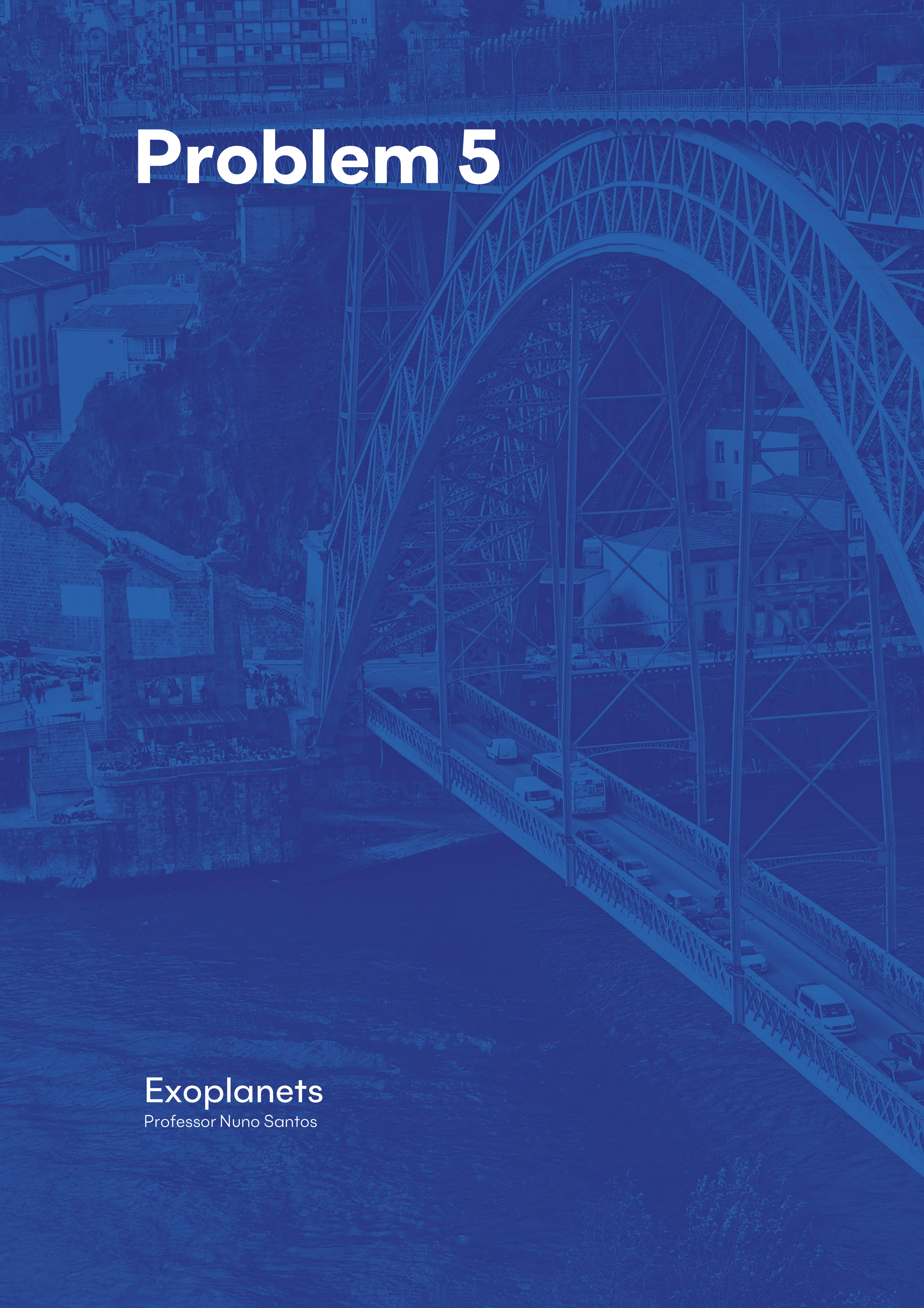


Figure 3-Polarization for $\gamma \neq 0$. (10 points)



Problem 5

Exoplanets

Professor Nuno Santos

Introduction

The 2019 Nobel Prize discovery of the planet orbiting a solar-like star marked, in 1995, the onset of a whole new area of modern astrophysics. Today, more than 4000 exoplanets have been detected orbiting other "suns". The results show that planets are ubiquitous in our Galaxy, but also that rocky planets are likely the most common among these. Complementary studies are allowing to characterize the planets in great detail. The measurement of accurate masses and radii (and thus mean densities) is setting strong constraints about their internal structure and composition. The detection of exoplanet atmospheres provides important clues about their nature and formation processes. The prospects of detecting and characterizing another Earth are now one of the main drivers for the development of new instruments and space missions by the main international agencies (ESO, ESA, NASA).

The following questions focus on some aspects of planet detection and characterization, but also on some interesting problems and challenges raised by planet formation models. All questions are independent and can be solved in any order.

Question 1 [25 points]

One of the main exoplanet detection and characterization methods (the so called Radial-Velocity method) is based on the measurement of the Doppler velocity of the star as it wobbles around the center of mass of the star-planet system.

Assume that you have a planet similar to Jupiter, with ~10^{-3} solar masses, orbiting a Sun-mass star in a circular orbit. Assume that the orbital period is 12 years, and the orbital radius of the planet is 5.2 AU (Astronomical Units; 1 AU = 150×10^6km). Using simple principles, derive an expression that relates the Mass ratio (M_{star}/M_{planet}) with the orbital velocity ratio (V_{star}/V_{planet}) of the two bodies around the common center-of-mass.

Compute the expected orbital velocity of the star (in units of km/s).

Question 2 [25 points]

Assuming that the planet is in thermal equilibrium with the star, and that the energy received by the planet is rapidly distributed on its surface (i.e., that the planet has uniform temperature), show that the temperature of the planet is proportional to $T_\star\sqrt{R_\star/D}$, where $T_\star$, $R_\star$, and D are the temperature of the star, its radius, and the distance between the star and the planet, respectively.

Comment on the physical nature of the proportionality factor.

Question 3 [25 points]

A significant fraction of the known exoplanets are giants, similar to Jupiter, but orbiting their host stars at very short distances. This raises several questions about planet formation and evolution, but also about the very existence of these worlds. Assume that the giant planet mentioned above is actually at 0.05 AU from a star similar to the Sun, such that its temperature is 1250 K.

Using simple principles, discuss if the atmosphere of this planet can survive against evaporation. For simplicity, assume that: 1) the atmosphere of a Jupiter-like planet is composed of hydrogen, 2) the planet has half the mass of Jupiter but the same average density (Data for Jupiter: M=1.898×10^{27} kg, R=70 000 km).

Question 4 [25 points]

Planets are formed in disks of gas and dust that are formed as the outcome of the star formation process. In these disks, solids are expected to grow though collisions over timescales of several million years, eventually leading to planet size objects.

One of the biggest challenges of the planet formation process is related with the fact that, in a disk, the gas (that composes 99% of the mass of the disk) and the solids do not rotate at the same velocity. This leads to a gas drag that will make small, meter-sized pebbles to fall into the star in timescales of a few thousands of years.

Assume you have a disk of gas and dust whose density and temperature decrease with distance to the star. Show that, in such circumstances, you expect the velocity of the gas to be given by

$$v_{dust} \times \left(1 - \frac{c^2}{r^2 \Omega^2}\right)^{1/2},$$

where $\Omega = \sqrt{GM/r^3}$ and c is the speed of sound.

Solutions

Question 1 [25 points]

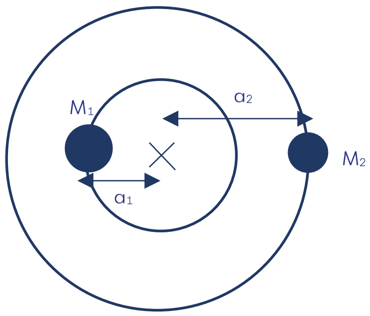


Figure 1-Problem scheme

$$M_1 a_1 = M_2 a_2 \Rightarrow \frac{a_1}{a_2} = \frac{M_2}{M_1}$$

$$\frac{v_1}{v_2} = \frac{a_1}{a_2} = \frac{M_2}{M_1} \Leftrightarrow \frac{v_1}{v_2} = \frac{M_2}{M_1}$$

If $M_1 = 1000M_2 \Rightarrow v_1 = 0{,}001v_2$

$v_2 = (5.2AU \times 150 \times 10^6\text{~km} \times 2 \times \pi)/(12\text{anos} \times 365 \times 24 \times 3600) = 12.9kms^{-1}$

Question 2 [25 points]

The power received by a planet is given by:

$$P_{in} \propto 4\pi R_*^2 T_*^4 \times \frac{\pi R_p^2}{4\pi D^2}$$

where R_* is the stellar radius and T_* the temperature, and R_p and D are the planet radius and the distance to the star, respectively.

$$P_{out} \propto 4\pi R_p^2 T_p^4$$

In equilibrium:

$$P_{in} = P_{out} \Rightarrow T_P \propto T_* \sqrt{\frac{R_*}{2D}}$$

The proportionality is related to the planet's Albedo.

Question 3 [25 points]

For a planet to keep its atmosphere:

$$v_{particles} < v_{esc}$$

Escape velocity (v_{esc}) at the limit:

$$\frac{1}{2} m v_{esc}^2 = \frac{GMm}{R}$$

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

For a $0.5M_{Jupyter}$ planet, if $\rho = \rho_{Jupyter}$, we can estimate:

$$R \approx 57 \times 10^3 km$$

$$\Rightarrow v_{esc} \approx 47km/s$$

Let's now assume that $T_{planet} = 1250K$:

$$E_k = \frac{3}{2}KT \approx \frac{1}{2}m < v_k >^2$$

If the atmosphere is made of H_2, $< v_k > \approx 4km/s$.

Since $< v_k > \ll v_{esc}$ we can expect the planet to retain its atmosphere.

Question 4 [25 points]

We have the following situation:

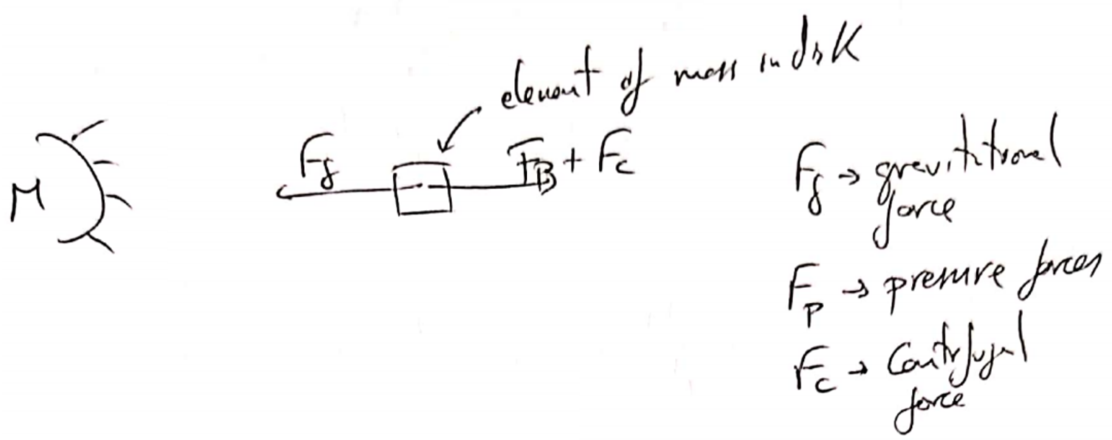


Figure 2-Problem scheme

The master equation is:

$$\frac{v^2}{r} = \frac{GM}{r^2} + \frac{1}{\rho}\frac{\partial P}{\partial r} \quad (1)$$

which is a sum of two effects: $\frac{v^2}{r} = \frac{GM}{r^2}$ takes into account the solid component and $\frac{v^2}{r} = \frac{1}{\rho}\frac{\partial P}{\partial r}$ accounts also for the gas. For solids, $\frac{v^2}{r} = \frac{GM}{r^2} \Rightarrow v = \Omega r$, where $\Omega = \sqrt{\frac{GM}{r^3}}$.

Knowing that $\Omega = \sqrt{\frac{GM}{r^3}}$ and $p = c^2\rho$, from equation 1:

$$\frac{v^2}{r} \approx \Omega^2 r + \frac{c^2}{r}$$

because $\frac{\partial P}{\partial r} \approx \frac{P}{r}$ (when doing the integration).

So:

$$\frac{v^2}{r} \approx \Omega^2 r \left(1 + \frac{c^2}{\Omega^2 r^2}\right)$$

and

$$v^2 \approx \Omega^2 r^2 \left(1 + \frac{c^2}{\Omega^2 r^2}\right)$$

making

$$v \approx \Omega r \left(1 + \frac{c^2}{\Omega^2 r^2}\right)^{1/2}$$

where $\Omega r = v_{dust}$.

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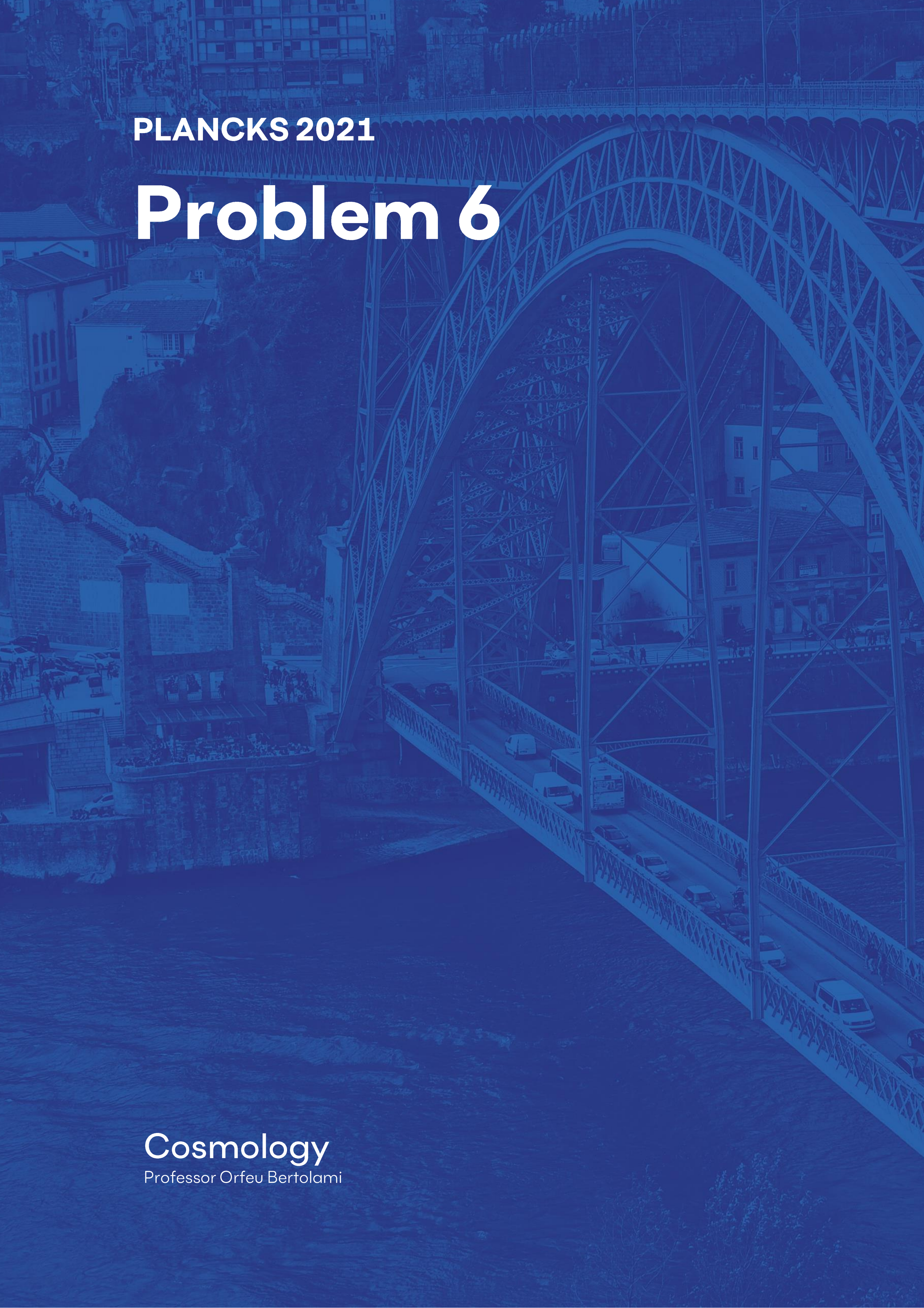
Problem 6



Cosmology

Professor Orfeu Bertolami

Introduction

The simplest cosmological model scientifically non-trivial can be obtained by Newtonian considerations assuming that a mass, M, is isotropically distributed in a volume, V_3, around an arbitrary origin. Suppose that, at a given time, a generic galaxy with mass m is at a distance $a(t)$ from the origin.

Question 1 [16 points]

Use Newton's Second law and the law of Universal Gravitation to obtain an equation of motion for $a(t)$.

Question 2 [16 points]

Show that this equation of motion admits the integral:

$$\frac{1}{2}\dot{a}^2 = \frac{GM}{a} - \frac{k}{2},$$

where k is an integration constant.

Question 3 [16 points]

Assume that the mass M is made up of homogeneously distributed pressureless dust with density, $\rho(t)$, within the radius, $a(t)$, of a sphere. Insert this mass into the previous equation to obtain an expression for the square of the expansion rate, $H = \dot{a}/a$, in terms of the density. This equation is known as Friedmann's equation and was obtained by the Russian polymath Alexander Friedmann (1888-1925) in 1922 in the context of the Theory of General Relativity, which means that this equation is more general than the Newtonian considerations assumed for the above derivation (small velocities, $v/c \ll 1$, and weak gravitational fields, $V/c^2 \ll 1$, where V is the gravitational potential).

Question 4 [16 points]

Assume that the mass M is constant to obtain a relationship between $\rho(t)$ and $a(t)$.

Question 5 [36 points]

In the Theory of General Relativity the integration constant, k, is associated to the spatial curvature of the Universe. There are three possible geometries to consider:

a. Closed Universe for $k > 0$, which corresponds to a Universe with a spatial geometry of a three-dimensional sphere, S^3, with radius of curvature, $R(t)$, related with the above model by

$$\frac{k}{a(t)^2} = \frac{1}{R(t)^2},$$

associated to the volume $V_3(t) = 2\pi^2 R(t)^3$;

b. Flat Universe for $k = 0$, which corresponds to a Universe with spatial geometry of a three-dimensional Euclidean space, E^3, for which $R \to \infty$;

c. Open Universe for $k < 0$, which corresponds to a Universe with spatial geometry of a three-dimensional hyperboloid, H^3.

Show that the evolution of the Universe can be understood for these three different cases rewriting Friedmann's equation as

$$\frac{1}{2}\dot{a}^2 = -V_{eff}(a) - \frac{k}{2},$$

where $V_{eff}(a) = -4\pi G\rho a^2/3$, drawing the diagram of $V_{eff}(a)$ as a function of a and discussing how $a(t)$ evolves as a function of the cosmic time, t .

Relevant constants:

Newton's gravitational constant: $G = 6.67 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$

Speed of light: $c = 3 \times 10^8\ m/s$

Solutions

Question 1 [16 points]

$$m\ddot{a} = -\frac{GMm}{a^2}$$

$$\ddot{a} = -\frac{GM}{a^2}$$

Question 2 [16 points]

$$\ddot{a}\dot{a} = -\frac{GM}{a^2}\dot{a}$$

$$\frac{d}{dt}\left[\frac{1}{2}\dot{a}^2\right] = \frac{d}{dt}\left[\frac{GM}{a}\right]$$

$$\frac{1}{2}\dot{a}^2 = \frac{GM}{a} - \frac{k}{2}$$

Question 3 [16 points]

$$M = \frac{4\pi}{3}a^3\rho$$

let be $H \equiv \frac{\dot{a}}{a}$,

$$\frac{1}{2}\dot{a}^2 = \frac{4\pi G}{3}\rho a^2 - \frac{k}{2}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

Hence,

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

Question 4 [16 points]

$$M = \text{const} \Rightarrow \rho_m \propto a^{-3}$$

Question 5 [36 points]

$$\frac{1}{2}\dot{a}^2 + V_{eff}(a) = -\frac{k}{2}$$

$$V_{eff}(a) = -\frac{4\pi G\rho}{3}a^2 = -\frac{4\pi G}{3}\frac{\text{const}}{a}$$

The potential corresponds to V_{eff} and the energy to $\frac{-k}{2}$.

Graphically:

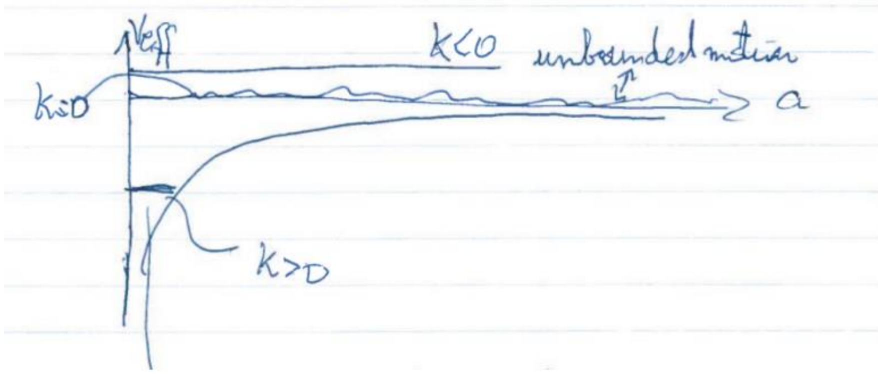


Figure 1- V_{eff} function of a.

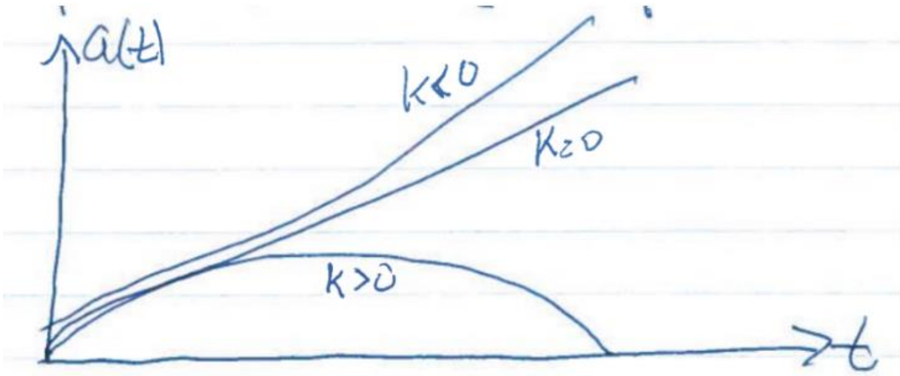
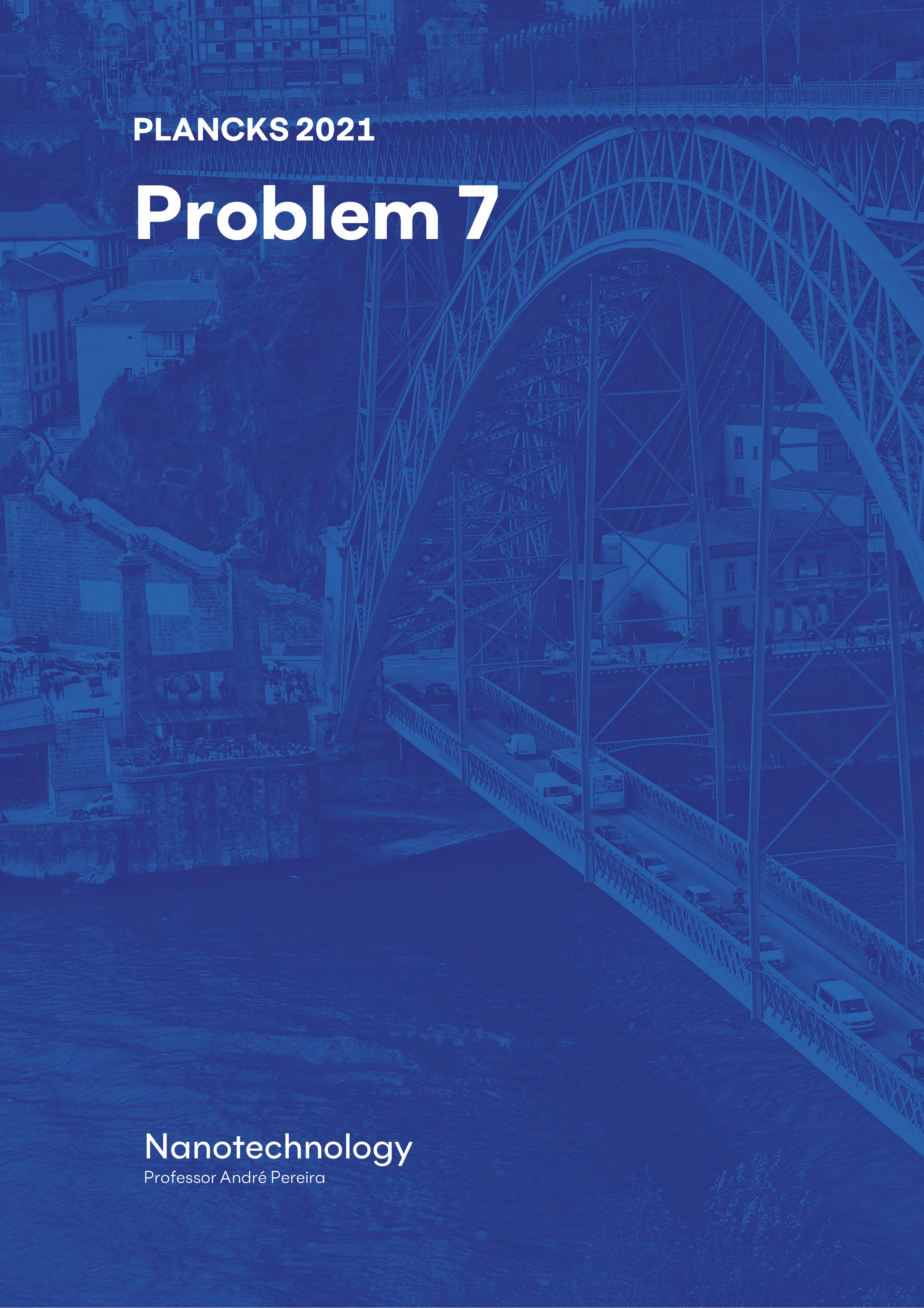


Figure 2-Expansion followed by collapse.



PLANCKS 2021

Problem 7

Nanotechnology

Professor André Pereira

Introduction

Nanotechnology is an emerging area that revolutionized the end and beginning of the XX and XXI Century, respectively. This field has already proved many advantages to improve the level of our society. In particular, Spintronics has a preponderant role in the growth and fast processors applied to several devices such as laptops, desktops, workstations, and servers.

One breakthrough was the invention of magnetic junctions constituted by the two electrodes that are ferromagnetic materials sandwiching a non-magnetic material, all at the nanometer thickness (Fig. 1). These devices can detect the binary unit (*bit*) in computing and digital communications, namely for information storage systems such as hard disk drive (HDD). The electric current in these devices consists of two partial currents in a ferromagnetic material, each with either spin-up or spin-down electrons. Moreover, they can present two different configurations related to the relative direction of the FM layers (these layers are usually metals).

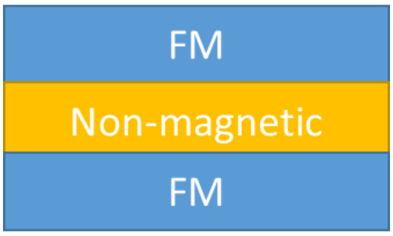


Figure 3- Two FM materials sandwiching a non-magnetic material

Question 1 [15 points]

Explain the working principle of these nanodevices and how they can measure a bit.

Question 2 [20 points]

Consider that the magnetic junction is a spin valve, e.g. the non-magnetic layer is a metal. What is the expression of the spin valve system's maximum sensitivity (magnetoresistance's maximum) using the simplest model? [Suggestion: consider the variation of resistance]

Question 3 [40 points]

Consider now a magnetic tunnel junction barrier (MTJ) in which the non-magnetic layer is an insulator. Disregarding the magnetic contribution considering the metal layers, determine the tunnel current's general expression through the MTJ.

Question 4 [25 points]

Quantum sensors are another technology gaining extreme relevance. An example is the quantum dots nanoparticles that can be used as sensors for biomedical applications.

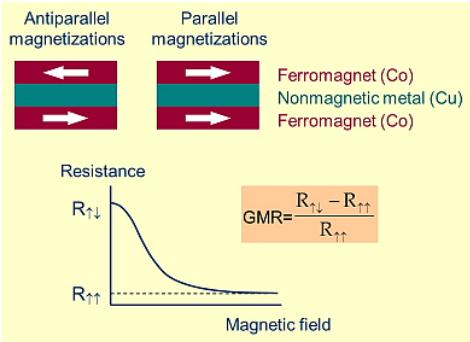
4.1 Determine the general expression between the bandgap (*Eg*) of quantum dots and its bulk counterpart.

4.2 What is expected for the quantum dot's *Eg* if the simple parabolic band's curvature presents a strongly curved parabola?

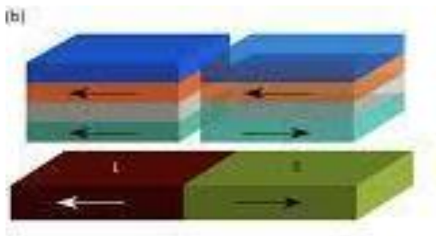
Solutions

Question 1 [15 points]

Model of 2 channels:



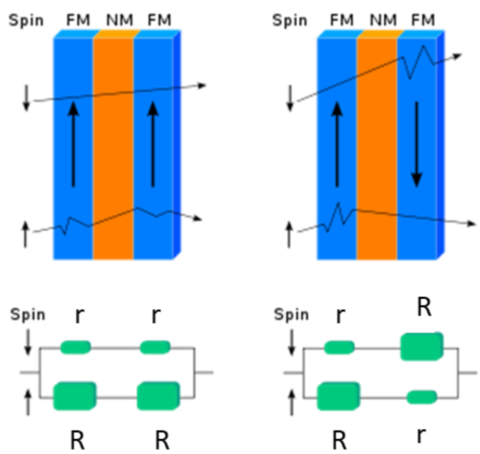
Two values of resistance: when they are parallel - state "0"; perpendicular - state "1". It works like in the following figure:



Notice that the bit needs to be in the same direction as the magnetic layers.

Question 2 [20 points]

Model of 2 channels



Parallel (left) $R_P = (r+R)/(2rR)$

Antiparallel (right) $R_{AP} = (R+r)/2$

GMR: $GMR = (R_{AP} - R_P)/R_P * 100\%$

Final Result

$$GMR = \frac{2rR\left(\frac{r+R}{2} - \frac{r+R}{2rR}\right)}{r+R}$$

Question 3 [40 points]

Consider two metal electrodes with an insulator of thickness L between them. If electrodes are under the same potential, the system is in thermodynamic equilibrium and the Fermi levels of electrodes coincide (Fig. 1).

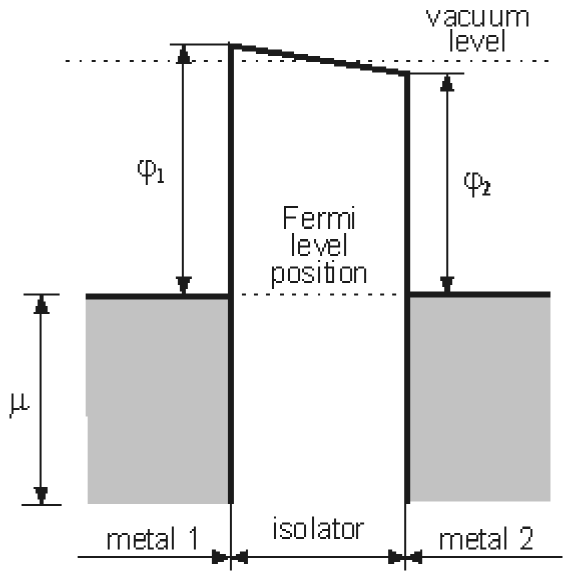


Fig. 1. Diagram of MIM system in equilibrium.
j1 and j2 – work function of the left and right metals, respectively.

Let's calculate the transparency of the rectangular barrier. Suppose that electrons of energy *E* are subjected to a potential barrier defined by

$$U(z) = \begin{cases} 0, & z < 0 \\ U_0, & 0 \le z \le L \\ 0, & z > L \end{cases} \qquad (1)$$

Assuming that the total energy E is less than U_0, we have:

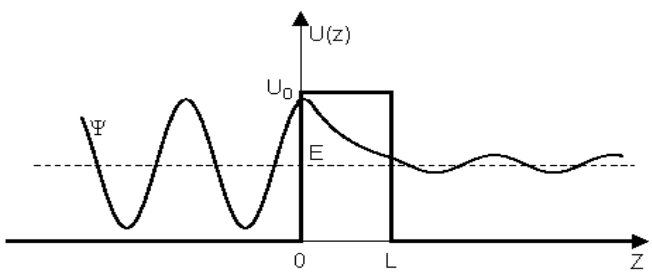


Fig. 2. Rectangular potential barrier and particle wave function Ψ

The stationary Schrödinger equations can be written as follows

$$\begin{cases} \ddot{\Psi} + k_1^2\Psi = 0, \; z < 0 \\ \ddot{\Psi} - k_2^2\Psi = 0, z \in [0; L] \\ \ddot{\Psi} + k_1^2\Psi = 0, \; z > L \end{cases} \quad (2)$$

where $k_1 = \frac{\sqrt{2mE}}{\hbar}$, $k_2 = \frac{\sqrt{2m(U_0-E)}}{\hbar}$ are wave vectors, and h is the Planck's constant.

The solution to the wave equation at $z < 0$ can be expressed as a sum of incident and reflected waves $\Psi = \exp(ik_1z) + a\exp(-ik_1z)$, while solution at $z > L$ as a transmitted wave $\Psi = b\exp(ik_1z)$. A general solution inside the potential barrier $0 < z < L$ is written as $\Psi = c\exp(ik_2z) + d\exp(-ik_2z)$. Constants $a, \; b, \; c, \; d$ are determined from the wavefunction Ψ and $\dot{\Psi}$ continuity condition at $z = 0$ and $z = L$.

The barrier transmission coefficient can be naturally considered as a ratio of the transmitted electrons probability flux density to that one of the incident electrons.

In the case under consideration this ratio is just equal to the squared wavefunction module at $z > L$ because the incident wave amplitude is assumed to be 1 and wave vectors of both incident and transmitted waves coincide.

$$D = bb^* = \left(\cosh^2(k_2L) + \frac{1}{4}\left(\frac{k_2}{k_1} - \frac{k_1}{k_2}\right)^2 \sinh^2(k_2L)\right)^{-1} \quad (3)$$

If $k_2L \gg 1$, then both $\cosh(k_2L)$ and $\sinh(k_2L)$ can be approximated to $\exp(k_2L)/2$ and (3) will be written as

$$D(E) = D_0 exp\left\{-\frac{2L}{\hbar}\sqrt{2m(U_0 - E)}\right\} \quad (4)$$

where $D_0 = 4\left[1 + \frac{1}{4}\left(\frac{k_2}{k_1} - \frac{k_1}{k_2}\right)^2\right]^{-1}$.

The approximation technique of the Schrödinger equation solution when quasiclassical conditions are met was first used by Wentzel, Kramers and Brillouin (WKB). This technique is known as WKB approximation or quasiclassical quantization method. With this method the barrier transparency is given by

$$D(E) \propto exp\left\{-\frac{2}{\hbar}\int_{z_1}^{z_2}\sqrt{2m(U(z) - E)}dz\right\} \quad (5)$$

For the number of electrons N_1 tunneling through the barrier from electrode 1 into electrode 2, we can write

$$N_1 = \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} \frac{p_z}{4\pi^3\hbar^3 m} f_1(E)\big(1 - f_2(E + eV)\big)D(E_z)dp_z = \int_0^{E_m} D(E_z)n(p_z)dE_z \quad (6)$$

where f_1 and f_2 are Fermi Dirac distributions in electrodes 1 and 2, respectively,

$$n(p_z) = \frac{1}{4\pi^3\hbar^3}\int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} f_1(E)\big(1 - f_2(E + eV)\big)D(E_z)dp_y \quad (7)$$

and E_m is the maximum energy of tunneling electrons.

Integration of expression (7) can be performed in polar coordinates. Because in the model under consideration $p_r^2 = p_x^2 + p_y^2$, $E_r = p_r^2/2m$ and the total energy is $E = E_z + E_r$, by changing variables $p_x = p_r cos\theta$, $p_y = p_r sin\theta$, we get

$$n(p_z) = \frac{1}{4\pi^3\hbar^3}\int_0^{2\pi} d\theta \int_0^\infty f_1(E)\big(1 - f_2(E + eV)\big)p_r dp_r = \frac{m}{2\pi^2\hbar^3}\int_0^\infty f_1(E)\big(1 - f_2(E + eV)\big)dE_r \quad (8)$$

Substituting (8) in (6), we obtain

$$N_1 = \frac{m}{2\pi^2\hbar^3}\int_0^E D(E_z)dE_z \int_0^\infty f_1(E_z + E_r)\big(1 - f_2(E_z + E_r + eV)\big)dE_r \quad (9)$$

The number of electrons N_2 tunneling back from electrode 2 into electrode 1 is calculated in the same way. The potential barrier transparency in the given case will be such as if positive voltage V is applied to electrode 1 relative to electrode 2.

In this case

$$N_2 = \frac{m}{2\pi^2\hbar^3}\int_0^{E_m} D(E_z)dE_z \int_0^\infty f_2(E_z + E_r + eV)\big(1 - f_1(E_z + E_r)\big)dE_r \quad (10)$$

Net electrons flow N through the barrier is obviously $N = N_1 - N_2$.

Let us denote

$$\xi_1(E_z) = \frac{me}{2\pi^2\hbar^3}\int_0^\infty f_1(E)\big(1 - f_2(E + eV)\big)dE_r,$$

$$\xi_2(E_z) = \frac{me}{2\pi^2\hbar^3}\int_0^\infty f_2(E + eV)\big(1 - f_1(E)\big)dE_r,$$

$$\xi(E_z, eV) = \xi_1 - \xi_2 = \frac{me}{2\pi^2\hbar^3}\int_0^\infty [f_1(E) - f_2(E + eV)]dE_r \quad (11)$$

Then, the tunneling current density J is

$$J = \int_0^{E_m} D(E_z)\xi(E_z, eV)dE_z \quad (12)$$

According to Fig. 1, $U(z)$ can be written in the form $U(z) = \mu + \varphi(z)$. Then, integrating (5) and using expression (A5) we get

$$D(E_z) \propto exp\left\{-A\delta_z\sqrt{\mu + \bar{\varphi}(z) - E_z}\right\} \quad (8)$$

where $\bar{\varphi}$ is the average barrier height relative to Fermi level of the negative electrode:

$$\bar{\varphi} = \frac{1}{\delta_z}\int_{z_1}^{z_2} \varphi(z)dz;$$

$$A = 2\beta\sqrt{\frac{2m}{\hbar^2}},$$

$\beta -$ dimensionless factor defined in the Appendix (A6) as:

$$\beta = 1 - \frac{1}{8\underline{f}^2\delta_z}\int_{z_1}^{z_2}\left[f(z) - \underline{f}\right]^2 dz$$

At $T = 0\,K$

$$\xi(E_z) = \frac{me}{2\pi^2\hbar^3}\begin{cases} eV, & E_z \in [0; \mu - eV] \\ \mu - E_z, & E_z \in [\mu - eV; \mu] \\ 0, & E_z > \mu \end{cases} \quad (9)$$

Combining (8) and (9) into (7), we obtain

$$J=\frac{me}{2\pi^2\hbar^3}\left\{eV\int_0^{\mu-eV}exp\left[-A\delta_z\sqrt{\mu+\bar{\varphi}-E_z}\right]dE_z+\int_{\mu-eV}^{\mu}(\mu-E_z)exp\left[-A\delta_z\sqrt{\mu+\bar{\varphi}-E_z}\right]dE_z\right\} \quad (10)$$

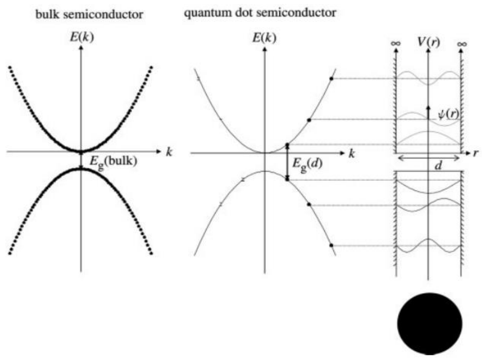
Integrating (10), we get Simmons current equation

$$J=\frac{\alpha}{\delta_z^2}\left\{\bar{\varphi}\ exp\left(-A\delta_z\sqrt{\bar{\varphi}}\right)-(\bar{\varphi}+eV)exp\left[-A\delta_z\sqrt{\bar{\varphi}+eV}\right]\right\} \quad (11)$$

where $\alpha=e/4\pi^2\beta^2\hbar$.

Question 4 [25 points]

4.1. In a quantum dot, the movement of electrons is confined in all three dimensions and there are only discrete (k_x,k_y,k_z) states in the k-space.



The charge carriers are confined in all three dimensions and this system can be described as an infinite three-dimensional potential well. The potential energy is zero everywhere inside the well but is infinite on its walls. We can also call this well a box. The simplest shapes for a three-dimensional box can be, for instance, a sphere or a cube. If the shape is cubic, the Schrödinger equation can be solved independently for each of the three translational degrees of freedom and the overall zero-point energy is simply the sum of the individual zero point energies for each degree of freedom:

$$E_{well,1d}=(1/8)h^2/md^2$$

$$E_{well,3d(cube)}=3E_{well,1d}=(3/8)h^2/md^2$$

where d is the size of the cube edge.

If the box is a sphere of diameter d, the Schrödinger equation can be solved by introducing spherical coordinates:

$$E_{well,3d(sphere)}=(1/2)h^2/md^2$$

A correction should be done considering that the strength of the screening coefficient depends on the dielectric constant of the semiconductor. An estimate of the Coulomb term yields

$$E_{Coul}=-1.8e^2/2\pi\varepsilon\varepsilon_0 d$$

The final equation is:

$$E_g(d) = E_g(bulk) + h^2/2m * d^2 - 1.8e^2/2\pi\varepsilon\varepsilon_0 d$$

4.2. The strongly curved parabola will change the effect mass to have a low effective mass and the E_g will increase.

Appendix

Let us integrate an arbitrary function $\sqrt{f(z)}$ from z_1 to z_2.

$$\int_{z_1}^{z_2} \sqrt{f(z)}\, dz \quad \text{(A1)}$$

Defining $\underline{f}$ as

$$\underline{f} = \frac{1}{\delta_z} \int_{z_1}^{z_2} \sqrt{f(z)} dz \quad \text{(A2)}$$

where $\underline{f}$ – average value of a function f on the interval from z_1 to z_2, $\delta_z = z_2 - z_1$. Then equation (A1) can be rewritten as

$$\int_{z_1}^{z_2} \sqrt{f(z)} dz = \sqrt{\underline{f}} \int_{z_1}^{z_2} \sqrt{1 + \frac{[f(z) - \underline{f}]}{\underline{f}}} dz \quad \text{(A3)}$$

Considering a Taylor series expansion of the integrand (A3) in and neglecting $\left[\left(f(z) - \underline{f}\right)/\underline{f}\right]^3$ and higher order members, we get

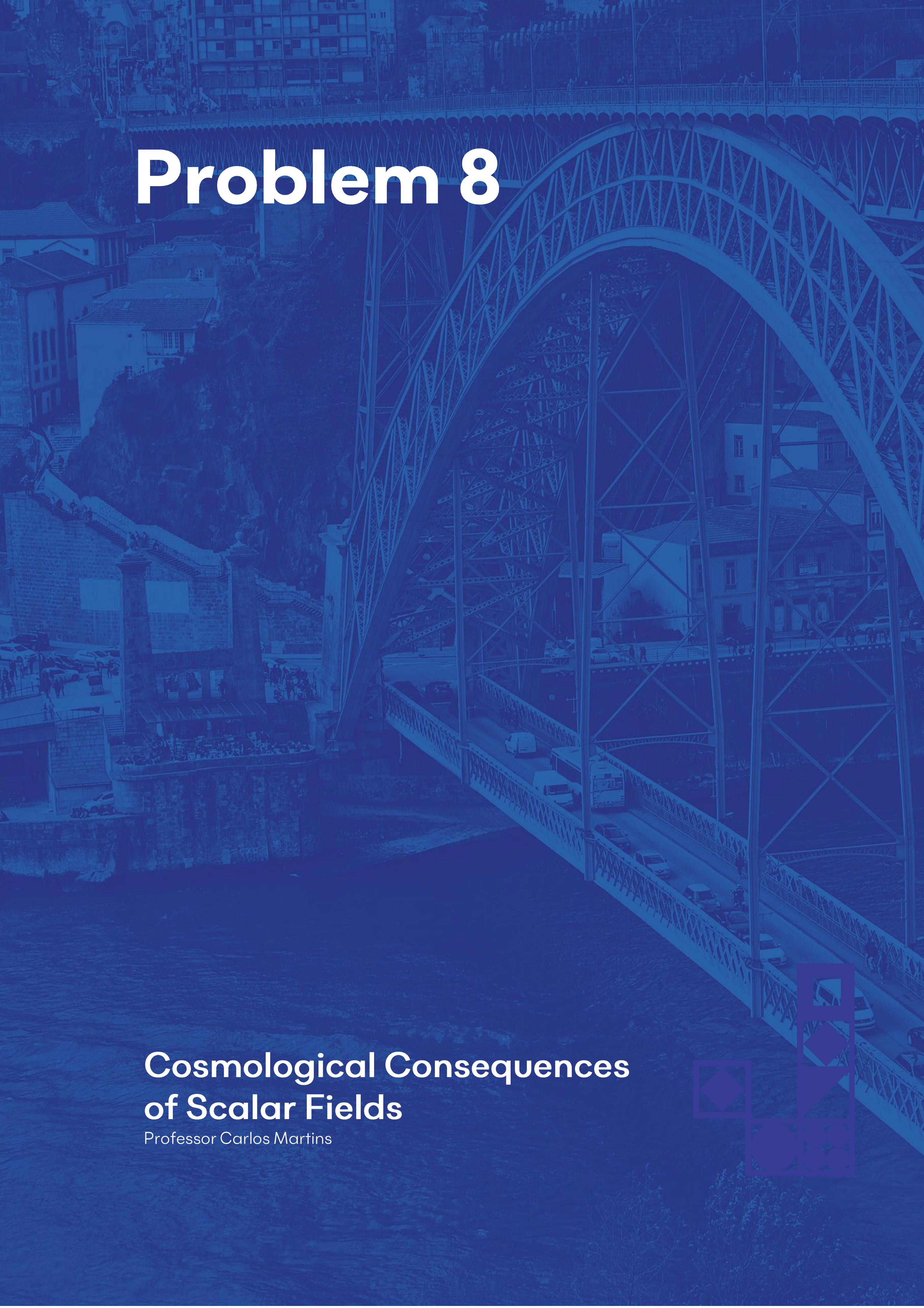
$$\int_{z_1}^{z_2} \sqrt{f(z)} dz = \sqrt{\underline{f}} \int_{z_1}^{z_2} \left\{1 + \frac{[f(z) - \underline{f}]}{2\underline{f}} - \frac{[f(z) - \underline{f}]^2}{8\underline{f}^2}\right\} dz \quad \text{(A4)}$$

The second term in (A4) vanishes upon integration, therefore (A4) can be expressed as

$$\int_{z_1}^{z_2} \sqrt{f(z)} dz = \beta \sqrt{\underline{f}\delta_z} \quad \text{(A5)}$$

where the correction factor is

$$\beta = 1 - \frac{1}{8\underline{f}^2 \delta_z} \int_{z_1}^{z_2} \left[f(z) - \underline{f}\right]^2 dz \quad \text{(A6)}$$



Problem 8

Cosmological Consequences of Scalar Fields

Professor Carlos Martins

Introduction

Since the 2012 discovery of a Higgs-like particle at the LHC, we know that fundamental scalar fields are among Nature's building blocks. Here we will explore some cosmological consequences of such scalar fields. We will assume homogeneous and isotropic universes, for which the Friedmann equation is

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho\,,(1)$$

where a is the scale factor, $H = \dot{a}/a$ is the Hubble parameter (the dot denotes a time derivative), k is the curvature parameter, and ρ is the total density (a sum of those of the constituents of the universe). We will work in units where $c = 1$. It is also useful to know the continuity equation

$$\dot{\rho} = -3H(\rho + p) = -3H(1 + w)\rho\,,(2)$$

where p is the total pressure and for convenience we also introduced $w = p/\rho$, the equation of state parameter.

Question 1 [20 points]

Consider a scalar field with

$$\rho_1 = \frac{1}{2}\dot{\phi}^2 + V_1(\phi)\,,\quad p_1 = \frac{1}{2}\dot{\phi}^2 - V_1(\phi)\,,(3)$$

where V_1 is a generic potential. Calculate the cosmological evolution equation for this scalar field. Then repeat the calculation for a scalar field with

$$\rho_2 = \frac{V_2(\phi)}{\sqrt{1-\dot{\phi}^2}}\,,\quad p_2 = -V_2(\phi)\sqrt{1-\dot{\phi}^2}\,,(4)$$

where again V_2 is a generic potential.

Under what conditions can each field dominate the universe and cause its recent acceleration?

Question 2 [20 points]

Consider the first of the scalar fields in Question 1. Show that if the field speed is small one can write, to first order,

$$w(z) = -1 + (1 + w_0)\frac{H_0^2}{H^2(z)}\,,(5)$$

where w_0 and H_0 denote present-day values.

Further assuming a flat universe containing only matter and the scalar field (with present-day fractional contributions to the energy density Ω_m and Ω_ϕ), show that the Friedmann equation has the form

$$\frac{H^2(z)}{H_0^2} = \Omega_m(1+z)^3 + \Omega_\phi \left[\frac{(1+z)^3}{\Omega_m(1+z)^3 + \Omega_\phi}\right]^{\frac{1+w_0}{\Omega_\phi}}. (6)$$

Question 3 [30 points]

Consider a flat universe containing matter and a cosmological constant, but also a scalar field which obeys the cosmological evolution equation

$$\ddot{\phi} + 3H\dot{\phi} = -3H_0^2 \left[\eta_m \Omega_m \left(\frac{a_0}{a}\right)^3 + \eta_\Lambda \Omega_\Lambda\right], (7)$$

where η_m and η_Λ are constant coupling parameters describing how the scalar field couples to the matter and dark energy sectors and a_0 is the present-day value of the scale factor (you can assume $\phi_0 = 0$).

Assuming that you can neglect the scalar field's contribution to the Friedmann equation, solve the above evolution equation and obtain the explicit form of the redshift evolution of the scalar field.

Hint: Under the above assumption there is an exact analytic solution, which is easiest to obtain through a carefully chosen change of variables. You should obtain as final result a combination of logarithmic functions, which is typical for many cosmological scalar fields.

Question 4 [30 points]

An interesting observational consequence of scalar fields is that they lead to a variation of the fine-structure constant α (a measure of the strength of the electromagnetic interaction), and thus also to a violation of the Einstein Equivalence Principle (the cornerstone of General Relativity). For a universe containing only matter and a homogeneous scalar field (which is also responsible for accelerating the universe), α has the redshift dependence

$$\frac{\Delta\alpha}{\alpha}(z) \equiv \frac{\alpha(z) - \alpha_0}{\alpha_0} = \zeta \int_0^z \sqrt{3 f_\phi(y)[1 + w_\phi(y)]} \frac{dy}{1+y}, (8)$$

where w_ϕ is the scalar field equation of state parameter,

$$f_\phi = \frac{\rho_\phi}{\rho_m + \rho_\phi} (9)$$

is the fractional contribution of the scalar field to the energy density of the universe, and ζ is another constant coupling parameter.

Calculate the generic explicit form (and redshift dependence) of the dark energy equation of state parameters that lead to a logarithmic dependence of α,

$$\frac{\Delta\alpha}{\alpha}(z) \propto \ln(1+z). (10)$$

You may again assume a flat universe. Then calculate the explicit redshift-dependent form of the Friedmann equation for that generic equation of state parameter.

Question 1 [20 points]

The scalar field evolution equations can be found by substituting the expressions for density and pressure (Eq.3 and Eq.4, for each of the two models) into the continuity equation (Eq.2). This leads, respectively, to

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV_1}{d\phi} = 0 \ (11)$$

and

$$\frac{\ddot{\phi}}{1-\dot{\phi}^2} + 3H\dot{\phi} + \frac{1}{V_2}\frac{dV_2}{d\phi} = 0 \ \ (12)$$

To answer the second part, one first needs to obtain the Raychaudhuri equation. The simplest way to do it is by differentiating both sides of the Friedmann equation (Eq.1). This will lead to a $\dot{\rho}$ term for which one can substitute the continuity equation (Eq.2). After some simplifications (including re-substituting the Friedmann equation) one finds

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \ \ (13)$$

The recent universe contains matter (with a density ρ_m), plus the assumed scalar field. For the scalar field to dominate the universe, the Friedmann equation requires $\rho_\phi > \rho_m$ while for the universe to be accelerating the Raychaudhuri equation requires $\rho_\phi + 3p_\phi + \rho_m < 0$ (recall that matter is pressureless). Therefore for the first model the conditions are

$$2V_1(\phi) > 2\rho_m - \dot{\phi}^2\,, \quad 2V_1(\phi) > \rho_m + 2\dot{\phi}^2\,, \ (14)$$

while for the second model they are

$$V_2(\phi) > \sqrt{1-\dot{\phi}^2}\rho_m\,, \quad V_2(\phi) > \frac{\sqrt{1-\dot{\phi}^2}}{1-3\dot{\phi}^2}\rho_m\,, \ (15)$$

Obtaining each of these is worth 4 points (i.e. obtaining both is worth 8 points).

Question 2 [20 points]

For the first of the fields in Part 1 (Eq.3) we can write

$$1 + w = 1 + \frac{p}{\rho} = \frac{\dot{\phi}^2}{1/2\dot{\phi}^2 + V}\,. \ (16)$$

For a slowly moving field $\dot{\phi}^2 \ll V$, and moreover V will be almost constant in time, so we can write

$$(1+w) \propto \dot{\phi}^2 \ \ (17)$$

Now consider the equation of motion for this field, which was obtained in Part 1 (Eq.11). For a slowly moving field one can neglect the $\ddot{\phi}$ term, and since V is almost constant, so is its derivative. Therefore we can also write

$$\dot{\phi} \propto \frac{1}{H} \ (18)$$

Together, these imply that $(1+w) \propto 1/H^2$. Normalizing with present-day values therefore yields

$$w(z) = -1 + (1 + w_0)\frac{H_0^2}{H^2(z)}, \quad (19)$$

Now let's consider a universe with matter and a scalar field. The redshift (or time) dependence of each component can be inferred using the continuity equation. It is useful to write this in terms of redshift rather than time

$$\frac{d\rho}{dz} = 3\frac{1+w(z)}{1+z}\rho \quad (20)$$

For matter one has $w = 0$ and therefore $\rho_m \propto a^{-3} \propto (1+z)^3$ while for the dark energy the behaviour will depend on the explicit form of $w(z)$. Again normalizing the Friedmann equation with present-day values, we can write

$$\frac{H^2(z)}{H_0^2} = \Omega_m(1+z)^3 + \Omega_\phi \exp\left[3\int_0^z \frac{(1+w(y))}{1+y}dy\right] \quad (21)$$

To calculate the integral we can now use

$$1 + w(z) = (1 + w_0)\frac{H_0^2}{H^2(z)}, \quad (22)$$

and again since the field is moving slowly (and to first approximation, dark energy can be assumed to be a constant, i.e. independent of redshift) one can write

$$1 + w(z) = \frac{1+w_0}{\Omega_m(1+z)^3 + \Omega_\phi}, \quad (23)$$

Finally, calculating the integral leads to

$$\frac{H^2(z)}{H_0^2} = \Omega_m(1+z)^3 + \Omega_\phi\left[\frac{(1+z)^3}{\Omega_m(1+z)^3+\Omega_\phi}\right]^{\frac{1+w_0}{\Omega_\phi}} \quad (24)$$

Question 3 [30 points]

In order to answer this one first needs an explicit solution for the Friedmann equation in a flat universe with matter and a cosmological constant. In this case the Friedmann equation can be written

$$\frac{H^2(z)}{H_0^2} = \Omega_m(1+z)^3 + \Omega_\Lambda = E^2(z) \quad (25)$$

where for later convenience we defined the function $E(z)$. For the moment, it is more convenient to rewrite this in terms of the scale factor

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\Omega_m\frac{a_0^3}{a^3} + \Omega_\Lambda\right) \quad (26)$$

This can be solved by the useful change of variables $y^2 = (\Omega_\Lambda/\Omega_m)(a/a_0)^3$, leading to the equation

$$\dot{y}^2 = \left(\frac{3}{2}H_0\right)^2 \Omega_\Lambda(1 + y^2), \quad (27)$$

which can be easily integrated, although one needs to bear in mind that the integration constant should be $a(0) = 0$. The result is

$$\frac{a(t)}{a_0} = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}H_0\sqrt{\Omega_\Lambda}t\right) \quad (28)$$

Using this solution, Eq.(7) can now be integrated analytically twice. The first integral is

$$\dot{\phi} = -3\Omega_m H_0^2 \frac{a_0^3}{a^3}\left[\eta_m t + \frac{\eta_\Lambda}{2b}(\sinh(bt) - bt)\right] \quad (29)$$

where for convenience we have defined $b = 3\sqrt{\Omega_\Lambda}H_0$ and the integration constant was set to zero since one physically expects that $\dot{\phi}(0) = 0$.

One then integrates again, being careful to use the relation $a_0/a = 1 + z$ to convert time and scale factor into redshift. This will finally lead to the solution

$$\phi(z) = 2\eta_m \ln(1+z) + \frac{2(\eta_\Lambda - 2\eta_m)}{3\sqrt{\Omega_\Lambda}}\left[\ln\left(\frac{1+\sqrt{\Omega_\Lambda}}{\sqrt{\Omega_m}}\right) - \sqrt{E^2(z)}\ln\left(\frac{\sqrt{\Omega_\Lambda}+\sqrt{E^2(z)}}{\sqrt{\Omega_m(1+z)^3}}\right)\right], \quad (30)$$

where ln is the natural logarithm and $E(z)$ is defined in Eq.25. Obtaining this equation is worth 10 points.

Question 4 [30 points]

The obvious insight here is that in order for the solution of the integral to be a logarithm, the term inside the square root must be a constant, i.e.

$$f_\phi(z)[1 + w(z)] = const. \quad (31)$$

where clearly it is beneficial to express them as a function of redshift. The redshift dependencies of the matter and scalar field components can be gathered from Eqs. 20 and 21 (for matter there is an explicit form, for the scalar field only an integral form). One then simply differentiates both sides of Eq.31, and after some algebra finds the following equation

$$\frac{dw}{dz} = -3(1 + w_0)\frac{w}{1+z}\left[\frac{1+w}{1+w_0} - \Omega_\phi\right]. \quad (32)$$

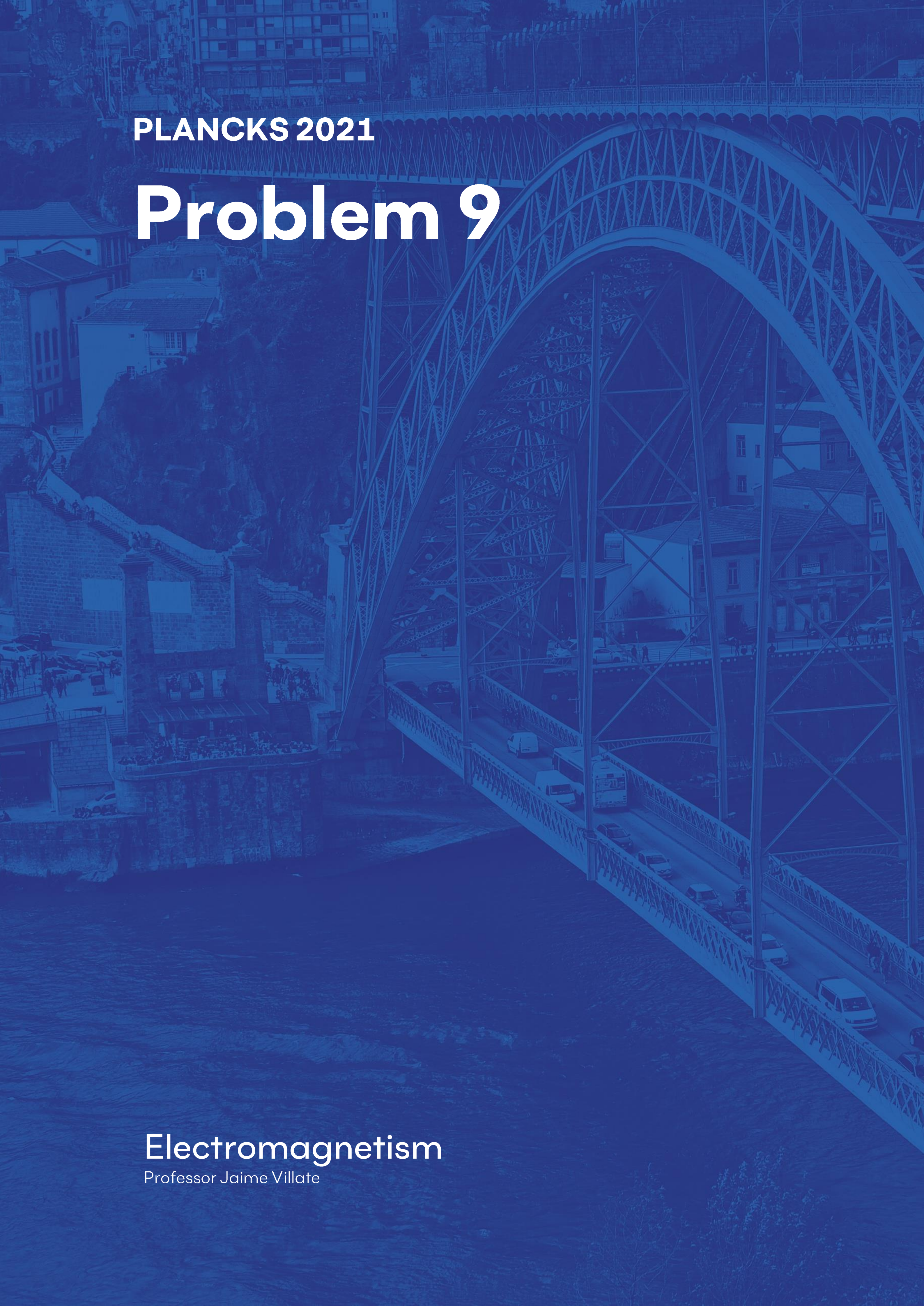
Equation [eq32] can be easily integrated, with the integration constant found bearing in mind the physical interpretation that w_0 must be the present-day value of the dark energy equation of state parameter. This leads to the solution

$$w(z) = \frac{[1-\Omega_\phi(1+w_0)]w_0}{\Omega_m(1+w_0)(1+z)^{3[1-\Omega_\phi(1+w_0)]} - w_0}, \quad (33)$$

where we are assuming that $\Omega_m + \Omega_\phi = 1$. Obtaining this result is worth 10 points.

To obtain the Friedmann equation one either uses Eq.9 to find the explicit form of ρ_ϕ, or alternatively substitutes Eq. 33 into Eq.21 and computes the integral. Either way, this leads to the explicit result

$$\frac{H^2(z)}{H_0^2} = \Omega_m(1+z)^3 + \frac{\Omega_\phi}{\Omega_m(1+w_0) - w_0}\left[\Omega_m(1+w_0)(1+z)^3 - w_0(1+z)^{3\Omega_\phi(1+w_0)}\right], \quad (34)$$



PLANCKS 2021

Problem 9

Electromagnetism

Professor Jaime Villate

Introduction

A proton moves in a vacuum chamber, entering a region with a uniform magnetic field $\vec{B}$ that points into the page, as shown in the figure. The proton passes through point P, at a distance l from the region, with velocity $\vec{v}$, on the plane of the page and making an angle θ with the perpendicular to the border of the region. After entering the region, it exits at a point which is at a distance d from the point where it entered, and with velocity that makes an angle ϕ with the perpendicular to the border. Assume that v is large enough so the effect of gravity can be neglected during the trajectory, but much smaller than the speed of light making relativistic effects irrelevant.

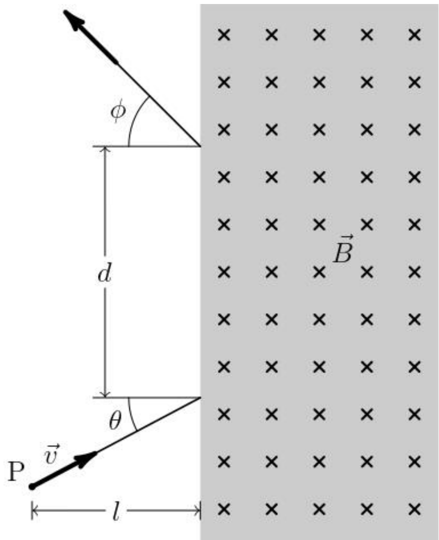


Figure 4 - Schematic representation of a proton entering a region with a magnetic field

Question 1 [10 points]

Find the exit angle ϕ.

Question 2 [20 points]

Find the distance d.

Question 3 [35 points]

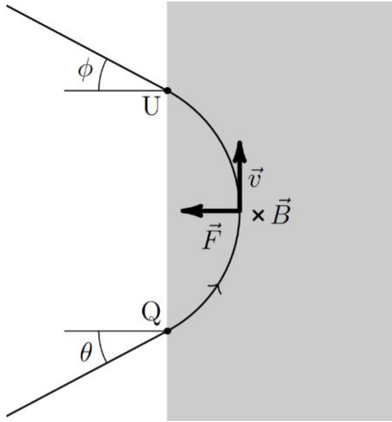
If the direction of the uniform magnetic field is reversed, making it point out of the page, find the angle θ that would make the proton return to point P after exiting the region.

Question 4 [35 points]

With the initial magnetic field, pointing into the page, if the region is kept at a constant electrostatic potential V different from the

potential equal to zero outside the region, find the distance d.

Solutions



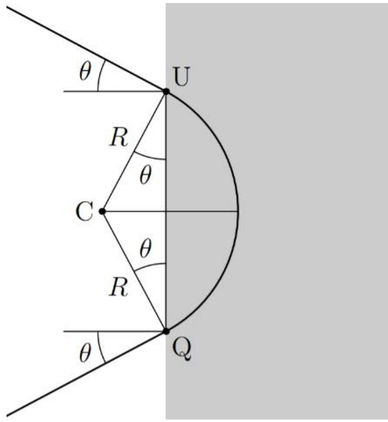
Question 1 [10 points] In the region with the magnetic field, the force acting on the proton is:

$$\vec{F} = e\,\vec{v} \times \vec{B}$$

where e is the charge of the proton.

That force will always be on the plane perpendicular to $\vec{B}$ (plane of the page). Since the proton enters the region at point P with velocity on that same plane, its trajectory will be on that plane. And since $\vec{F}$ is also perpendicular to $\vec{v}$ and it has constant module, the proton has uniform circular motion, following an arc of circle in anti-clockwise direction, as shown in the figure.

The velocity of the proton in the entry point Q and in the exit point U are both tangent to that arc. The symmetry of the circle implies that ϕ is equal to θ.



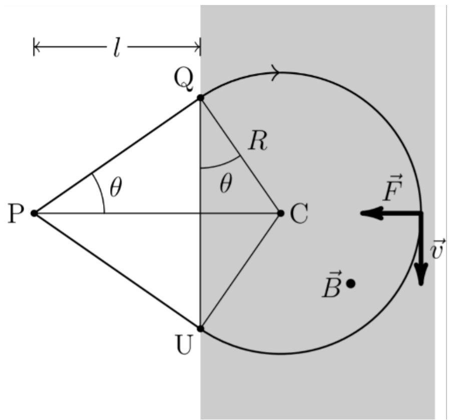
Question 2 [20 points] The module of the magnetic force is the centripetal force responsible for the proton's circular motion. Therefore:

$$e\,v\,B = \frac{m\,v^2}{R} \quad \Rightarrow \quad R = \frac{m\,v}{e\,B}$$

where R is the radius of the arc and m is the mass of the proton.

As shown in the figure, the triangle QUC, where C is the center of the arc and Q and U are the points where the proton enters and exits the region, has two sides of length R and two angles equal to θ. Therefore, the distance d between Q and U will be:

$$d = 2\,R\cos\theta = \frac{2\,m\,v\cos\theta}{e\,B}$$



Question 3 [35 points] The direction of the circular motion will now be clockwise as shown in the figure. The radius of the arc, R, is given by the same expression obtained in the previous item.

Observing the figure we see that

$$R\cos\theta = l\tan\theta$$

Thus, we obtain a quadratic equation for $\sin\theta$:

$$R\sin^2\theta + l\sin\theta - R = 0$$

with solutions:

$$\sin\theta = -\frac{l}{2\,R} \pm \sqrt{\left(\frac{l}{2\,R}\right)^2 + 1}$$

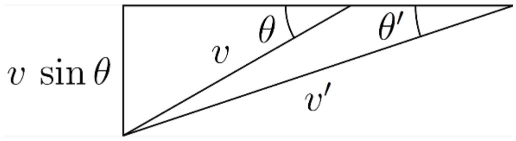
Since $0 \leq \theta \leq \pi/2$, the sine of θ must be positive and we are only interested in the positive solution. Substituting the expression found for the radius, we conclude that the incidence angle must be:

$$\theta = \sin^{-1}\left(\sqrt{\left(\frac{l\,e\,B}{2\,m\,v}\right)^2 + 1} - \frac{l\,e\,B}{2\,m\,v}\right)$$

Question 4 [35 points] The effect of the potential V is to modify the velocity $\vec{v}$ of the proton outside the region, into $\vec{v}\,'$ inside the region. By conservation of mechanical energy

$$\frac{m}{2}v^2 = \frac{m}{2}v'^2 + e\,V \quad \Rightarrow \quad v'^2 = v^2 - \frac{2\,e\,V}{m}$$

The electric force acting at the boundary of the region is perpendicular to it. Therefore, the components of $\vec{v}$ and $\vec{v}\,'$ parallel to the boundary of the region will be equal, as shown in the following figure



which leads to:

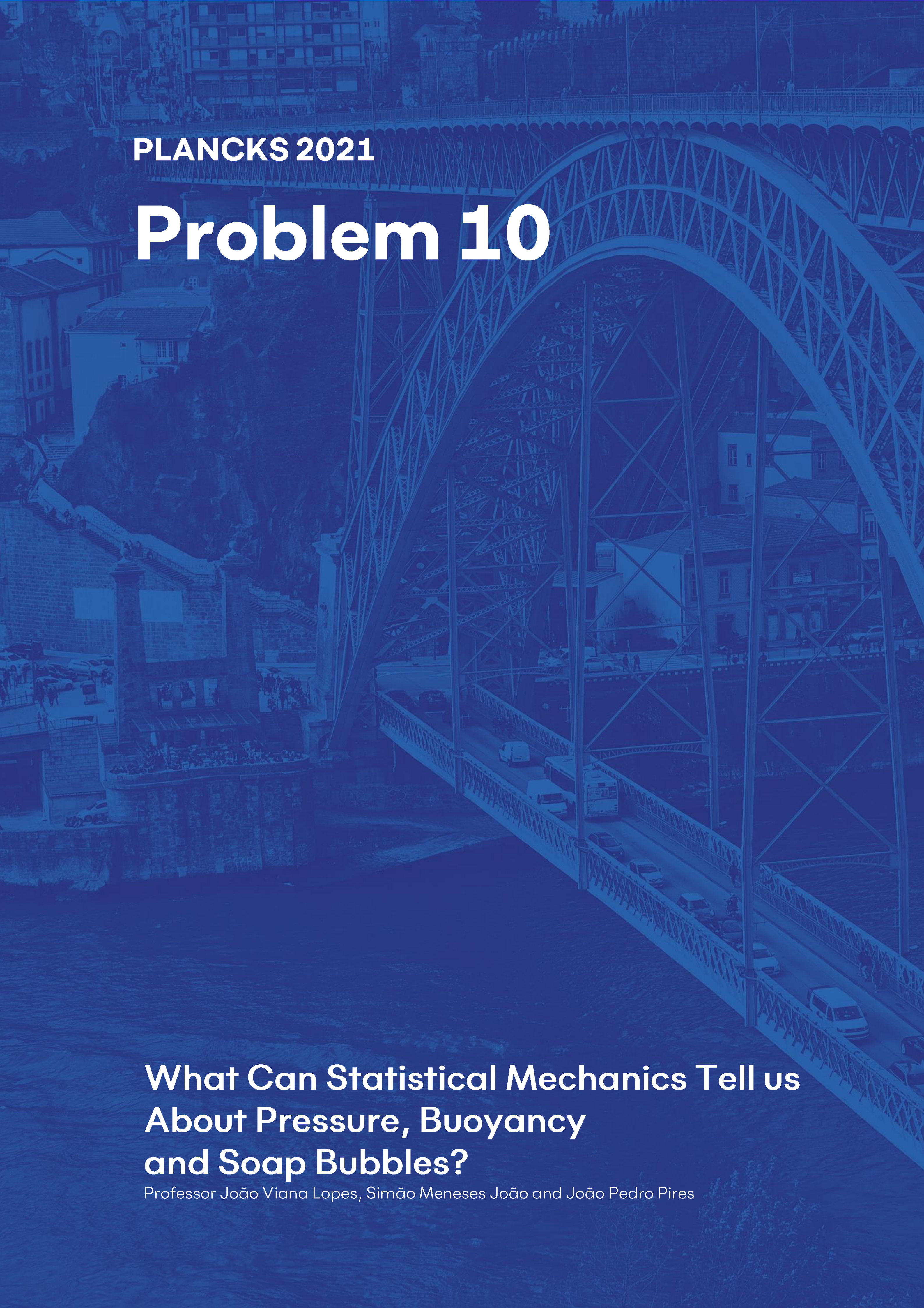
$$\cos\theta' = \frac{\sqrt{v'^2 - v^2\sin^2\theta}}{v'}$$

The expression for d is the same as in item (b), replacing v by v' and θ by θ'

$$d = \frac{2\,m\,v'\cos\theta'}{e\,B} = \frac{2\,m}{e\,B}\sqrt{v'^2 - v^2\sin^2\theta}$$

And substituting the expression for v'^2 we finally obtain

$$d = \frac{2\,m}{e\,B}\sqrt{v^2\cos^2\theta - \frac{2\,e\,V}{m}}$$



PLANCKS 2021

Problem 10

What Can Statistical Mechanics Tell us About Pressure, Buoyancy and Soap Bubbles?

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Introduction

Albeit being a central quantity in the physics of fluids, hydrostatic pressure dresses itself in rather diverse definitions depending on the physical context. In thermodynamics, the pressure of a gas confined inside a container is defined in terms of the work done when the confining volume is changed. In contrast, in the kinetic theory of gases, pressure is seen as the rate at which free particles transfer momentum to the wall due to elastic collisions. In both cases, the very definition of pressure is a non-local one, that seemingly requires the introduction of a macroscopic hard-wall.

Meanwhile, in statistical physics thermodynamic state variables (e.g. internal energy, temperature, chemical potential, electric polarization, magnetization, etc...) can be described in terms of averages of microscopic observables with respect to an equilibrium probability distribution for the micro-state of the many-particle system. In order to define pressure in this language, one must find a way which does not depend on the existence of a container's wall, onto which the gas can exert force. Instead, we will consider a system of N independent particles in free space, in equilibrium at a temperature T and subjected to a general confinement potential $-V(\mathbf{r})$. Our aim is to provide a local definition of pressure in this setup using purely statistical arguments.

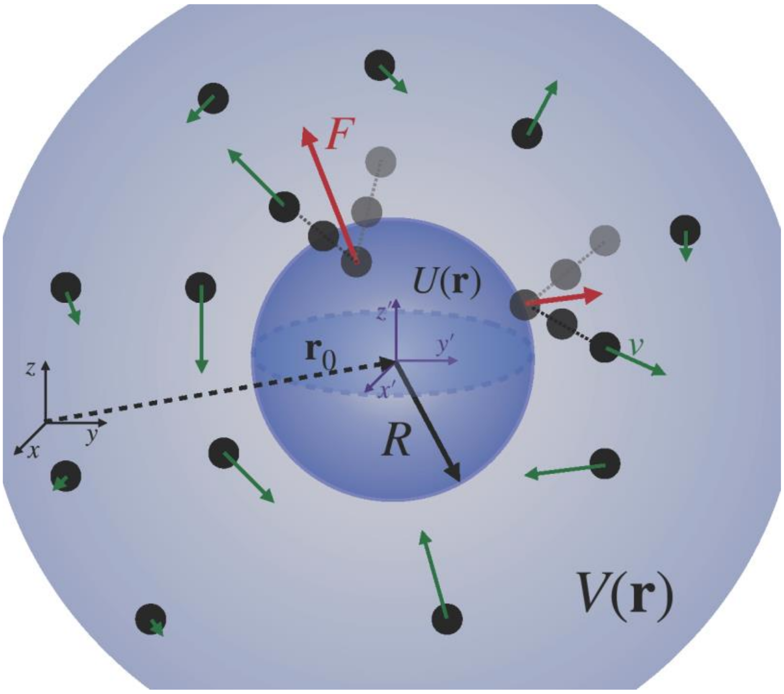


Figure 5 – Sketch of the physical situation.

1. Particle Density in Different Confining Geometries

The Hamiltonian for each particle of a non-interacting gas is simply

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r})$$

for a general external confining potential $V(\mathbf{r})$.

1.1. [8 points] Using the canonical ensemble, show that the particle density is

$$\rho(\mathbf{r}) = Ce^{-\beta V(\mathbf{r})}$$

and determine the constant C.

1.2. Sketch and characterize the particle density profiles for the following confinement $V(\mathbf{r})$:

c. [4 points] A cubic box (of side L) with hard walls

d. [4 points] A semi-infinite cylinder with hard walls defined by $x^2 + y^2 \leq R^2$ and $z > 0$, with a gravitational potential mgz.

e. [4 points] A three-dimensional harmonic well $\mathrm{V}(\mathbf{r}) = \frac{1}{2}\mathrm{k}|\mathbf{r}|^2$.

2. Local Pressure as a Statistical Quantity

To define a local pressure, let us introduce a small perturbation to (probe) the gaseous system — a small (radius R) and immovable sphere centered in position $\mathbf{r} = \mathbf{r}_0$. With no loss of generality, we can use a coordinate system centered in the probing sphere $\mathbf{r}' = \mathbf{r} - \mathbf{r}_0$ to make the calculations easier. The independent particles will repeatedly bounce off this sphere, as shown in Fig. 1, transferring momentum in each collision. This is the physical origin of gas pressure. Actually, only the momentum component normal to the surface will contribute to this pressure. To calculate it from such a microscopic description, we take the sphere's surface, not as a rigid wall, but rather as a steep potential which allows the particles to penetrate, whilst repelling them strongly in the outwards radial direction. Formally, this spherically symmetric potential can be modeled as

$$U_{\text{sph}}(\mathbf{r}') = U_0 f\left(\frac{r' - R}{R}\right)\Theta(R - r')$$

being included into the Hamiltonian for each gas particle. Here, $U_0 > 0$, $\Theta(x)$ is the Heaviside function and $f(x) \approx -x$ for $x \approx 0$. This way, when $U_0 \to +\infty$, we re-obtain the rigid wall scenario and, in that limit the interior details of the potential will not matter. Within this framework, answer the following questions:

2.1. [5 points] As the walls of the central sphere are not perfectly rigid, the particles are allowed to penetrate it. When doing so, each will feel a force $\mathbf{F}(\mathbf{r}_i')$ propelling it outwards ($\mathbf{r}_i'$ being the position of the particle i in the coordinate system centered in the sphere). Find an expression for this force, in terms of the $U_{\text{sph}}(\mathbf{r}')$ potential.

2.2. [5 points] Using the canonical ensemble, show that the statistical distribution of $\mathbf{F}(r')$ is

$$\mathbf{F}(r') = N \frac{r'^2 \left[-\frac{dU_{\text{sph}}}{dr'}\hat{\mathbf{r}}\right] \exp\left(-\beta U_{\text{sph}}(r')\right) \mathcal{I}(r')}{\int d^{(3)}\mathbf{r}' \exp\left(-\beta\left[U_{\text{sph}}(r') + V(\mathbf{r}_0 + \mathbf{r}')\right]\right)}$$

where $\mathcal{I}(r')$ is a radial function. Determine the form of $\mathcal{I}(r')$ and interpret it.

2.3. [5 points] Assuming the gas to be in thermodynamic equilibrium, what is the physical meaning of the average $\langle \mathbf{F} \cdot \hat{\mathbf{r}}' \rangle$? How does it relate to the pressure felt by the central sphere? Determine the pressure on the surface of the sphere.

2.4. [5 points] The integrals above can be done in the limit of rigid walls ($U_0 \to +\infty$). In this limit, prove that

$$\int_0^{\infty} dr'\, r'^2 \left[-\frac{dU_{\text{sph}}}{dr'}\right] \exp\left(-\beta U_{\text{sph}}(r')\right) \mathcal{I}(r') = \frac{1}{\beta} R^2 \mathcal{I}(R).$$

2.5. [5 points] Assuming that $V(\mathbf{r})$ is a slowly-varying function at scale of the central sphere. Show that the expression for the pressure simplifies to:

$$P(\mathbf{r}_0) = k_{\text{B}} T \rho(\mathbf{r}_0).$$

Retrieve the ideal gas law. Which confining potential will you use?

2.6. [5 points] Consider the case in which the sphere is centered on an harmonic well potential — $\mathrm{V}(r) = \frac{1}{2}\mathrm{k}|r|^2$. Obtain the exact expression of the pressure exerted on the sphere as a function of temperature and k. Derive the ideal gas law in the limit $k \to 0^+$ and the first correction due to a finite k. Physically justify the sign of this first correction. [**Hint**: Despite there being no container, the confining potential allows one to define an effective volume, where the gas gets concentrated in equilibrium.]

3. A Statistical Derivation of Archimedes' Principle

Since the classical antiquity, it is known that a solid body immersed in a fluid is acted upon by a vertical Buoyancy force which precisely equals the weight of the displaced volume of fluid (*Archimedes' Principle*). Using the framework devised above, one can re-obtain this result using a microscopic theory.

3.1 [10 points] Derive the average force in the $\hat{\mathbf{u}}$ direction $\langle \mathbf{F} \cdot \hat{\mathbf{u}} \rangle = \bar{F}_u$ caused by the gas **on the sphere**, i.e.

$$\bar{F}_u = -N \frac{\int_0^\infty dr'\, r'^2 \left[-\frac{dU_{\text{sph}}}{dr'} \right] \exp\left(-\beta U_{\text{sph}}(r') \right) \mathcal{J}(r')}{\int d^{(3)}\mathbf{r} \exp\left(-\beta V(\mathbf{r}) \right)},$$

where $\mathcal{J}(r' = |\mathbf{r} - \mathbf{r}_0|) = \int_0^{2\pi} d\phi' \int_0^\pi d\theta' \cos\theta' \sin\theta' \exp\left(-\beta V(r', \theta', \phi') \right)$ by aligning the z' axis of the spherical coordinates along $\hat{\mathbf{u}}$.

3.2. [10 points] If $V(\mathbf{r})$ is considered constant across the sphere, the Buoyancy force is zero. Based on the hydrostatic interpretation of this force, provide a physical justification for this result.

3.3. [10 points] In the rigid wall limit $U_0 \to +\infty$, obtain the following expression for the Buoyancy force:

$$\mathbf{F} = -\frac{4}{3}\pi R^3 \rho(\mathbf{r}_0)\left(-\nabla V(\mathbf{r}_0) \right).$$

If $V(\mathbf{r})$ is an uniform gravitational potential, show that the previous expression reduces to Archimedes' principle

$$F_{\text{A}} \quad = \quad V_b \rho(z) m g$$

where V_b is the volume of the sphere.

[Hint: Approximate $\exp\left(-\beta V(\mathbf{r}) \right)$ by a multivariable Taylor series around $\mathbf{r}_0$ (up to first order) and express it in spherical coordinates. Set $\hat{\mathbf{u}}$ as the direction of the gradient of V]

4. How can a Soap Bubble be used a Barometer?

Previously, we have used a small spherical probe as a theoretical device to define the pressure at any point inside a gas, in a statistical sense. Now, we will see that such device can actually be put to practice and used as a local barometer.

Consider a soap bubble of radius R suspended in the gas. This bubble is stable thanks to the mutual equilibrium of its internal gas pressure, the surface tension of the thin soap film and the pressure due to the external gas. If the external pressure is increased, the radius of the bubble must decrease accordingly, in order to achieve a new equilibrium state. Therefore, the radius R can be used as an indirect measure of the local pressure of the gaseous environment. The change in energy due to a change in area of a surface held together by surface tension is given by $dE = \gamma dA$, where $\gamma > 0$, so the system tends to smaller areas. Assuming the gas inside the bubble to be an ideal gas in thermodynamic equilibrium, answer the following questions.

4.1. [10 points] Obtain Laplace's equation,

$$P_{out} - P_{in} = \frac{-2\gamma}{R}$$

for the equilibrium of forces acting on the soap bubble.

4.2. [10 points] Show that the external pressure is related to the bubble's radius (R) through

$$P_{out} = \frac{N k_B T}{\frac{4}{3}\pi R^3} + \frac{-2\gamma}{R}$$

where N is the number of gas particles inside the bubble and T is the temperature. Verify that this expression allows for a one-to-one correspondence between R and P_{out}, across any physical regime.

Problem 1: Particle Density in Different Confining Geometries

1.1 General expression

The quantity that has to be averaged over in order to calculate the statistical average of the density of particles is

$$\rho(\mathbf{r}) = \sum_{i=1}^{N} \delta\left(\mathbf{r} - \mathbf{r}_i\right)$$

Considering only the gas subject to an external potential, the Hamiltonian of one particle is

$$H(\mathbf{r}, \mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r})$$

and the Hamiltonian of the whole system is

$$\mathcal{H}(\{\mathbf{r}_i, \mathbf{p}_i\}) = \sum_{i=1}^{N} H\left(\mathbf{r}_i, \mathbf{p}_i\right) = \sum_{i=1}^{N} \left(\frac{\mathbf{p}_i^2}{2m} + V(\mathbf{r}_i)\right)$$

The partition function of N particles is

$$Z_N = \frac{1}{N!\, h^{3N}} \int d^{3N}\mathbf{r} \int d^{3N}\mathbf{p} \exp(-\beta\mathcal{H})$$

which has the $N!$ to correct for Gibbs' paradox. Since the particles are independent, Z reduces to

$$Z_N = \frac{1}{N!}\left[\frac{1}{h^3}\int d^3\mathbf{r} \int d^3\mathbf{p} \exp\big(-\beta H(\mathbf{r}, \mathbf{p})\big)\right]^N$$

the integral in the momentum can be done explicitly

$$Z_N = \frac{1}{N!}\left[\left(\frac{2\pi m}{\beta h^2}\right)^{3/2} \int d^3\mathbf{r} \exp\big(-\beta V(\mathbf{r})\big)\right]^N$$

The average of the density of particles is

$$\langle\rho(\mathbf{r})\rangle = \frac{\frac{1}{N!\, h^{3N}} \int d^{3N}\mathbf{r} \int d^{3N}\mathbf{p} \sum_{i=1}^{N} \delta\left(\mathbf{r} - \mathbf{r}_i\right) \exp(-\beta\mathcal{H})}{Z_N}$$

since the integrals are independent, we get the simplification

$$\langle\rho(\mathbf{r})\rangle = N \frac{\frac{1}{N!\, h^{3N}} \int d^3\mathbf{r} \int d^3\mathbf{p}\, \delta(\mathbf{r} - \mathbf{r}_i) \exp(-\beta H)}{\frac{1}{N!}\left[\frac{1}{h^3}\int d^3\mathbf{r}' \int d^3\mathbf{p} \exp\big(-\beta H(\mathbf{r}', \mathbf{p})\big)\right]}$$

which simplifies further because the momentum integrals also factor out

$$\langle\rho(\mathbf{r})\rangle = N\frac{\int d^3\mathbf{r}'\delta(\mathbf{r}-\mathbf{r}')\exp\big(-\beta V(\mathbf{r}')\big)}{\int d^3\mathbf{r}'\exp\big(-\beta V(\mathbf{r}')\big)}$$

Acting with the Dirac delta yields the final expression

$$\langle\rho(\mathbf{r})\rangle = N\frac{\exp\big(-\beta V(\mathbf{r})\big)}{\int d^3\mathbf{r}'\exp\big(-\beta V(\mathbf{r}')\big)}$$

1.1.1 Small cubic box

$$V(\mathbf{r}) \quad = \quad \begin{cases} 0 & \text{inside the box} \\ +\infty & \text{outside the box} \end{cases}$$

therefore

$$\langle\rho(\mathbf{r})\rangle_{\text{box}} = \begin{cases} \dfrac{N}{V_b} & \text{inside the box} \\ 0 & \text{outside the box} \end{cases}$$

1.1.2 Cylinder

$$V(\mathbf{r}) \quad = \quad \begin{cases} mgz & \text{inside the cylinder} \\ +\infty & \text{outside the cylinder} \end{cases}$$

we just have to solve the integral

$$\begin{aligned} \int d^3\mathbf{r}'\exp\big(-\beta V(\mathbf{r}')\big) &= \int_0^R d\,rr\int_0^{2\pi} d\,\phi\int_0^{\infty} d\,z\exp(-\beta mgz) \\ &= \pi R^2\int_0^{\infty} d\,z\exp(-\beta mgz) \\ &= \pi R^2\left.\frac{\exp(-\beta mgz)}{-\beta mg}\right|_0^{\infty} \\ &= \frac{\pi R^2}{\beta mg} \end{aligned}$$

therefore

$$\langle\rho(\mathbf{r})\rangle_{\text{box}} = \begin{cases} N\dfrac{\pi R^2}{\beta mg} & \text{inside the cylinder} \\ 0 & \text{outside the cylinder} \end{cases}$$

1.1.3 Harmonic confinement

The harmonic potential extends to all space $V(r) = \frac{1}{2}k|r|^2$ and integrates to

$$\begin{aligned}\int d^3\mathbf{r}'\exp\left(-\beta V(\mathbf{r}')\right) &= \int_0^{+\infty} d\,rr^2\int_0^{2\pi} d\,\phi\int_0^{\pi} d\,\theta\sin(\theta)\exp\left(-\beta\frac{1}{2}kr^2\right)\\ &= 4\pi\int_0^{+\infty} d\,rr^2\exp\left(-\frac{\beta k}{2}r^2\right)\\ &= 2\pi\int_{-\infty}^{+\infty} d\,rr^2\exp\left(-\frac{\beta k}{2}r^2\right)\\ &= 2\pi\frac{\sqrt{\pi}}{2}\left(\frac{\beta k}{2}\right)^{-\frac{3}{2}}\\ &= \left(\frac{2\pi}{\beta k}\right)^{\frac{3}{2}}\end{aligned}$$

which provides the final value for the density

$$\langle\rho(\mathbf{r})\rangle = N\left(\frac{2\pi}{\beta k}\right)^{-\frac{3}{2}}\exp\left(-\frac{\beta k}{2}r^2\right)$$

Problem 2: Local Pressure as a Statistical Quantity

2.1 Expression for the force

The force each particle feels is simply minus the gradient of the potential to which they are subjected. This case is particularly simple because the potential is spherically symmetric and so the force only has radial component:

$$\mathbf{f} = -\nabla U(\mathbf{r}) = -\frac{dU}{dr}\hat{\mathbf{r}}$$

2.2 Distribution of the force

The total force is a function of all the coordinates

$$\mathbf{f}_{\text{total}}(\{\mathbf{r}_i\}) = \sum_{i=1}^{N}\mathbf{f}\,(\mathbf{r}_i)$$

but the force at a point $\mathbf{r}$ depends on how many particles are in that position

$$\mathbf{F}(\mathbf{r}) = \sum_{i=1}^{N}\mathbf{f}\,(\mathbf{r}_i)\delta(\mathbf{r}_i-\mathbf{r})$$

The force distribution is therefore

$$\begin{aligned}\langle\mathbf{F}(\mathbf{r})\rangle &= \frac{\frac{1}{N!\,h^{3N}}\int d^{3N}\mathbf{r}\int d^{3N}\mathbf{p}\mathbf{F}(\mathbf{r})\exp(-\beta\mathcal{H})}{Z_N}\\ &= \frac{\frac{1}{N!\,h^{3N}}\int d^{3N}\mathbf{r}\int d^{3N}\mathbf{p}\sum_{i=1}^{N}\mathbf{f}\,(\mathbf{r}_i)\delta(\mathbf{r}_i-\mathbf{r})\exp(-\beta\mathcal{H})}{Z_N}\\ &= \frac{\frac{1}{N!\,h^3}N\int d^3\mathbf{r}_1\int d^3\mathbf{p}_1\mathbf{f}(\mathbf{r}_1)\delta(\mathbf{r}_1-\mathbf{r})\exp\left(-\beta H(\mathbf{r}_1,\mathbf{p}_1)\right)}{\frac{1}{N!}\frac{1}{h^3}\int d^3\mathbf{r}_1\int d^3\mathbf{p}_1\exp\left(-\beta H(\mathbf{r}_1,\mathbf{p}_1)\right)}\end{aligned}$$

Now the Hamiltonian of one particle has an extra term stemming from the spherical potential in the middle

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}) + U(r)$$

so that, after integrating the momentum, we get

$$\langle \mathbf{F}(\mathbf{r}) \rangle = N \frac{\left[-\frac{dU}{dr}\hat{\mathbf{r}}\right] \exp(-\beta[V(\mathbf{r}) + U(r)])}{\int d^3\mathbf{r}_1 \exp(-\beta[V(\mathbf{r}_1) + U(r_1)])}$$

Since we're only interested in the radial distribution of the force, we can integrate its angular component in spherical coordinates

$$\begin{aligned}
\langle \mathbf{F}(r) \rangle &= \int_0^{2\pi} d\,\phi \int_0^{\pi} d\,\theta \sin(\theta) r^2 \langle \mathbf{F}(\mathbf{r}) \rangle \\
&= N \frac{r^2 \left[-\frac{dU}{dr}\hat{\mathbf{r}}\right] \exp\big(-\beta U(r)\big) \int_0^{2\pi} d\,\phi \int_0^{\pi} d\,\theta \sin(\theta) \exp\big(-\beta V(\mathbf{r})\big)}{\int d^3\mathbf{r}_1 \exp(-\beta[V(\mathbf{r}_1) + U(r_1)])} \\
&= N \frac{r^2 \left[-\frac{dU}{dr}\hat{\mathbf{r}}\right] \exp\big(-\beta U(r)\big) \mathcal{I}(r)}{\int d^3\mathbf{r}_1 \exp(-\beta[V(\mathbf{r}_1) + U(r_1)])}
\end{aligned}$$

where we defined

$$\int_0^{2\pi} d\,\phi \int_0^{\pi} d\,\theta \sin(\theta) \exp\big(-\beta V(\mathbf{r})\big) = \mathcal{I}(r)$$

2.3 Pressure

The quantity $\langle \mathbf{F} \cdot \hat{\mathbf{r}} \rangle$ is the average radial force that the particles exert on the surface of the sphere. Dividing by the surface area, we get the pressure

$$P = \frac{1}{4\pi R} \langle \mathbf{F} \cdot \hat{\mathbf{r}} \rangle$$

It can now be calculated

$$\begin{aligned}
P &= \frac{1}{4\pi R^2} \langle \mathbf{F} \cdot \hat{\mathbf{r}} \rangle \\
&= \frac{1}{4\pi R^2} N \int_0^{\infty} d\,r \hat{\mathbf{r}} \cdot \langle \mathbf{F}(r) \rangle \\
&= \frac{1}{4\pi R^2} N \int_0^{\infty} d\,r \hat{\mathbf{r}} \cdot \frac{r^2 \left[-\frac{dU}{dr}\hat{\mathbf{r}}\right] \exp\big(-\beta U(r)\big) \mathcal{I}(r)}{\int d^3\mathbf{r}_1 \exp(-\beta[V(\mathbf{r}_1) + U(r_1)])} \\
&= \frac{1}{4\pi R^2} N \frac{\int_0^{\infty} d\,r r^2 \left[-\frac{dU}{dr}\right] \exp\big(-\beta U(r)\big) \mathcal{I}(r)}{\int d^3\mathbf{r}_1 \exp(-\beta[V(\mathbf{r}_1) + U(r_1)])}
\end{aligned}$$

2.4 Taking the limit

The numerator can be simplified by noting that the potential is zero when $r > R$ and letting the derivative act on the exponential

$$
\begin{aligned}
&= \int_0^\infty d\,r r^2 \left[-\frac{dU}{dr}\right] \exp\big(-\beta U(r)\big)\mathcal{I}(r) \\
&= \frac{1}{\beta}\int_0^\infty d\,r \frac{d}{dr}\exp\big(-\beta U(r)\big) r^2 \mathcal{I}(r) \\
&= \frac{1}{\beta}\int_0^R d\,r \frac{d}{dr}\exp\big(-\beta U(r)\big) r^2 \mathcal{I}(r)
\end{aligned}
$$

this can be integrated by parts

$$
\begin{aligned}
&= \frac{1}{\beta}\int_0^R d\,r \frac{d}{dr}\exp\big(-\beta U(r)\big) r^2 \mathcal{I}(r) \\
&= \frac{1}{\beta}\left[\left[\exp\big(-\beta U(r)\big) r^2 \mathcal{I}(r)\right]_0^R - \int_0^R d\,r \exp\big(-\beta U(r)\big)\frac{d}{dr}\big(r^2\mathcal{I}(r)\big)\right]
\end{aligned}
$$

the last term vanishes in the limit $U \to +\infty$ and the first one is zero at $r = 0$ so we reach

$$
\int_0^\infty d\,r r^2 \left[-\frac{dU}{dr}\right] \exp\big(-\beta U(r)\big)\mathcal{I}(r) = \frac{1}{\beta} R^2 \mathcal{I}(R)
$$

2.5 Slowly varying potential

If we assume the potential $V(\mathbf{r})$ is slowly varying in the scale of the sphere, then we can approximate it by its value at the origin

$$
\mathcal{I}(R) = \int_0^{2\pi} d\,\phi \int_0^\pi d\,\theta \sin(\theta) \exp\big(-\beta V(R,\theta,\phi)\big) = 4\pi \exp\big(-\beta V(\mathbf{0})\big)
$$

The denominator in the pressure also simplifies, because for very large U, there will be no contribution to the integral, and this lack of contribution is very small. Therefore, we can simply ignore U

$$
\int d^3\mathbf{r}_1 \exp(-\beta[V(\mathbf{r}_1) + U(r_1)]) = \int d^3\mathbf{r}_1 \exp(-\beta[V(\mathbf{r}_1)])
$$

With these simplifications, the pressure

$$
\begin{aligned}
P &= \frac{1}{4\pi R^2} N \frac{\int_0^\infty d\,r r^2 \left[-\frac{dU}{dr}\right]\exp\big(-\beta U(r)\big)\mathcal{I}(r)}{\int d^3\mathbf{r}_1 \exp(-\beta[V(\mathbf{r}_1) + U(r_1)])} \\
&= \frac{1}{4\pi R^2} N \frac{\frac{1}{\beta} R^2 \mathcal{I}(R)}{\int d^3\mathbf{r}_1 \exp\big(-\beta V(\mathbf{r}_1)\big)} \\
&= \frac{1}{4\pi R^2} N \frac{\frac{1}{\beta} R^2 4\pi \exp\big(-\beta V(\mathbf{0})\big)}{\int d^3\mathbf{r}_1 \exp\big(-\beta V(\mathbf{r}_1)\big)} \\
&= \frac{N}{\beta} \frac{\exp\big(-\beta V(\mathbf{0})\big)}{\int d^3\mathbf{r}_1 \exp\big(-\beta V(\mathbf{r}_1)\big)}
\end{aligned}
$$

the last term is exactly the density of particles, so

$$
P = k_B T \rho(\mathbf{0})
$$

This is the pressure of the gas at the origin, because the sphere is assumed to be very small. This expression is actually valid for any point because the origin is arbitrary:

$$P(\mathbf{r}) = k_B T \rho(\mathbf{r})$$

Using the density of particles derived in the first part of the problem, $\rho(\mathbf{r}) = N/V_b$, we get

$$P(\mathbf{r}) = k_B T N/V_b$$

the ideal gas law.

2.6 Harmonic potential

The general expression obtained before yields the pressure in the rigid walls of the central bubble (of radius R) as follows:

$$P_{\text{sph}} = \frac{N k_{\text{B}} T}{4\pi} \times \frac{\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \exp\big(-\beta V(R)\big)}{\int_0^{\infty} dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta r^2 \exp\big(-\beta V(r)\big)},$$

where we already assumed a spherically symmetric form for the confinement potential, $V(\mathbf{r})$. For the case in point, we have $V(r) = kr^2/2$. The integral in the denominator is then simply a gaussian integral in a three-dimensional infinite volume, whilst the numerator integrates trivially for a spherically symmetric potential. Hence, we arrive at

$$P_{\text{sph}} = N k_{\text{B}} T \left[\frac{2\pi k_{\text{B}} T}{k}\right]^{-\frac{3}{2}} \times e^{-\frac{kR^2}{2k_{\text{B}}T}}.$$

Although there is no fixed volume container, the harmonic confinement potential naturally defines an effective spherical volume, which reads

$$V_{\text{eff}} = \frac{4\pi}{3}\left[\frac{k_{\text{B}} T}{k}\right]^{\frac{3}{2}}$$

and allows the equation for P_{sph} to be written as

$$P_{\text{sph}} = \sqrt{\frac{2}{9\pi}} \frac{N k_{\text{B}} T}{V_{\text{eff}}}\left[1 - \frac{1}{2} \times \frac{R^2}{k_{\text{B}} T/k} + \cdots\right].$$

Note that in lowest order, we have an ideal gas law, with the correct dependence on both temperature and the effective volume of the gas cloud. The first correction to this law is negative in the radius of the sphere — this makes sense as the potential energy at the boundary of the central sphere is now larger than zero, leading to a decreased kinetic energy of the gas and, hence, a reduced momentum transfer to the surface.

Problem 3: A Statistical Derivation of Archimedes' Principle

3.1 Average vertical force

To calculate $\langle \mathbf{F} \cdot \hat{\mathbf{z}} \rangle = \bar{F}_z$ we can use the distribution of the force we calculated previously

$$
\begin{aligned}
\bar{F}_z &= \langle \mathbf{F}\cdot\hat{\mathbf{z}}\rangle \\
&= \int d^3\mathbf{r}\langle \mathbf{F}(\mathbf{r})\rangle\cdot\hat{\mathbf{z}} \\
&= N\frac{\int d^3\mathbf{r}\left[-\frac{dU}{dr}\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}\right]\exp(-\beta[V(\mathbf{r})+U(r)])}{\int d^3\mathbf{r}_1\exp(-\beta[V(\mathbf{r}_1)+U(r_1)])}
\end{aligned}
$$

in spherical coordinates $\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}=\cos(\theta)$ so we get

$$
\begin{aligned}
\bar{F}_z &= \langle \mathbf{F}\cdot\hat{\mathbf{z}}\rangle \\
&= \int d^3\mathbf{r}\langle \mathbf{F}(\mathbf{r})\rangle\cdot\hat{\mathbf{z}} \\
&= N\frac{\int_0^R d\,rr^2\int_0^{2\pi}d\,\phi\int_0^{\pi}d\,\theta\sin\theta\left[-\frac{dU}{dr}\cos(\theta)\right]\exp(-\beta[V(\mathbf{r})+U(r)])}{\int d^3\mathbf{r}_1\exp(-\beta[V(\mathbf{r}_1)+U(r_1)])}
\end{aligned}
$$

The denominator is simplified like before, and the numerator becomes

$$
\begin{aligned}
&\int_0^R d\,rr^2\int_0^{2\pi}d\,\phi\int_0^{\pi}d\,\theta\sin\theta\left[-\frac{dU}{dr}\cos(\theta)\right]\exp(-\beta[V(\mathbf{r})+U(r)]) \\
=&\int_0^R d\,rr^2\left[-\frac{dU}{dr}\right]\exp\big(-\beta U(r)\big)\int_0^{2\pi}d\,\phi\int_0^{\pi}d\,\theta\sin\theta\cos(\theta)\exp\big(-\beta V(\mathbf{r})\big) \\
=&\int_0^R d\,rr^2\left[-\frac{dU}{dr}\right]\exp\big(-\beta U(r)\big)\mathcal{J}(r)
\end{aligned}
$$

where we defined

$$
\mathcal{J}(r)=\int_0^{2\pi}d\,\phi\int_0^{\pi}d\,\theta\sin\theta\cos(\theta)\exp\big(-\beta V(\mathbf{r})\big)
$$

3.2 Constant potential

If $V(\mathbf{r})$ is constant, then the integral $\mathcal{J}$ is zero:

$$
\begin{aligned}
\mathcal{J}(r) &= \int_0^{2\pi}d\,\phi\int_0^{\pi}d\,\theta\sin\theta\cos(\theta)\exp(-\beta V) \\
&= \exp(-\beta V)2\pi\int_0^{\pi}d\,\theta\sin\theta\cos(\theta) \\
&= \exp(-\beta V)2\pi\left(\frac{\sin^2\theta}{2}\right)_0^{\pi}=0
\end{aligned}
$$

The buoyancy comes from a local imbalance of pressure. If there is a higher pressure on one side of an object than another, there will be a net force acting on that object. If the pressure is constant, no net force will exist.

3.3 Derivation

To derive Archimedes' principle, we let V vary slowly. Then, we can say

$$
\exp\big(-\beta V(\mathbf{r})\big)=\exp\big(-\beta V(\mathbf{0})\big)\left[1-\beta\frac{\partial V}{\partial x}\bigg|_{\mathbf{r}=\mathbf{0}}x-\beta\frac{\partial V}{\partial y}\bigg|_{\mathbf{r}=\mathbf{0}}y-\beta\frac{\partial V}{\partial z}\bigg|_{\mathbf{r}=\mathbf{0}}z\right]
$$

which in spherical coordinates is

$$
\exp\big(-\beta V(\mathbf{r})\big)=\exp\big(-\beta V(\mathbf{0})\big)-\beta\exp\big(-\beta V(\mathbf{0})\big)\nabla V(\mathbf{0})\cdot(r\sin\theta\cos(\phi),r\sin\theta\sin(\phi),r\cos\theta)
$$

Only the term in the z direction will contribute to the integral

$$
\begin{aligned}
\mathcal{J}(r) &= \int_0^{2\pi} d\,\phi \int_0^{\pi} d\,\theta \sin\theta \cos(\theta) \exp\left(-\beta V(\mathbf{r})\right) \\
&= -\beta \left.\frac{\partial V}{\partial z}\right|_{\mathbf{r}=\mathbf{0}} \exp\left(-\beta V(\mathbf{0})\right) \int_0^{2\pi} d\,\phi \int_0^{\pi} d\,\theta \sin\theta \cos(\theta) r \cos\theta \\
&= -\beta \left.\frac{\partial V}{\partial z}\right|_{\mathbf{r}=\mathbf{0}} \exp\left(-\beta V(\mathbf{0})\right) \frac{4\pi}{3} r
\end{aligned}
$$

and so the net force is

$$
\begin{aligned}
\bar{F}_z &= N \frac{\frac{1}{\beta}\int_0^R d\,r r^2 \delta(r-R)}{\int d^3\mathbf{r}_1 \exp\left(-\beta V(\mathbf{r}_1)\right)} \left[-\beta \left.\frac{\partial V}{\partial z}\right|_{\mathbf{r}=\mathbf{0}} \exp\left(-\beta V(\mathbf{0})\right) \frac{4\pi}{3} r\right] \\
&= -\frac{4\pi}{3} R^3 N \frac{\exp\left(-\beta V(\mathbf{0})\right)}{\int d^3\mathbf{r}_1 \exp\left(-\beta V(\mathbf{r}_1)\right)} \left[-\left.\frac{\partial V}{\partial z}\right|_{\mathbf{r}=\mathbf{0}}\right]
\end{aligned}
$$

In reality, since this z axis was arbitrary for this result, this is valid in a general direction and for a general point, so we get

$$
\mathbf{F} = -\frac{4\pi}{3} R^3 \rho(\mathbf{r}_0)\left(-\nabla V(\mathbf{r}_0)\right)
$$

In the case of Archimedes, the potential V only depends on z and so does ρ:

$$
\begin{aligned}
F_A &= -\frac{4\pi}{3} R^3 \rho(z)(-mg) \\
&= V_b \rho(z) mg
\end{aligned}
$$

Problem 4: How can a Soap Bubble be used a Barometer?

4.1 Laplace's equation

Consider a soap bubble with radius R filled by an ideal gas of N_1 particles at temperature T_1 and surrounded by another gas of N_2 particles at temperature T_2 in a container of volume V_2. The bubble is held together by surface tension, which has an associated energy $E_{\text{surf}} = \gamma A$ where A is the surface area of the bubble, where $\gamma > 0$. The change in energy due to a variation in the radius dR can be calculated as

$$
dE = \gamma 8\pi R dR
$$

and the force is

$$
F = -\frac{dE}{dR} = -\gamma 8\pi R
$$

The pressure at the surface is therefore

$$
P_{surf} = \frac{F_{surf}}{A} = -\gamma 8\pi R \frac{1}{4\pi R^2} = \frac{-2\gamma}{R}
$$

Equilibrium of forces dictates $P_{out} + P_{surf} = P_{in}$, so we obtain

$$P_{out} - P_{in} = \frac{-2\gamma}{R}$$

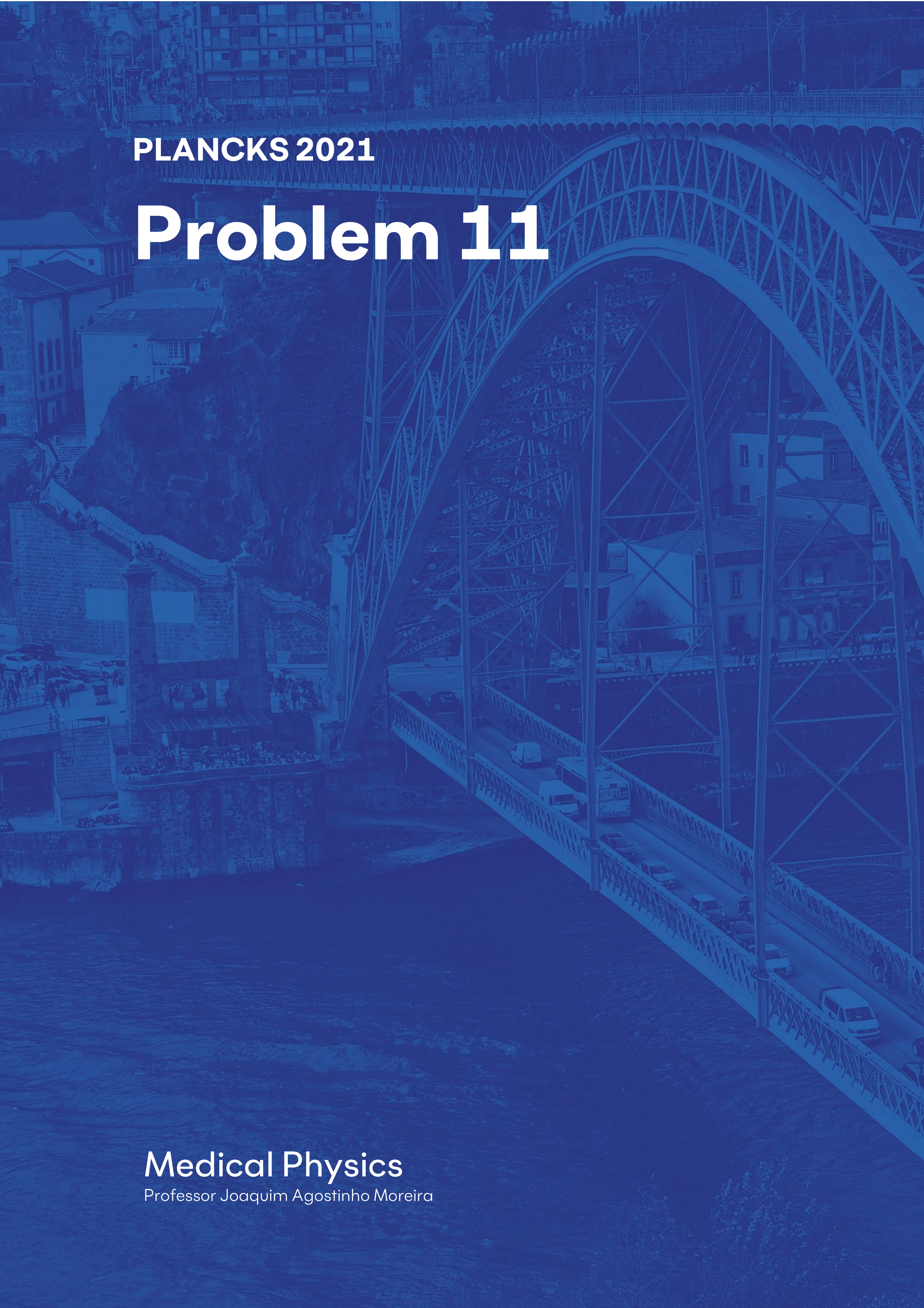
which is Laplace's equation.

4.2 Barometer

With this expression, we can get a formula for the pressure as a function of the bubble's radius. Assuming the gas in the middle is an ideal gas, we obtain

$$\begin{aligned} P_{out} &= P_{in} + \frac{-2\gamma}{R} \\ &= \frac{N_1 k_B T_1}{V_1} + \frac{-2\gamma}{R} \\ &= \frac{N_1 k_B T_1}{\frac{4}{3}\pi R^3} + \frac{-2\gamma}{R} \end{aligned}$$

When $P_{out} > 0$, the function is injective and so by measuring the radius of the bubble, we can deduce the pressure inside the large container. The bubble can then work as a barometer to measure the pressure of the container as a function of position.



PLANCKS 2021

Problem 11

Medical Physics

Professor Joaquim Agostinho Moreira

Introduction

The Magnetic Resonance phenomenon is the basis of a medical tool used to obtain anatomic and functional images of the human body, widely spread around the world.

To do so, powerful superconducting solenoids or Helmholtz coils produce strong magnetic fields that create a macroscopic magnetization arising from the nuclear magnetic momenta alignment. To obtain suitable signals to build the image, the magnetization must be perturbed from its equilibrium direction, which can be done using radiofrequency pulses. When this perturbation is removed, the magnetization relaxes towards the equilibrium value. During this process, the signal for imaging is recorded.

Consider a proton under the influence of a uniform magnetic field $\vec{B}_o = B_o\hat{z}$. The magnetic moment $\vec{\mu}$ of proton is related with the total angular momentum $\vec{J}$ following the equation $\vec{\mu} = \gamma\vec{J}$, where γ is the gyromagnetic ratio. Assume that no dissipative mechanism is considered and consider a fixed reference frame (x, y, z).

Question 1 [30 points]

Write the equation of motion of the magnetic moment under the influence of the magnetic field, assuming that the angle between these two vectors is θ_0, and show that each magnetic momentum precesses around the direction of the external magnetic field $\vec{B}_o$.

Question 2 [5 points]

Determine the precession frequency ω_o of the magnetic moment.

Question 3 [25 points]

A second reference frame (x', y', z') is now considered which rotates with angular velocity $\vec{\Omega} = -\omega_0\hat{z}$, in such a way that $z = z'$. Write the equation of motion of the magnetic moment referring to the rotating reference frame.

Question 4 [30 points]

Assume now that a known radiofrequency magnetic field described by $\vec{B}_1(t) = B_1(t)\cos(\omega t)\hat{x} - B_1(t)\sin(\omega t)\hat{y}$, with $B_1(t)$ a rectangular function, applied during a time interval τ. Show that the magnetic moment precesses around the x' axis with angular frequency $\omega_1 = \gamma B_1$ if the condition $\omega = \omega_o$ holds. This is the case of resonance conditions.

Question 5 [10 points]

Calculate τ to flip the magnetic moment onto the y'-axis.

Solutions

Question 1 [30 points]

The student should recognize that a magnetic moment, $\vec{\mu}$ when inside a magnetic field $\vec{B}$, is acted upon by a torque $\vec{\tau}$, given by

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

Using the relation $\vec{\mu} = \gamma\vec{J}$ and $\vec{\tau} = \frac{d\vec{J}}{dt}$ he should arrive at the conclusion that:

$$\frac{d\vec{J}}{dt} = \vec{\mu} \times \vec{B}$$
$$\frac{d\vec{\mu}}{dt} = \gamma\vec{\mu} \times \vec{B}$$

Note that this should be considered as the equation of motion.

The student should now show that the magnetic moment precesses around $\overrightarrow{B_0}$. One way to show this is to prove that $\frac{d|\vec{\mu}|}{dt} = 0$ and that $d\vec{\mu} \perp \vec{\mu}, \overrightarrow{B_0}$ and $d\vec{\mu} \neq \vec{0}$. The student also has the liberty to use other methods, however he should show that the precession phenomena appears around any $\overrightarrow{B_0}$.

Question 2 [5 points]

The student should arrive at the conclusion that $\omega_0 = \gamma B_0$. One way to do so, is to recognize that $d\mu = \mu \sin\theta_0 d\phi$ where $d\phi$ is the angle swept by the magnetic moment in a time dt. Since $d\mu = \gamma\mu B_0 \sin\theta_0 dt$, we can equate both expressions and show that $\frac{d\phi}{dt} = \gamma B_0 = \omega_0$.

Question 3 [25 points]

The student has to be able to recognize that for a rotating frame with $\vec{\Omega} = -\omega_0\hat{z}$, $\vec{\mu}$ is constant. It's important to note that there are at least two valid cases:

The student recognizes that the magnetic moment precesses in a clockwise fashion around $\overrightarrow{B_0}$through verbal arguments, with angular velocity ω_0, and that the rotating frame with $\vec{\Omega} = -\omega_0\hat{z}$ also results in a clockwise rotating frame with the same angular velocity, resulting in a constant $\vec{\mu}$;

The student is able to mathematically show that:

$$\left(\frac{d}{dt}\right)_{inertial} = \left(\frac{d}{dt}\right)_{rotational} + \vec{\Omega} \times$$

which will allow him to prove that $\left(\frac{d\vec{\mu}}{dt}\right)_{rotational} = \vec{0}$.

Question 4 [30 points]

The student should show that:

$$\left(\frac{d}{dt}\right)_{inertial} = \left(\frac{d}{dt}\right)_{rotational} + \vec{\Omega} \times$$

With this tool he should be able to show that:

$$\left(\frac{d\vec{\mu}}{dt}\right)_r = \gamma\left[\vec{\mu} \times \left(\vec{B} + \frac{\vec{\Omega}}{\gamma}\right)\right]_r$$

He should now recognize that the magnetic field $\overrightarrow{B_1}(t) = B_1(t)\widehat{x'}$, for $\omega = \omega_0$.

Finally, he should be able to show that:

$$\frac{d\vec{\mu}}{dt} = \gamma\vec{\mu} \times B_1\hat{x}'$$

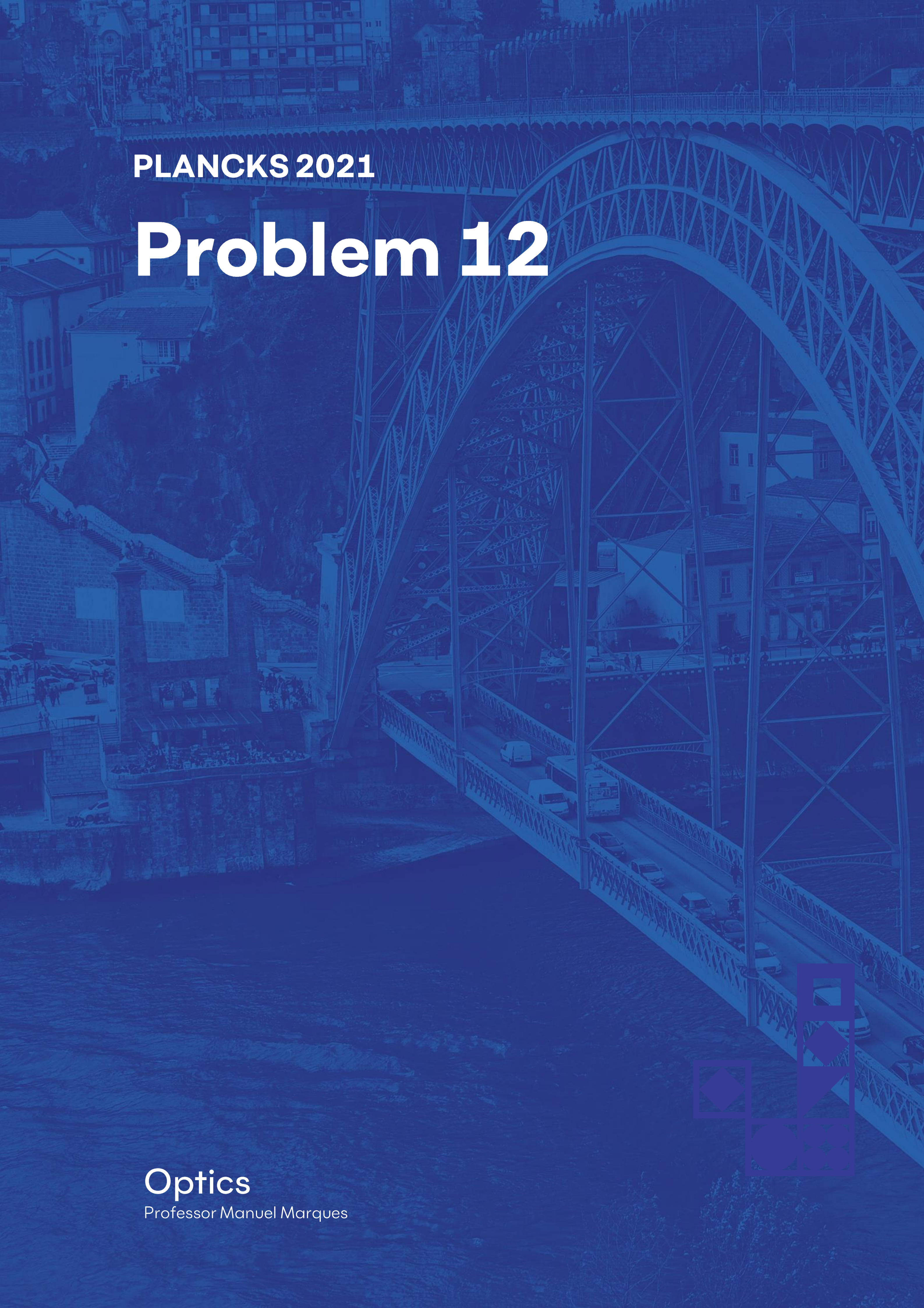
and prove that this is a precession around the $\hat{x}'$ axis with angular velocity $\omega_1 = \gamma B_1$.

Question 5 [10 points]

The student should recognize that the rotation happens in the $y'z$ plane and that the magnetic moment $\vec{\mu}$ has an angle of θ_0 with the $\hat{z}$ axis.

Then, he should acknowledge that the rotation needed is $\theta = \frac{\pi}{2} - \theta_0$. Therefore

$$\frac{\pi}{2} - \theta_0 = \omega_1\tau \rightarrow \tau = \left(\frac{\pi}{2} - \theta_0\right)\frac{1}{B_1\gamma}$$



PLANCKS 2021

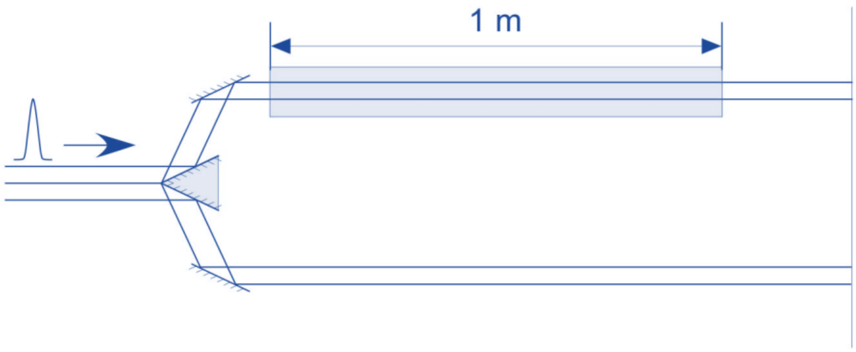
Problem 12

Optics

Professor Manuel Marques

Question 1 [20 points]

A short light pulse ($\lambda_0 = 590nm$) is split in two by a prism mirror. One of the beams travels in the air (refractive index 1.000) and the other travels through one meter of glass (silica) with a refractive index of 1.458.



Compute the time difference of the two pulses arriving at the target.

Note: For simplicity, assume $c = 3.000 \times 10^8$ m/s in all questions of this problem.

Question 2 [20 points]

Let's assume that the two beams in the previous system interfere constructively at the target (mirrors are used to superpose them). What would be the minimum temperature variation of the one-meter silica bar to change the interference from constructive to destructive interference?

The temperature coefficient of the refractive index of silica is 8.7×10^{-6} K^{-1}.

Question 3 [20 points]

A short pulse of light cannot be considered monochromatic. The limitation in the time will truncate the sinusoidal oscillation of the electromagnetic wave. The resulting spectrum depends on the temporal shape of the pulse (the shape that makes the envelope of the ideal electromagnetic sinusoidal wave). For the particular mathematical Gaussian shape envelope, the resulting envelope of the frequencies shape is also a gaussian, and the product of the two Gaussian envelopes is equal to one ($\Delta t \times \Delta f = 1$). This means that a gaussian very short pulse with 10 fs (1 fs = 10^{-15} s) will have a gaussian spectrum with 10^{14} Hz width (corresponding to 83 nm for a pulse centered at 500 nm).

Consider a 300 km fiber optic link with gaussian digital pulses carrying the information. The fiber optic link is operated at the 1550 nm window ($\lambda_0 = 1550$ nm), where the refractive index is 1.468 and has a linear variation with wavelength 5.4×10^{-6} nm^{-1}.

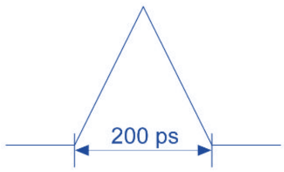
To allow some simplification of the problem, we will assume a gaussian pulse with width τ in a time slot of 2τ (meaning that for a transmission frequency of 1 MHz, τ is equal to half a period of the data frequency, $\tau = 1\mu s / 2$), and that the pulse can spread in time by 25% (τ can increase 1.25 times during the propagation).

What would be the maximum frequency of transmission for this simplified system, due to the different speeds of the pulse spectral components?

Note: with these simplifications, the maximum data frequency computed will be smaller than what an optical fiber can carry. (The refractive index variation with wavelength crosses zero at the second transmission window, around 1310 nm).

Question 4 [20 points]

For simplification, consider now a triangular-shaped pulse centered at 1550 nm wavelength, and with a temporal duration of 200 ps.



The pulse carries an energy of 200 nanojoules and has a diameter of 10 micrometers in the optical fiber (for simplification assume a uniform distribution of power).

One of the effects that limit the performance of fiber optical systems is the nonlinear refractive index. The speed of light inside a material can be changed by the irradiance of light (power per unit area), in an optical fiber, this effect has a value of $3.0 \times 10^{-20} \mathrm{m}^2/\mathrm{W}$. In the leading part of the pulse, as the power rises in time, the refractive index increases, decreasing the speed of light (this is similar to the distance to the entrance of the fiber increasing with time). The effect reverses at the back of the pulse. This variation of the velocity shifts the frequencies of the pulse in an analogous way to the Doppler shift.

Compute the new spectral width of this pulse after 1 m propagation, considering this nonlinear effect.

Note: assume for the initial spectrum the one obtained with a Gaussian pulse with the 200 ps width.

Question 5 [20 points]

The dispersion of the fiber at around 1550 nm is anomalous, i.e., the longer wavelengths travel slower than the shorter ones. Since the nonlinear refractive index shifts the wavelengths in the leading edge of the pulse to longer wavelengths, and the ones in the trailing edge to shorter wavelengths, the shifting of wavelengths can be compensated by the different propagation velocities. This is indeed used for the soliton propagation of pulses (a hyperbolic secant shape is one solution for these pulses).

Compute the pulse energy, for a triangular pulse with 200 ps duration (like the one considered in Question 4) that keeps the same pulse duration after 50 km, using the simplifications below.

Notes: As said, this is only possible for a special shape of the pulse, so we will consider a simplified problem. First, we will consider, like in Question 3, that the original pulse spectrum is equal to the one of a Gaussian pulse of 200 ps. Second, we consider that the nonlinear effect is only present at the first meters of the fiber (no nonlinear behavior on the rest of the propagation to separate the two effects). We will consider only the points at half height (separated by 100 ps). We will consider only the maximum shifted frequencies (the slower at the leading edge, and the faster at the trailing edge) for calculation. The equal pulse duration to be computed shall be simplified to be that the slower wavelength generated at the leading edge is 100 ps slower than the faster wavelength generated at the trailing edge.

Note: At 1550 nm the refractive index is 1.468 and has a linear variation with wavelength of 5.4×10^{-6} nm^{-1}. The nonlinear refractive index is 3.0×10^{-20}m^2/W. Assume a uniform spot with 10 micrometer diameter.

Solutions

Question 1 [20 points] The time needed to the light travel a distance L in a medium with refractive index $n_i$is:

$$t = \frac{L \times n_i}{c}$$

The difference in time between the two pulse will be:

$$\Delta t = \frac{L\times n_g}{c} - \frac{L\times n_{air}}{c} = \frac{L\times(n_g - n_{air})}{c} = 1.527 \text{ ns} = 1.527 \times 10^{-9} \text{ s}$$

Question 2 [20 points] If the two beams have initially a constructive interference, that means that the optical path length difference is an integer multiple of the wavelength (or that the phase difference is an integer multiple of 2π)

$$o.p.d. = L \times n_g - L \times n_{air} = m\lambda_0 \quad \text{or} \quad L \times k_g - L \times k_{air} = p2\pi \quad \text{where } k_i = \frac{2\pi}{\lambda} = \frac{2\pi n_i}{\lambda_0}$$

The next destructive interference occurs when the optical path difference increases by half a wavelength (or the phase difference changes by π). So:

$$\Delta(o.p.d.) = \left(L \times \left(n_g + \frac{\delta n_g}{\delta T}\Delta T\right) - L \times n_{air}\right) - \left(L \times n_g - L \times n_{air}\right) = L \times \frac{\delta n_g}{\delta T}\Delta T = \frac{\lambda_0}{2}$$

$$\Delta T = \frac{\lambda_0}{2L\times\frac{\delta n_g}{\delta T}} = \frac{590\times10^{-9}}{2\times1\times8.7\times10^{-6}} = 0.03391 \text{ K}$$

Or, using phase,

$$\Delta\Phi = \left(L \times \frac{2\pi\left(n_g + \frac{\delta n_g}{\delta T}\Delta T\right)}{\lambda_0} - L \times \frac{2\pi n_{air}}{\lambda_0}\right) - \left(L \times \frac{2\pi n_g}{\lambda_0} - L \times \frac{2\pi n_{air}}{\lambda_0}\right) = L \times \frac{2\pi\frac{\delta n_g}{\delta T}\Delta T}{\lambda_0} = \pi$$

$$\Delta T = \frac{\pi\times\lambda_0}{2\pi\times L\times\frac{\delta n_g}{\delta T}} = \frac{590\times10^{-9}}{2\times1\times8.7\times10^{-6}} = 0.03391 \text{ K}$$

Question 3 [20 points] Assuming a transmission frequency F with the pulse occupying half-period, the gaussian pulse width (τ) will be $\tau = \frac{1}{2F}$. A Gaussian pulse of width τ will wave a bandwidth of $\Delta f = \frac{1}{\tau} = 2F$, centered at a frequency $f = \frac{c}{\lambda_0} = 1.935 \times 10^{14}$ Hz.

We now have to choose using the bandwidth in frequency, and convert the variation of the refractive index with wavelength to the variation with frequency, or convert the bandwidth to wavelength and use the variation of the refractive index with wavelength.

$$\frac{dn}{df} = -\frac{dn}{d\lambda}\frac{c}{f^2} \quad \text{or} \quad \Delta\lambda = -\Delta f\frac{c}{f^2} = -\Delta f\frac{\lambda_0^2}{c}$$

The faster wavelengths at the leading edge of the pulse will be the first to arrive at the rear end of the fiber. The slower wavelength at the trailing edge of the pulse will be the last to arrive. The new pulse temporal width will be the initial pulse width increased by the time difference on arrival between the faster and slower wavelengths.

The time to travel is equal to the length of the fiber multiplied by the refractive index at that wavelength and divided by the speed of light in vacuum $t_{travel} = \frac{Ln}{c}$. The time difference, at arrival, between the faster and slower wavelengths will be:

$$\Delta t_{travel} = \frac{Ln_{max}}{c} - \frac{Ln_{min}}{c} = \frac{L}{c}(n_{max} - n_{min}) = \frac{L}{c}\left(\frac{dn}{d\lambda} \times \Delta\lambda\right) = \frac{L}{c}\left(\frac{dn}{df} \times \Delta f\right)$$

In order to limit the pulse width spread to be smaller than 25% the initial pulse width, we need to have:

$$\Delta t_{travel} < 0.25\,\tau = \frac{0.25}{2F}$$

or

$$\Delta t_{travel} = \frac{L}{c}\left(\frac{dn}{df} \times \Delta f\right) = \frac{L}{c}\left(\frac{dn}{d\lambda}\frac{c}{f^2} \times 2F\right) < \frac{0.25}{2F}$$

Rearranging:

$$F^2 < \frac{0.25}{4\frac{dn}{d\lambda}} \times \frac{f^2}{L} = \frac{0.25}{4 \times 5.4 \times 10^3} \times \frac{(1.935 \times 10^{14})^2}{3 \times 10^5} = 1.445 \times 10^{18}\ \text{Hz}^2$$

So, the maximum data frequency would be the square root of this value $= 1.700 \times 10^9$ Hz = 1.7 GHz

Question 4 [20 points] Auxiliary data:

- Beam area: $A = \pi\frac{D^2}{4} = 7.854 \times 10^{-11}$ m^2
- Peak power: $P_p = \frac{2E}{\tau} = 2.000 \times 10^3$ W
- Peak Irradiance: $I_p = \frac{P_p}{A} = 2.546 \times 10^{13}$ W/ m^2
- Peak refractive index change: $\Delta n_p = n_2 I_p = 7.3639 \times 10^{-7}$

Time variation of the refractive index change on the pulse: $\frac{d\Delta n}{dt} = \frac{\Delta n_p}{10^{-10}} = 7.639 \times 10^3$ s^{-1}

Assuming a 200 ps gaussian pulse, and using $\Delta t \times \Delta f = 1$, we get $\Delta f = 5 \times 10^9$ Hz for the spectral width without nonlinear effects. This is centered at $f = \frac{c}{\lambda} = 1.935 \times 10^{14}$ Hz.

We can compute the optical path length (equivalent to the distance traveled in vacuum in the same time) as the physical distance multiplied by the refractive index ($opl = L \times n$). The rate of change of the optical path length is equivalent to a source speed.

This equivalent speed of the source moving away in the leading edge of the pulse is: $v = \frac{d\Delta n}{dt} \times L = 7.639 \times 10^3$ m s^{-1}

This 'velocity' of the source, moving away, shifts the spectrum to lower frequencies by an amount given by the Doppler formula (as the 'speed' is small, can be used either the relativistic, or the non-relativistic, Doppler equation). The Doppler shift will be:

$$\Delta f = -\left(1 - \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\right) \times f = -4.929 \times 10^{9} \text{ Hz}$$

($\Delta\lambda = 0.03947$ nm)

In the trailing edge of the pulse, the Doppler shift will be symmetrical (frequency increases).

The total width of the spectrum will be the original width (5×10^{9}), plus two times the calculated Doppler shift. This gives a total spectrum width of 1.486×10^{10} Hz. (0.119 nm)

Question 5 [20 points] Auxiliary data, from previous resolutions:

Assuming a 200 ps gaussian pulse, and using $\Delta t \times \Delta f = 1$, we get $\Delta f = 5 \times 10^{9}$ Hz for the spectral width without nonlinear effects. This is centered at $f = \frac{c}{\lambda} = 1.935 \times 10^{14}$ Hz.

Beam area: $A = \pi \frac{D^2}{4} = 7.854 \times 10^{-11}$ m^2

Refractive index variation with frequency $\frac{dn}{df} = -\frac{dn}{d\lambda}\frac{c}{f^2}$

Peak Irradiance: $I_p = \frac{2E}{\tau A}$

Peak refractive index change: $\Delta n_p = n_2 I_p = \frac{2\, n_2\, E}{\tau A}$

Time variation of the refractive index change on the pulse:

$$\frac{d\Delta n}{dt} = \frac{\Delta n_p}{\frac{\tau}{2}} = \frac{4\, n_2\, E}{A \times \tau^2}$$

The optical path length is ($opl = L \times n$). The rate of change of the optical path length is equivalent to a source speed. This equivalent speed of the source moving away in the leading edge of the pulse is:

$$v = \frac{d\Delta n}{dt} \times L = \frac{4\, n_2\, E}{A \times \tau^2} \times L$$

This 'velocity' of the source, moving away, shifts the spectrum to lower frequencies by an amount given by the Doppler formula (as the 'speed' is small, can be used either the relativistic, or the non-relativistic, Doppler equation). The Doppler shift will be:

$$\Delta f = -\left(1 - \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\right) \times f$$

As v is much smaller than c, we can use the non-relativistic Doppler shift:

$$\Delta f \cong -\left(\frac{c}{c+v}\right) \times f = -\frac{f}{1+\frac{v}{c}} = -\frac{f \times c}{c + \frac{4\, n_2\, E}{A \times \tau^2} \times L}$$

In the trailing edge of the pulse, the Doppler shift will be symmetrical (frequency increases).

The faster frequency in the pulse is separated to the slower frequency by an amount equal to pulse bandwidth added with two times the frequency shift in the leading edge (symmetrical to the one generated at the trailing edge) The pulse bandwidth after the nonlinear effect will be:

$$\Delta f' = 5 \times 10^9 + \frac{2 \times f \times c}{c + \frac{4\, n_2\, E}{A \times \tau^2} \times L}$$

The time difference between the extreme frequencies after propagating 50 km is:

$$\Delta t_{travel} = \frac{L}{c}\left(\frac{dn}{df} \times \Delta f'\right) = \frac{L}{c} \times \frac{dn}{df}\left(5 \times 10^9 + \frac{2 \times f \times c}{c + \frac{4\, n_2\, E}{A \times \tau^2} \times L}\right)$$

This must be equal to 100 ps. So:

$$\frac{L}{c} \times \frac{dn}{df}\left(5 \times 10^9 + \frac{2 \times f \times c}{c + \frac{4\, n_2\, E}{A \times \tau^2} \times L}\right) = 10^{-10}$$

We can rearrange, replace all variables by their numerical value and obtain E:

$$\frac{2 \times f \times c}{c + \frac{4\, n_2\, E}{A \times \tau^2} \times L} = \frac{c \times 10^{-10}}{L \times \frac{dn}{df}} - 5 \times 10^9$$

$$E = \frac{\dfrac{2 \times f \times c}{\frac{c \times 10^{-10}}{L \times \frac{dn}{df}} - 5 \times 10^9} - c}{\frac{4\, n_2}{A \times \tau^2} \times L}$$

The pulse energy (E)needed to keep the same width after 50 km, is 6.852 mJ.