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Absorption and emission of electromagnetic radiation by cosmic dust

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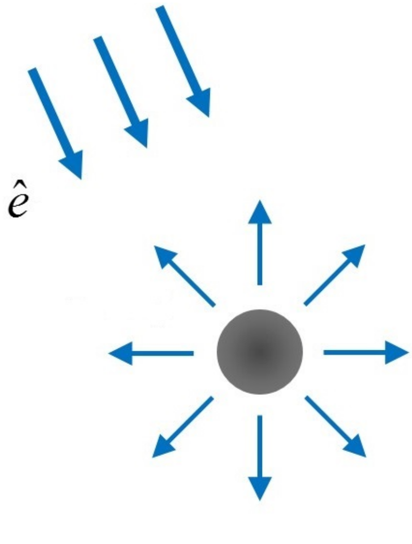
This problem studies the force associated with the action of a beam of electromagnetic radiation on a moving dust particle, (a small, homogeneous, spherical body of mass m moving with velocity $\vec{v}$ in some reference frame). Let us call Φdt the energy of the radiation that illuminates the particle during a time dt coming in the direction of the unitary vector $\hat{e}$. We start with the simplifying assumption that the particle absorbs all the incoming radiation and isotropically re-emits a fraction Q of it in the form of thermal radiation[1].



1 How much momentum does the re-emitted radiation take? Does the particle accretes mass? (0.5 points)
2 Compute the acceleration of the particle on the incident direction and the drag along $\vec{v}$ that would slow down the particle. (1 point)
3 In the case where the particle moves in the Solar System, it is illuminated by the Sun. Determine Φ in terms of the particle radius R, its distance to the Sun r, and the power $L_\odot$ emitted by the Sun. (0.5 points)
4 Determine the net attraction force toward the Sun. For what radii R will the resulting radial force overcome gravity, pushing homogeous particles of density ρ away from the Sun? Will the angular momentum $\vec{J}$ of the particle remain constant? (1 point)

You may have found that the rate at which the mass of the particle changes depends on its velocity. Strange, right? Mass - rest mass! - becoming a velocity-dependent property. Something analogous happens with the isotropy of the re-emitted radiation; if it occurs in the particle's rest frame, then it will not occur in others that are not at rest relative to it.

From now on we will re-analyse the problem in the light of relativistic mechanics. Our starting point here is that the isotropy of the re-emitted radiation only holds in the proper reference frame S' in which the particle is at rest at the instant of analysis. You are asked to determine the motion of the particle in the Sun's rest frame S where it moves with velocity $\vec{v}$. From here on we will ignore the gravitational force (as if $G = 0$). Also, the notation can be too cumbersome and the problem too long to handle in the allotted time. For this reason we will consider that all radiation is re-emitted (Q or $Q' = 1$) and there is no scattering.

The particle is instantaneously at rest in frame S' that moves with velocity $\vec{v}$ with respect to S, so that for any four-vector $V^\mu = (V^0, \vec{V})$,

$$V'^0 = \gamma\left(V^0 - \frac{\vec{v}\cdot\vec{V}}{c}\right) \quad \text{or,} \quad V^0 = \gamma\left(V'^0 + \frac{\vec{v}\cdot\vec{V}'}{c}\right) \tag{1}$$

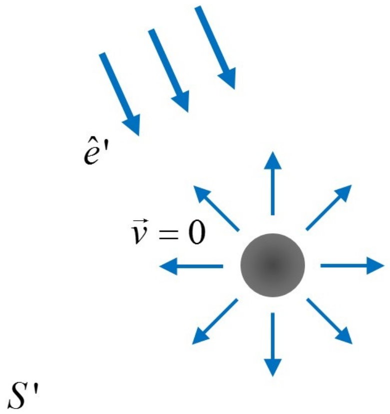
$$\vec{V}' = \vec{V} + \left[(\gamma - 1)\frac{\vec{v}\cdot\vec{V}}{\vec{v}^2} - \gamma\frac{V^0}{c}\right]\vec{v}, \quad \text{or} \quad \vec{V} = \vec{V}' + \left[(\gamma - 1)\frac{\vec{v}\cdot\vec{V}'}{\vec{v}^2} + \gamma\frac{V'^0}{c}\right]\vec{v} \tag{2}$$

[1] Part of the incoming radiation would also be scattered, a phenomenon that we will ignore in this problem for simplicity, i. e., only absorption and isotropic re-emission here.

where $\gamma = (1 - \vec{v}^2/c^2)^{-1/2}$. In words, the transverse components of $\vec{V}$ are unchanged while the longitudinal ones experience the Lorentz boost. Also, for an arbitrary interval

$$(cd\tau)^2 = (cdt')^2 - d\vec{x}'^2 = (cdt)^2 - d\vec{x}^2, \quad (3)$$

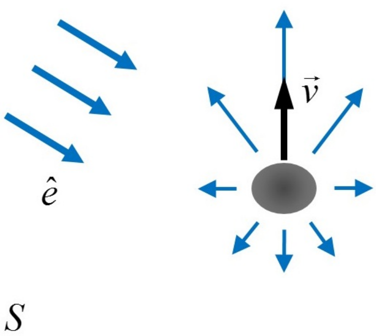
where τ is the particle's proper time, that is invariant. Note that using τ instead of t or t' in your equations will save precious time.



5 In S' the incoming radiation that illuminates the particle with a four-momentum $dp'_{in}/d\tau$ per unit time, comes in the direction $\hat{e}'$, while re-emission occurs isotropically. Write $dp'_{in}/d\tau$ and $dp'_{out}/d\tau$ in terms of $\hat{e}'$ and the power Φ' of the solar radiation that illuminates the particle as observed in S'. (0.5 points)

6 Write the equation of motion for the particle four-momentum of components $p'^{\mu} = (p'^0, \vec{p}')$ in S'. (0.5 points)

7 Work out the transformation that connects the four-forces $F = dp/d\tau$ and $dF' = dp'/d\tau$ in both frames S and S'. Use this result to write the components of $dp/d\tau$ in terms of those of $dp_{in}/d\tau$. (2 points)



8 Write the components of

$$\frac{dp_{in}}{d\tau} = (\frac{dp^0_{in}}{d\tau}, \frac{d\vec{p}_{in}}{d\tau}), \text{ and } \frac{dp_{out}}{d\tau} = \frac{(dp^0_{out}}{d\tau}, \frac{d\vec{p}_{out}}{d\tau})$$

in terms of the Solar System quantities Φ and $\hat{e}$. Then explicitly determine

$$\frac{dp'^0_{in}}{d\tau}, \frac{d\vec{p}'_{in}}{d\tau}, \text{ and } \frac{dp'^0_{out}}{d\tau}, \frac{d\vec{p}'_{out}}{d\tau}$$

in terms of those components and $\vec{v}$, i. e., in terms of $\Phi, \hat{e}$ and $\vec{v}$. Ignore any effect of the Lorentz contracted geometry of the sphere. (2 points)

9 Using the relations obtained so far, determine the force on the particle, $\vec{F} = d\vec{p}/d\tau$, observed in the solar reference frame in terms of Φ, $\vec{v}$, $\hat{e}$, and the dimensionless quantity $\Gamma = (1 - \hat{e} \cdot \vec{v}/c)$. (1 point)

10 Finally, give the particle acceleration in the solar reference frame, S, and check whether or not the mass becomes velocity independent. (1 point)

Solution

1. The energy E and momentum p of the particle change due to the net balance between what goes in and what goes out. The incoming radiation delivers Φ energy per unit time, while the outcoming removes $Q\Phi$, hence

$$\frac{dE}{dt} = \frac{dE_{in}}{dt} - \frac{dE_{out}}{dt} = \frac{dE_k}{dt} + \frac{d(mc^2)}{dt} \Rightarrow \frac{d(mc^2)}{dt} = \Phi(1-Q) - \vec{v}\cdot\frac{d\vec{p}}{dt} \tag{4}$$

Where $\frac{dE_k}{dt}$ is the change in kinetic energy of the particle. A similar argument applies to the momentum. However, since the re-radiation is isotropic, the total momentum carried by the outcoming radiation vanishes; hence

$$\frac{d\vec{p}}{dt} = \frac{d\vec{p}_{in}}{dt} - \frac{d\vec{p}_{out}}{dt} = \frac{\Phi}{c}\hat{e}, \tag{5}$$

where the right hand side is the momentum per unit time given to the particle by the incoming radiation. Combining both equations gives

$$\frac{d(mc^2)}{dt} = \Phi\left[(1-Q) - \hat{e}\cdot\vec{v}/c\right] = \Phi\left[\Gamma - Q\right] \tag{6}$$

so that the sign of $(1-Q) - \hat{e}\cdot\vec{v}/c$ determines whether the mass increases of decreases. $\Gamma = 1 - \hat{e}\cdot\vec{v}/c$ is a widely used notation.

2. Using (5) and (6)

$$\frac{d\vec{p}}{dt} = \frac{dm}{dt}\vec{v} + m\frac{d\vec{v}}{dt} \Rightarrow \frac{d\vec{v}}{dt} = \frac{\Phi}{mc}\left(\hat{e} - (\Gamma - Q)\frac{\vec{v}}{c}\right). \tag{7}$$

so the radiation not only pushes the particle in its incoming direction $\hat{e}$, it also slows or pushes it in the direction of velocity $\vec{v}$. Since $|\hat{e}\cdot\vec{v}/c| < 1$ is generally small and $Q \leq 1$, a slowdown of the particle is to be expected in most cases.

3. The power density radiated at a distance r from the Sun is $L_\odot/4\pi r^2$, so that the total power on the particle will be $\Phi = L_\odot \pi R^2/4\pi r^2 = L_\odot R^2/4r^2$.

4. Now $\hat{e} = \vec{r}/r$ is in the Sun$\rightarrow$particle direction. Both, gravity and the radiation pressure, act on the radial direction; the resulting radial force is

$$F_r = -\frac{GM_\odot m}{r^2} + \frac{\Phi}{c} = -\frac{GM_\odot m}{r^2}(1-\alpha) \tag{8}$$

where $\alpha = L_\odot R^2/(4GM_\odot m) = 3L_\odot/(16\pi GM_\odot \rho R)$. The radial force is in the direction away from the Sun if $\alpha > 1$, i. e., for small particles with radius $R < 3L_\odot/(16\pi GM_\odot \rho)$. The angular momentum $\vec{J}$ of the particle remains constant:

$$\dot{\vec{J}} = d(\vec{r}\wedge\vec{p})/dt = \dot{\vec{r}}\wedge\vec{p} + \vec{r}\wedge\dot{\vec{p}} = 0. \tag{9}$$

which vanishes in the solar system since in it $\dot{\vec{p}} = \frac{\Phi}{c}\hat{e}$ goes in the radial direction, (even though there is a component of $\dot{\vec{v}}$ in the direction of $\vec{v}$).

5.

$$dp'_{in}/d\tau = \left(dp'^0_{in}/d\tau,\ d\vec{p'}_{in}/d\tau\right) = \frac{\Phi'}{c}(1, \hat{e}') \quad (10)$$
$$dp'_{out}/d\tau = \left(dp'^0_{out}/d\tau,\ d\vec{p'}_{out}/d\tau\right) = \frac{\Phi'}{c}(1, \vec{0}) \quad (11)$$

where we used that $Q' = 1 \Rightarrow dp'^0_{in}/d\tau = dp'^0_{out}/d\tau$, and the isotropy of the outgoing radiation,

6. The equation $(dp/d\tau) = (dp_{in}/d\tau) - (dp_{out}/d\tau)$ is valid in any frame as these objects are four vectors (the $p's$ are four-vectors and τ , the proper time, is a scalar). Specifically, in the instantaneous proper frame S' one has

$$dp'/d\tau = \frac{\Phi'}{c}(1, \hat{e}') - \frac{\Phi'}{c}(1, \vec{0}) = \frac{\Phi'}{c}(0, \hat{e}') = (0, d\vec{p'}_{in}/d\tau). \quad (12)$$

7. Taking into account the four vector character of the objects, we can use (1) and (2) to write

$$dp^0/d\tau = \gamma\left(dp'^0/d\tau + \frac{\vec{v}\cdot d\vec{p'}/d\tau}{c}\right) \quad (13)$$
$$d\vec{p}/d\tau = d\vec{p'}/d\tau + \left[(\gamma-1)\frac{\vec{v}\cdot d\vec{p'}/d\tau}{\vec{v}^2} + \gamma\frac{dp'^0/d\tau}{c}\right]\vec{v}. \quad (14)$$

According to (12) $dp'^0/d\tau = 0$ and the above reduces to

$$dp^0/d\tau = \gamma\frac{\vec{v}\cdot d\vec{p'}/d\tau}{c} \quad (15)$$
$$d\vec{p}/d\tau = d\vec{p'}/d\tau + \left[(\gamma-1)\frac{\vec{v}\cdot d\vec{p'}/d\tau}{\vec{v}^2}\right]\vec{v}. \quad (16)$$

We need to get the scalar product $\vec{v}\cdot d\vec{p'}/d\tau$ which is on the rhs of these expressions. Using (12), (1), and (2) again,

$$d\vec{p'}/d\tau = d\vec{p'}_{in}/d\tau = d\vec{p}_{in}/d\tau + \left[(\gamma-1)\frac{\vec{v}\cdot d\vec{p}_{in}/d\tau}{\vec{v}^2} - \gamma\frac{dp^0_{in}/d\tau}{c}\right]\vec{v}$$
$$\vec{v}\cdot d\vec{p'}/d\tau = \vec{v}\cdot d\vec{p'}_{in}/d\tau = \gamma\left[\vec{v}\cdot d\vec{p}_{in}/d\tau - \frac{\vec{v}^2 dp^0_{in}/d\tau}{c}\right]. \quad (17)$$

Putting (17) into (15) and (16), and after some algebra, we arrive at

$$dp^0/d\tau = \gamma^2\left[\frac{\vec{v}}{c}\cdot d\vec{p}_{in}/d\tau - \frac{\vec{v}^2}{c^2}dp^0_{in}/d\tau\right] \quad (18)$$
$$d\vec{p}/d\tau = d\vec{p}_{in}/d\tau + \gamma^2\left[\frac{\vec{v}\cdot d\vec{p}_{in}/d\tau}{c} - dp^0_{in}/d\tau\right]\frac{\vec{v}}{c} \quad (19)$$

Everything on the rhs of (18) and (19) is written in terms of S frame (unprimed) quantities, so we have formally solved the problem. However, it remains to determine those unprimed quantities from the data given in the text, which corresponded to quantities of S'.

8. Taking into account (1), (2), (10) and (11) and $d\vec{p'}_{out}/d\tau = 0$, that transforms in $dp^0_{out}/d\tau = \gamma\,(dp'^0_{out}/d\tau)$ (or $\Phi = \gamma\Phi'$), and also in $d\vec{p}_{out}/d\tau = \gamma\ \ (dp'^0_{out}/d\tau)\vec{v}/c = \Phi\ \vec{v}/c^2$ we get

$$dp_{in}/d\tau = (dp^0_{in}/d\tau,\ d\vec{p}_{in}/d\tau) = \frac{\Phi}{c}(1,\hat{e}) \tag{20}$$
$$dp_{out}/d\tau = (dp^0_{out}/d\tau,\ d\vec{p}_{out}/d\tau) = \frac{\Phi}{c}(1,\vec{v}/c) \tag{21}$$

9. Using these values in the rhs of (18) and (19) we get

$$dp^0/d\tau = \frac{\Phi}{c}(1-\gamma^2\Gamma) \tag{22}$$
$$d\vec{p}/d\tau = \frac{\Phi}{c}\left[\hat{e}-\gamma^2\Gamma\frac{\vec{v}}{c}\right]. \tag{23}$$

10. Differentiating $(p^0,\vec{p}) = m\gamma(c,\vec{v})$ with respect to time as we did in (7) for $m\vec{v}$, we get

$$\gamma\frac{dm}{d\tau} = 0 \tag{24}$$
$$\gamma\frac{d\vec{v}}{d\tau} = \frac{\Phi}{mc}\left[\hat{e}-\frac{\vec{v}}{c}\right]. \tag{25}$$

The end result is an increasing acceleration in the direction of the incoming radiation and a braking in the direction of $\vec{v}$. As expected, the change in mass occurs at a rate independent of speed; it vanishes here because we chose $Q = 1$, all the energy that comes in goes out. For $Q < 1$, we would obtain $\gamma dm/d\tau = (1-Q)\Phi/c^2$. A full account of these questions can be found in Ref. [4].

Historical note. The problem we are dealing with here is the so-called Poynting-Robertson effect. The effects of absorption and re-emision of solar radiation on the motion of small bodies were first analysed by Poynting[1] at the beginning of last century, still in the ether paradigm (it can be read in page 541 of that paper ”The ether, or whatever we term the lightbearing medium, is material...”, etc.). Three decades later, Robertson[2] reexamined this question from the standpoint of the theory of relativity. Section 2 of this article provides a deep and beautiful overview of the relativistic equations of motion of the system. Much later, Burns and his colleagues[3] wrote a paper designed to describe the problem for the astronomical community, which became the standard reference on the subject. Finally, Klacka and its group gave the most complete treatment of this phenomenon. The essay[4] collects relativistic and non-relativistic descriptions of the effects of scattering, absorption, thermal emission, anisotropy etc., and the competing phenomena of the solar wind.

1. J. H. Poynting, *Radiation in the Solar System: its Effect on Temeprature and its Pressure on Small Bodies*, Phil. Trans. Roy. Soc., A, 202, 525, 1903.

2. H. P. Robertson, DYNAMICAL EFFECTS OF RADIATION IN THE SOLAR SYSTEM, Mon. Not. Roy. Astron. Soc. 97, 423.

3. J. A. Burns, Ph. L. Lamy and S. Soter, Radiation Forces on Small Particles in the Solar System, Icarus 40, (1979) 1.

4. J. Klačka, J. Petržala, P. Pástor and L. Kómar, The Poynting–Robertson effect: A critical perspective, Icarus 232 (2012) 249.