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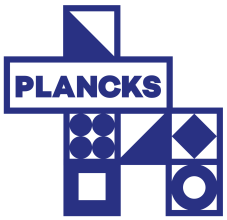

PLANCKS 2021

Preliminares de PLANCKS 2021
Fase Española

Light-matter Interactions

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Light-matter Interactions

I. Classical electromagnetic fields.

In this problem we will analyze the interaction of an atom with the electromagnetic field. We talk of a two level atom when only two atomic levels, say $|i\rangle$ and $|f\rangle$, are relevant for the physical process under study. This is the case when the field frequency ω is comparable to the transition frequency ω_{fi} between these two atomic levels.

Specially important is the case where the electromagnetic field wavelength is long compared with the atomic dimensions, so that the variation of the field over the atom can be neglected. Then, we can take $\mathbf{E} = \mathbf{E_0}\sin(\omega t)$. This is the long wavelength approximation, where the atom-field interaction Hamiltonian is[1] $H_I = -\mathbf{d}\cdot\mathbf{E}$.

The transition probability $|i\rangle \to |f\rangle$ can be given as a function of time as $P_{i\to f}(t) = |C_f(t)|^2$, where $C_f(t)$ is the transition amplitude between both states.

In what follows we will work to lowest order in perturbation theory where

$$C_f(t) = \frac{1}{2\hbar}(\mathbf{d}\cdot\mathbf{E_0})_{fi}\left\{\frac{(e^{i(\omega+\omega_{fi})t}-1)}{(\omega+\omega_{fi})} - \frac{(e^{-i(\omega-\omega_{fi})t}-1)}{(\omega-\omega_{fi})}\right\}, \tag{1}$$

and $(\mathbf{d}\cdot\mathbf{E_0})_{fi} = \langle f|\mathbf{d}\cdot\mathbf{E_0}|i\rangle$ is the transition matrix element.

1. When the field is near resonant with the atomic transition, i.e., $\omega \sim \omega_{fi}$, so that $\Delta = \omega - \omega_{fi} << \omega_{fi}$, one of the two terms in Eq. (1) can be neglected. Apply this simplification hereinafter keeping only the term which survives.
2. Obtain $P_{i\to f}(t)$ for the case of exact resonance where $\Delta = 0$. (1 point).
3. Obtain the off-resonance, $\Delta \neq 0$, transition probability $P_{i\to f}(t) = |C_f(t)|^2$ in terms of a function involving both $\Delta t/2$ and Δ. (3 points).
4. Obtain the maximum of the expression you have achieved in the previous question, $(P_{i\to f})_{max}$, for $\Delta \neq 0$. This should not depend on time. (1 point).

II. Quantum electromagnetic fields.

The interaction Hamiltonian of a two-level atom with a quantum electromagnetic field mode is given, under the long-wavelength approximation, by

$$H^{(I)} = -\mathbf{d}\cdot\mathbf{E} = -i\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{1/2}(\mathbf{d}\cdot\mathbf{e})(a-a^\dagger), \tag{2}$$

where ϵ_0 is the electric permittivity, V is the quantization volume of the electromagnetic field, $\mathbf{e}$ is the electromagnetic field polarization vector, while a, $a^\dagger$ are the annihilation and creation operators of the electromagnetic field mode.

[1] $\mathbf{d}$ is the electric dipole moment of the atom, that does not depend explicitly of time

5 Assuming the initial combined state of atom and field is $|i\rangle = |A\rangle|n\rangle$, where $|A\rangle$ is the internal initial state of the atom and $|n\rangle$ is the Fock state with n excitations, demonstrate that to lowest order in perturbation theory the system can only make transitions, via Eq. (2), to states $|f_1\rangle = |B\rangle|n-1\rangle$ and $|f_2\rangle = |C\rangle|n+1\rangle$, where $|B\rangle,\ |C\rangle$ are other atomic internal states. (1 point).

6 Compute the energies of states $|i\rangle$, $|f_1\rangle$, and $|f_2\rangle$, in absence of interaction, knowing that the energy of state $|A\rangle$ is E_A, the energy of state $|B\rangle$ is E_B, the energy of state $|C\rangle$ is E_C, and the frequency of the electromagnetic mode is ω. (1 point).

The light-matter interaction Hamiltonian with a quantum electromagnetic field mode, under the rotating-wave approximation, takes the form under a resonant driving,

$$H_{JC} = \hbar g(\sigma^+ a + \sigma^- a^\dagger), \tag{3}$$

where g is the light-matter coupling and $\sigma^\pm$ are spin excitation and deexcitation operators, $\sigma^+|g\rangle = |e\rangle,\ \sigma^+|e\rangle = 0$, and $\sigma^-|e\rangle = |g\rangle,\ \sigma^-|g\rangle = 0$, where $|e\rangle$, $|g\rangle$ are the excited and ground internal states of the atom under the two-level approximation.

7 Prove that the solution to the time-dependent Schrödinger equation $i\hbar\partial_t|\psi(t)\rangle = H_{JC}|\psi(t)\rangle$ can be obtained in terms of $\{|i\rangle \equiv |e\rangle|n\rangle, |f\rangle \equiv |g\rangle|n+1\rangle\}$ via the expression $|\psi(t)\rangle = C_i(t)|i\rangle + C_f(t)|f\rangle$, where we have assumed that the initial state of the system is $|i\rangle$, $C_i(0) = 1$, $C_f(0) = 0$. Obtain the expressions for $C_i(t)$, $C_f(t)$ for all t. (3 points).

Solution

1. We first keep the sizable term

$$C_f(t) \simeq \frac{1}{2\hbar}(\mathbf{d}\cdot\mathbf{E_0})_{fi}\left\{\frac{(1-e^{-i(\omega-\omega_{fi})t})}{(\omega-\omega_{fi})}\right\}, \tag{4}$$

2. We consider $\Delta = 0$, and expand the exponential term in a power series, keeping up to first order,

$$C_f(t) \simeq \frac{1}{2\hbar}(\mathbf{d}\cdot\mathbf{E_0})_{fi}\left\{\frac{(1-1+i\Delta t}{\Delta}\right\} = \frac{i}{2\hbar}(\mathbf{d}\cdot\mathbf{E_0})_{fi}t. \tag{5}$$

   Therefore, $P_{i\to f}(t) \equiv |C_f(t)|^2 = \frac{|(\mathbf{d}\cdot\mathbf{E_0})_{fi}|^2}{4\hbar^2}t^2$.

3. Starting with

$$C_f(t) \simeq \frac{1}{2\hbar}(\mathbf{d}\cdot\mathbf{E_0})_{fi}\left\{\frac{(1-e^{-i(\omega-\omega_{fi})t})}{(\omega-\omega_{fi})}\right\}, \tag{6}$$

   we take the modulus square, and express the resulting equation in terms of a sine function, considering that $\sin(x) = [\exp(ix) - \exp(-ix)]/(2i)$.

$$|C_f(t)|^2 \simeq \frac{|(\mathbf{d}\cdot\mathbf{E_0})_{fi}|^2}{4\hbar^2}\left|\frac{(1-e^{-i(\omega-\omega_{fi})t})}{(\omega-\omega_{fi})}\right|^2 = \frac{|(\mathbf{d}\cdot\mathbf{E_0})_{fi}|^2}{\hbar^2}\frac{\sin^2(\Delta t/2)}{\Delta^2}. \tag{7}$$

4. From previous expression we can obtain the maximum for $\Delta \neq 0$, which will happen when the sine is maximized (equal to 1): $(P_{i\to f})_{max} = \frac{|(\mathbf{d}\cdot\mathbf{E_0})_{fi}|^2}{\hbar^2}\frac{1}{\Delta^2}$.

5. Due to the properties of the annihilation and creation operators, $a|n\rangle \propto |n-1\rangle$, $a^\dagger|n\rangle \propto |n+1\rangle$, the action of $H^{(I)}$ of Eq. (2) on state $|i\rangle$ only allows transitions to states of type $|f_{1,2}\rangle$ for a certain atomic internal states $|B\rangle$ and $|C\rangle$.

   H action only allows transitions to states of type f1 or f2

6. The energies in absence of interaction are additive, such that they will correspond to

   $E_i = E_A + E_n = E_A + \hbar\omega n$, $E_{f_1} = E_B + \hbar\omega(n-1)$, and $E_{f_2} = E_C + \hbar\omega(n+1)$.

7. Substituting $|\psi(t)\rangle = C_i(t)|i\rangle + C_f(t)|f\rangle$ into the Schrödinger equation, we arrive to the coupled equations,

$$\dot{C}_i = -ig\sqrt{n+1}C_f \tag{8}$$
$$\dot{C}_f = -ig\sqrt{n+1}C_i. \tag{9}$$

   Eliminating C_f, we arrive to the second order, linear differential equation,

$$d^2C_i/dt^2 + g^2(n+1)C_i = 0. \tag{10}$$

   The solution to the equation verifying the initial conditions is

$$C_i(t) = \cos(gt\sqrt{n+1}). \tag{11}$$

From this and Eq. (8), we obtain $C_f(t) = -i\sin(gt\sqrt{n+1})$.