PLANCKS 2021

Preliminares de PLANCKS 2021
Fase Española

The Hubble Space Telescope

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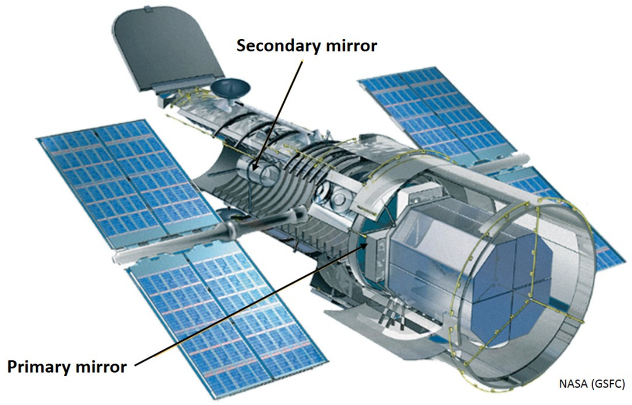
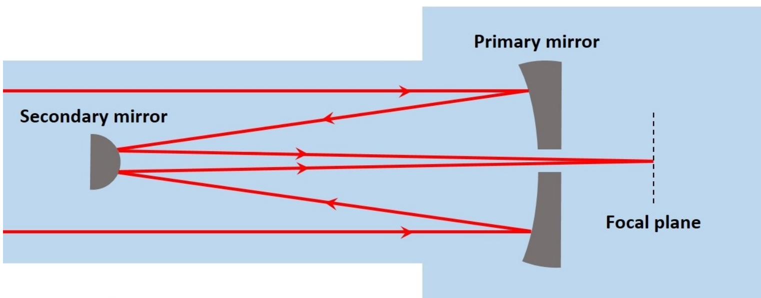
The Hubble Space Telescope is one of the most important and well known scientific instruments in history. Since its launch in 1990, it has detected hundreds of thousands of celestial objects, provided unprecedented astronomical images of high quality and beauty, and played a key role for the better understanding of the universe. The telescope was designed to be visited in space by astronauts on servicing missions to perform repairs, update the technology and install new instruments. Shortly after it was deployed, two flaws were detected that limited its performance, although both were corrected afterwards. We will deal with those issues here.



The Hubble weighs 11500 kg and operates in a low Earth circular orbit, with inclination of 28.5° to the equator, at altitude $h = 547$ km.

a) Calculate the Hubble's orbital period (in minutes), and calculate the time it spends in Earth's shadow during an orbit on the equinox days (i.e., the days of the year when the sunlight is parallel to the orbital plane). Data: Earth radius $R_\oplus = 6371$ km, ecliptic angle 23.5°. (1 point)



The Hubble's optical system is a two-mirror Cassegrain reflecting telescope of Ritchey-Chrétien design. The primary mirror is a $\Phi_1 = 2.4$ m diameter concave hyperboloid with a $R_1 = 11040$ mm curvature radius and a 0.6 m central opening. The secondary mirror is a $\Phi_2 = 0.3$ m diameter convex hyperboloid with a $R_2 = 1358$ mm curvature radius. The separation between mirrors is $d = 4906$ mm.

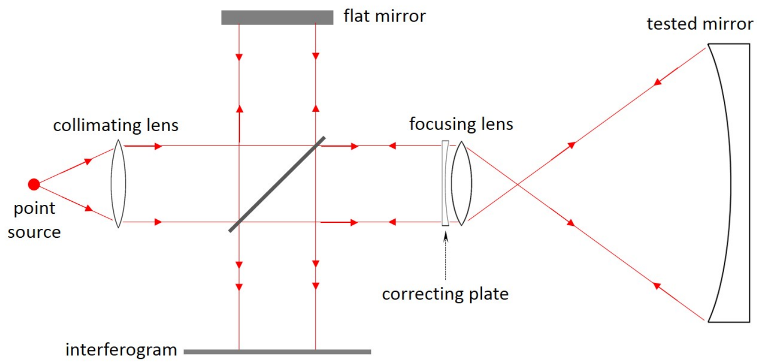
b) Calculate the focal length of the two mirrors, the effective focal length of the system and the f-number. Obtain the angular resolution of the telescope diffraction-limited at $\lambda = 632.8$ nm. (1 point)

Conic constants of the mirrors were calculated so that the optical aberrations compensated each other. However, due to an error in polishing the shape of the primary mirror was not correct, resulting a conic constant $Q^* = -1.0139$ instead of $Q = -1.0023$ as planned. The equation for a conic surface is $(1 + Q)z^2 - 2Rz + r^2 = 0$, where z is the surface elevation from a plane tangential to the vertex, r is the polar coordinate from the axis passing through the vertex, R is the radius of curvature, and $Q = -e^2$ is the conic constant, being e the eccentricity.

c) Give an expression for $z(r)$ up to 4th order (i.e., neglecting terms higher than r^4). Make, on the same graph, an approximate plot of the real and the planned elevations of the primary mirror. Calculate the error $\Delta z_{max}(r_H)$ (in microns) at the edge of the mirror between the elevation of the real and the planned surface. (1 point)

Paraxial rays reflected from a conical mirror converge at the focus, i.e., at the paraxial focal length f_p. However, the marginal rays intersect the axis at points located at a distance $f(r) = f_p + A_{\text{sa}} \cdot r^2$ from the mirror's vertex, where the term A_{sa} is the longitudinal spherical aberration.

d) Obtain the spherical aberration coefficient A_{sa}. What is the geometry of the aberration-free conical mirror? Calculate the value of the longitudinal spherical aberration for a ray reflected at the edge of the Hubble's primary mirror. (1.5 points)



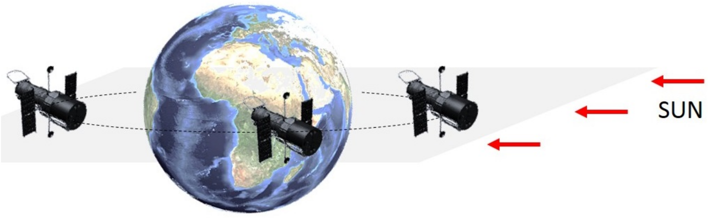
The optical quality of the primary mirror is tested by a Twyman–Green interferometer. A reference plane wavefront is interfered with the wavefront reflected from the Hubble's mirror, whose curvature center is located at the focus of a lens. In front of the lens is a correcting plate designed to compensate the spherical aberration of the planned mirror after the double pass of the light, so if the mirror's shape were correct the returning wavefront towards the interference screen would be plane.

e) Make a freehand drawing of the fringes of the resulting two-dimensional interferogram due to the wrong real shape of the Hubble's mirror. Shade the dark areas of the pattern. At least you must accurately locate the relative position of the first minimum. (1 point)

The power for Hubble comes from two rectangular solar arrays 12.2 x 2.5 m^2 in size. The efficiency of the solar cells to convert light energy into electric is $\eta = 12\%$. The temperature of the Sun is $T_\odot = 5780$ K, its diameter is $\Phi_\odot = 1391016$ km and the distance to Earth is $d = 150 \cdot 10^6$ km. Hubble consumes $P = 2.1$ kW. The night power is supplied by 6 batteries of 32 V and 88 A·h each, connected in parallel. Data: Stefan-Boltzmann constant $\sigma = 5.67 \cdot 10^{-8}$ W/(m^2·K^4).

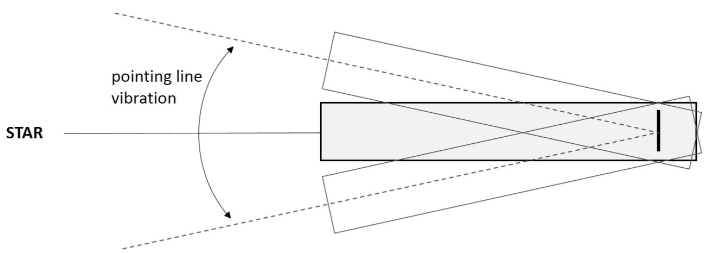
f) Calculate the value of the solar irradiance on the Hubble and, then, obtain the electric power produced by the solar panels when the solar incidence angle is 60°. What is the excess energy (in kJ) produced by the panels during one orbit? (1 point)

g) If the solar panels broke, how many orbits could Hubble travel with the energy stored in its batteries? (0.5 points)



The panels are subject to strong thermal variations due to the interruption of the solar flux when the satellite is in Earth's shadow. We will use the following simplified model to estimate this thermal effect. During sunlight, the panels face perpendicular to the sun. During the eclipse, the panels face the center of the Earth and they receive the blackbody radiation of the planet. Both emission and absorption occurs only from the side of the panels exposed to radiation (the other side is insulated).

h) Determine the equilibrium temperature reached by the panels in the sunlight and in the eclipse. Make an approximate plot of the temperature profile for the heating and cooling cycles. Data: Earth's temperature $T_\oplus = 252$ K, panels' absorptance $\alpha = 0.67$, panels' emissivity $\varepsilon = 0.85$. (1.5 points)



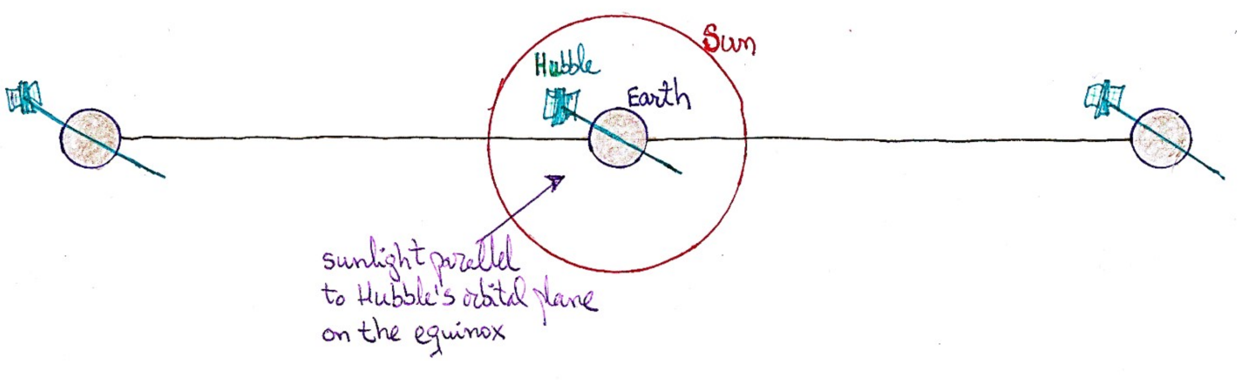
The thermal expansions and contractions of some structural elements of the solar panels cause mechanical disturbances in the telescope that excite vibrations of the pointing system (designed to hold an image stable at the focal plane) at a main frequency $\nu = 0.1$ Hz and amplitude $A = 0.1$ arcsec.

i) Calculate the RMS of the vibration, averaged over an oscillation period. Since this vibration results in the loss of pointing lock on the targeted star, what is the resolution of the telescope due to this problem? (0.75 points)

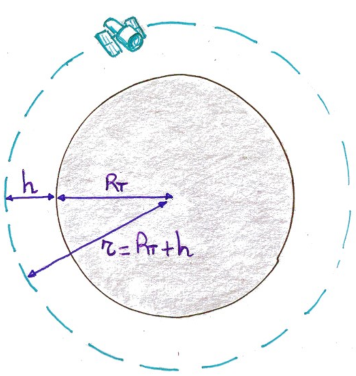
j) A possible solution could be the addition, through the Hubble positioning control system, of an external oscillation of the same frequency as the thermal disturbance and phase shift of π rad relative to it. Determine the amplitude of the external oscillation to reduce by a factor of 10 the amplitude of the resulting vibration. (0.75 points)

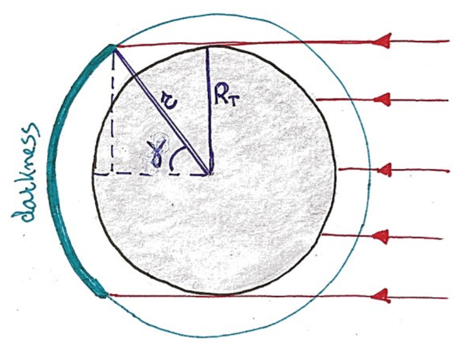
Solution
a) **Orbit data**
Orbital period.
$R_\oplus = 6371$ km and $h = 547$ km give $r = R_\oplus + h = 6918$ km.

From Kepler's 3^{rd} Law $T^2 = \frac{4\pi^2}{GM_\oplus} r^3 \Rightarrow T = 5732.3\,\text{s} = 95.5$ min.



Time eclipsed by Earth: The eclipse duration changes seasonally. A full calculation would require to take orbit's inclination, 25.5°, into account. However, on the equinox the sunlight is parallel to the Hubble's orbital plane.





The shadow cast by the Earth on the Hubble spacecraft is determined by the angle γ

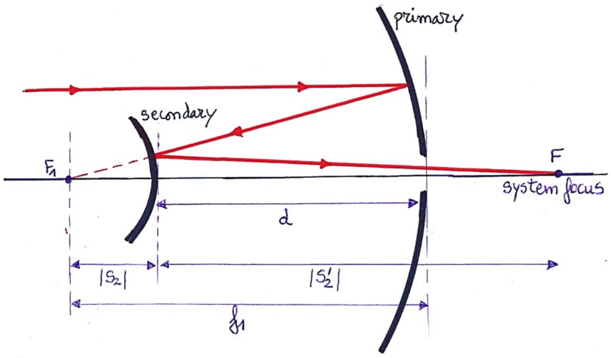
$$\sin\gamma = \frac{R_\oplus}{R_\oplus + h} \Rightarrow \gamma \simeq 67^\circ \simeq 1.17\,\text{rad}.$$

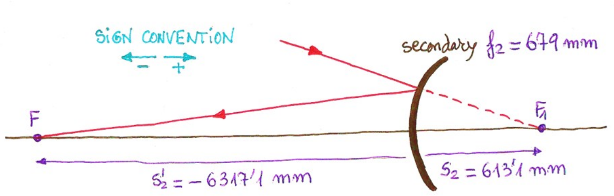
A simple rule of three gives the time t it spents in the Earth shadow

$$\frac{t}{2\gamma} = \frac{T}{2\pi} \Rightarrow t = 35.5\,\text{min},$$

with the remaining 60 minutes illuminated by the Sun.

b) **Focal distances**
For the primary mirror with $R_1 = 11040\,\text{mm},\ f_1 = R_1/2 = 5520\,\text{mm}$.
For the secondary mirror $R_2 = 1358\,\text{mm},\ f_2 = 679$ mm.
These are absolute values, they have opposite signs since one mirror is concave while the other is convex.





Given the separation $d = 4906$ mm between mirrors, we have $|s_2| = f_1 - d = 614$ mm, hence

$$\frac{1}{s_2'} + \frac{1}{s_2} = \frac{1}{f_2} \Rightarrow s_2' = -6414\,\text{mm}$$

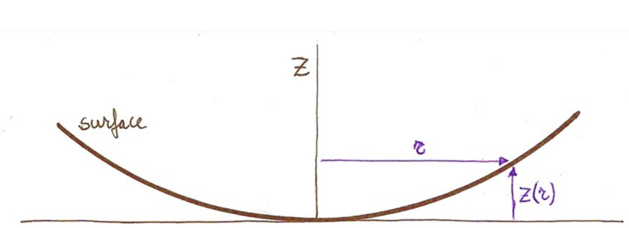
giving a second mirror lateral magnification $\beta_2 = -s_2'/s_2 = 10.44$ and an effective focal lenght $f = f_1\beta_2 \approx 57629$ mm.
This could also be directly computed using the systems theory formula:

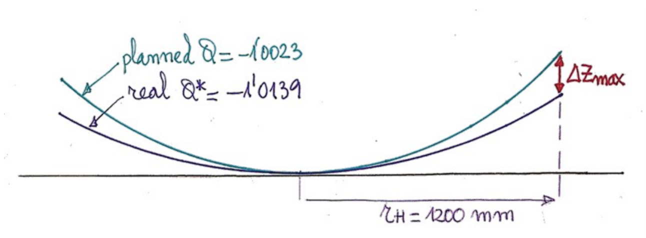
$$f = -\frac{f_1 f_2}{d - f_1 + f_2}$$

Given the diameter $\Phi_1 = 2400$ mm, the f-number becomes $N = f/\Phi \simeq 24$. The angular resolution is obtained applying the Rayleigh criterion, $\theta = 1.22(\lambda/\phi)$, gives $\theta = 3.2\ 10^{-7}$ rad $= 0.066$ arcsec for the wavelenght $\lambda = 632.8$ nm.

c) **Surface elevation** $z(r)$
From the text, $(1+Q)z^2 - 2Rz + r^2 = 0 \Rightarrow \frac{z(r)}{R} = \frac{1}{(1+Q)}\left[1 - \sqrt{1-(1+Q)\frac{r^2}{R^2}}\right]$ A Taylor series expansion up to fourth order in $(r/R) << 1$ gives

$$\frac{z(r)}{R} = \frac{r^2}{2R^2} + \frac{(1+Q)r^4}{8R^4}.$$

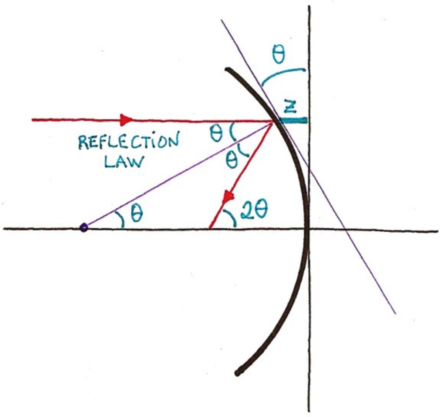


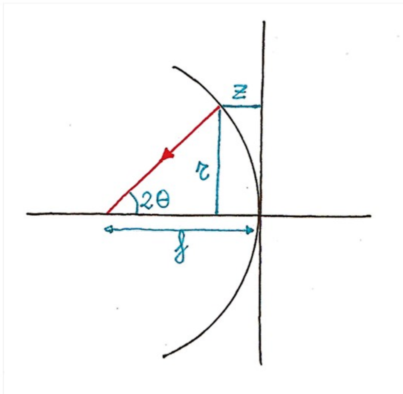


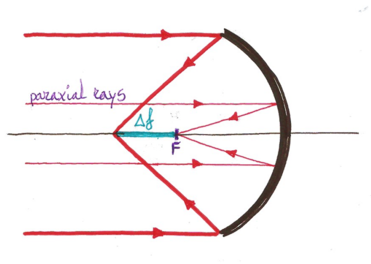
Hence, $\Delta z_{max} = z(r_H) - z^*(r_H) = \frac{Q-Q^*}{8R^3} r_H^4 = 2.2\mu m$, where we used that for the primary mirror surface $R = R_1 = 11040\,\text{mm}$, and $r_H = \phi_H/2 = 1200\,\text{mm}$.

d) **Spherical aberration coefficient**
We combine the reflection law with conic surface geometry to get

$$\tan\theta = \frac{dz}{dr} = z', \quad \frac{z(r)}{R} = \frac{r^2}{2R^2} + \frac{(1+Q)r^4}{8R^4} \to z'(r) = \frac{r}{R} + \frac{(1+Q)r^3}{2R^3}.$$







From the figure we note that $\tan 2\theta = \frac{r}{f-r}$, but also $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$, so that

$$f = z + \frac{r}{2}\left(\frac{1}{z'} - z'\right) \simeq \frac{R}{2}\frac{1}{1+\frac{(1+Q)r^2}{2R^2}}.$$

Obviously this is valid only up to order $(r/R)^4$. A Taylor series expansion gives

$$f \simeq \frac{R}{2} - \frac{(1+Q)r^2}{4R},$$

so to the paraxial focal distance $f = R/2$ we must add a spherical aberration term $A_{\text{sa}} \cdot r^2$ where $A_{\text{sa}} = -\frac{(1+Q)}{4R}$ is the longitudinal spherical aberration coefficient.
With $Q = -1.0139$, $R = 11040$ mm, and at the edge $r = r_H = 1200$ mm we get

$$\Delta f = -\frac{(1+Q)}{4R} r_H^2 = 453\,\mu\text{m}.$$

The aberration term should vanish for an aberration free conical mirror, i.e., $1+Q = 0 \Rightarrow z = \frac{r^2}{2R}$, a parabola.

e) **Interferogram**
No interference fringes would be produced if the shape of the tested primary mirror were as planned. However, the correcting plate does not compensate the additional spherical aberration that appears because of the sag difference between the real and the planned surface: $\Delta z = \frac{\Delta Q}{8R^3} r^4$, where $\Delta Q = Q - Q^* = 0.0116$.

Due to the reflection, the light travels twice this distance. Thus, the wavefront error with respect to the plane wavefront is

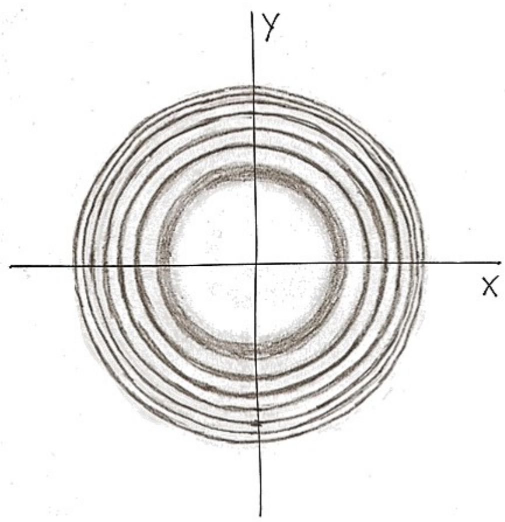
$$\Delta W(r) = 2\Delta z(r) = \frac{\Delta Q}{4R^3} r^4.$$

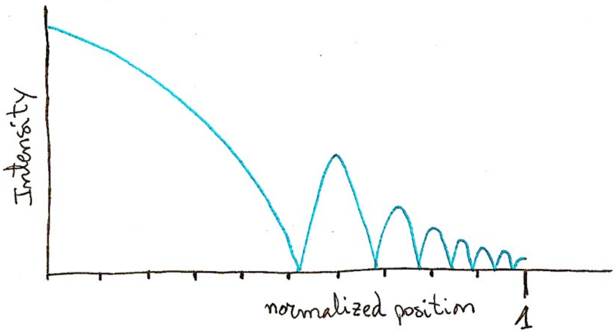
The minima (destructive interference) will occur for radial positions that satisfy $\Delta W(r) = (2m-1)\frac{\lambda}{2}$, with $m = 1, 2, 3\ldots$

Then, $\frac{\Delta Q}{4R^3}r^4 = (2m-1)\frac{\lambda}{2} \ \rightarrow \ r_m = (2m-1)^{1/4}$, where $r_1 = \left(\frac{2R^3\lambda}{\Delta Q}\right)^{1/4} = 619$ mm.

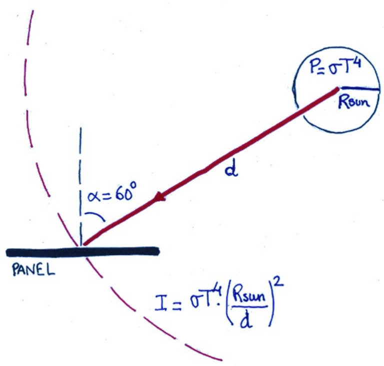
For the other minima: $r_2 = 815$ mm, $r_3 = 926$ mm ,$r_4 = 1007$ mm, etc.

These values are relative to the dimension of the mirror that has circular section of radius $r_H = 1200$ mm. Transforming to normalized coordinates at the interferogram plane, we get the minima at positions: 0.52, 0.68, 0.77, 0.84, 0.89, 0.94, and 0.98 relative to the radius of the interferogram.





f) **Electric power**



We will use the Stefan-Boltzmann Law $P_\odot = \sigma T_\odot^4$, the Luminosity of the Sun $L_\odot = 4\pi R_\odot^2 P_\odot$, and the Solar constant $I = L_\odot/(4\pi d^2)$, where d is the Sun-Earth distance. Putting the values given in the text, we get $I = 1360$ W m^{-2}. Considering the incidence angle $\theta = 60^\circ$ of the solar radiation, an electric efficiency $\eta = 12\%$, and the panels' area $A = 61$ m^2, we get
$P = \eta I A \cos\theta = 5$ kW

The Hubble is illuminated during a time $t = 60$ min per orbit. The energy produced during this time is $E = Pt = 18000$ kJ. On the other hand, the orbital period is $T = 95.5$ min, therefore to keep Hubble running with the power consumption $P = 2.1$ kW needs an energy $E = PT = 12000$ kJ per orbit. There is an energy excess of 6000 kJ.

g) **Broken solar panels**
Under these circumstances all the energy comes from the 6 batteries that can provide $Q = It = 6 \cdot 88$ A·h of charge. At a potential of 32 volts, the total energy E_Q available to the satellite will be

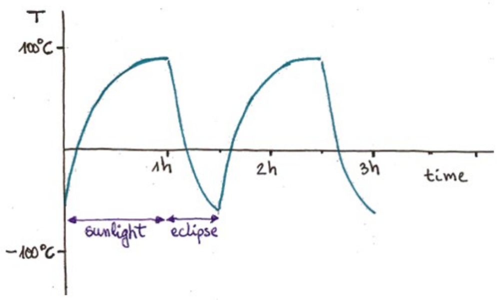
$E_Q = VQ = 60825.6$ kJ. As the satellite consumption is $P = 2.1$ kW, the possible operation time will be $t = E_Q/P = 8$ h allowing 5 orbits.

h) **Temperature of the panels**
At thermal equilibrium $\dot{Q} = C\frac{dT}{dt} = 0$, in general Q will increase due to absorption of that part of the incoming radiation from the Earth and the Sun that is not used to produce electricity. Besides, Q will also be lost due to emission. The overall balance is
$\dot{Q} = (\alpha - \eta)IA - \varepsilon A\sigma T^4$
where $\alpha = 0.67$ is the absorptance, $\eta = 0.12$ the electric efficiency, $\varepsilon = 0.85$ the emissivity of the panels of area A, and I the intensity of the radiation incident on the panels. In equilibrium $\frac{dT}{dt} = 0 \Rightarrow (\alpha - \eta)I = \varepsilon\sigma T^4$.

Sunlight equilibrium temperature:
$I = I_\odot \rightarrow T_S = \left[\frac{(\alpha-\eta)I_\odot}{\varepsilon\sigma}\right]^{1/4}$
Darkness equilibrium temperature:
$I = I_\oplus \rightarrow T_D = \left[\frac{(\alpha-\eta)I_\oplus}{\varepsilon\sigma}\right]^{1/4}$
We got the solar energy flux $I_\odot = 1360\,\text{Wm}^{-2}$ in question f). The Earth energy flux at height h above the surface is given by

$$I_\oplus = \sigma T_\oplus^4 \left(\frac{R_\oplus}{R_\oplus + h}\right)^2 = 194\,\text{Wm}^{-2},$$

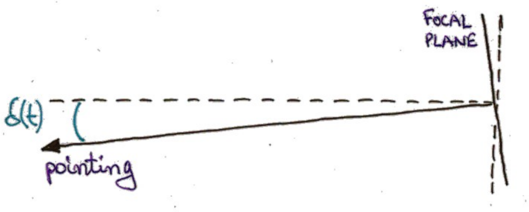
finally $T_S = 353$ K = 80 °C and $T_D = 217$ K= - 56 °C.

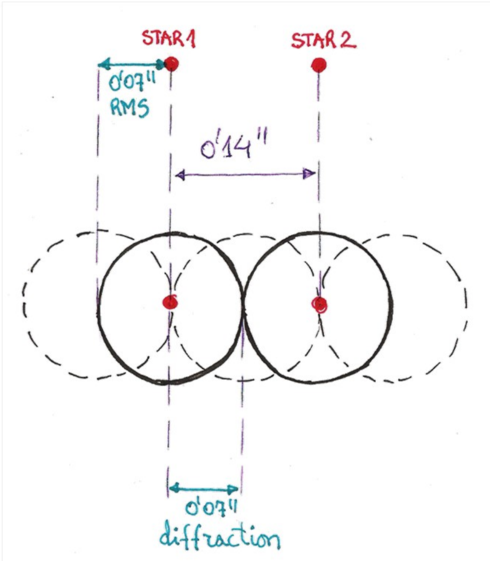


Temperature profile for the heating and cooling cycles

i) **Vibrations**
The angular amplitude $A = 0.1$ arcsec at frequency $\nu = 0.1$ Hz produce angular displacements $\delta(t) = A\cos 2\pi\nu t$. The corresponding root mean square RMS can be given as

$$\text{RMS}^2 = \overline{\delta^2} - \left(\bar{\delta}\right)^2 = \frac{1}{T}\int_0^T dt A^2 \cos^2(2\pi\nu t) = \frac{A^2}{2} \Rightarrow \text{RMS} = \frac{A}{\sqrt{2}} = 0.07\,\text{arcsec}.$$





These vibrations cause a loss of resolution. To the angular resolution limit of the Rayleigh criterion computed in b) above $\theta = 0.066 \simeq 0.07$ arcsec, we must add the RMS inaccuracy due to the vibrations. The new resolution will be 0.07+0.07 = 0.14 arcsec.

j) **Reducing vibrations**
The proposed solution consists of adding an external vibration $\delta_{ext}(t) = B\cos(2\pi\nu t + \varphi)$ of the same frequency, amplitude B, and with a relative phase shift φ, to the thermal and mecanical vibrations $\delta(t)$ of the telescope. The aim is to get a reduced amplitude; let's see how it works:

$$\delta_{reduced}(t) = \delta(t) + \delta_{ext}(t) = A\cos 2\pi\nu t + B\cos(2\pi\nu t + \varphi) = C\cos(2\pi\nu t + \phi)$$

This gives $C = \sqrt{A^2 + B^2 + 2AB\cos\varphi}$, hence for $\varphi = \pi \Rightarrow C = A - B$. To reduce the amplitude of the resulting vibration by a factor of 10, $C = A/10$, we need $B = 9\,A/10 = 0.09$ arcsec.