Two beads in a ring

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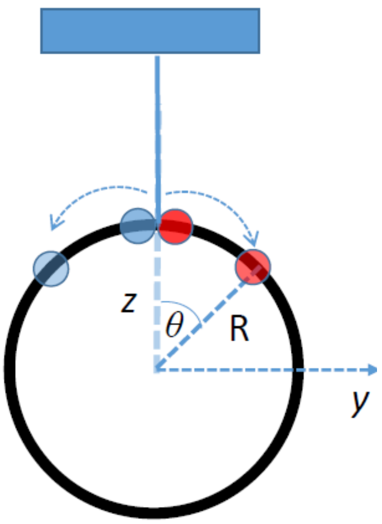
Consider a ring of radius R and mass M hanging vertically from a thread attached to the ceiling. Two small beads, of mass m each, move without friction along the ring under the action of gravity. The small beads will be considered as point particles of negligible radius. Different physical situations are shown in the figure corresponding to parts I, II and III below.

Set $R = 1$ m, $M = 1$ kg, $m = 0.1$ kg and $g = 10$ m/s^2 for the numerical calculations below.

Part I

First, suppose that the system, consisting of the ring and the two beads, lies in a vertical plane that does not rotate.



(1) Draw the relevant forces in the problem, namely, the weight of each bead mg, the reaction N of the ring on the beads, and the tension in the thread T. Write Newton's second law separately for the tangential and normal components (only for the right bead) and obtain from the first the modulus of the bead's velocity v as a function of the angle $\theta \in [0, \pi]$. (0.5 pts)

(2) Write the reaction N of the ring on the right bead as a function of m, g, R and θ. Focus on the case where the motion starts initially at rest from the top of the ring and determine the angle $\tilde{\theta}$ for which N vanishes. (1 pt)

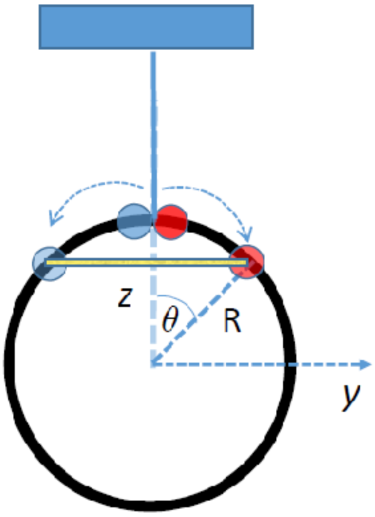
(3) Write the total reaction $\vec{N}_T(\theta_L, \theta_R)$ of the ring on both beads for the case considered in (2). Compute its value for $\theta = 0$ and $\theta = \pi/2$. (0.5 pts)

(4) What are the most general initial conditions that ensure that the ring will remain still? Now choose vanishing values for the initial conditions and give for them the tension T on the thread. (0.5 pts)

(5) Determine the angular intervals where the tension T on the thread obtained above becomes smaller than Mg, the maximum value of T, and at what angles it it is reached. Make a plot of T vs θ showing those cases. (1.5 pts)

Part II

Now suppose an elastic band (or soft spring) of constant k is stretched between the beads as they fall. When both beads are at the top of the ring the elastic force between them vanishes, while increasing according to Hooke's law as their relative distance increases.

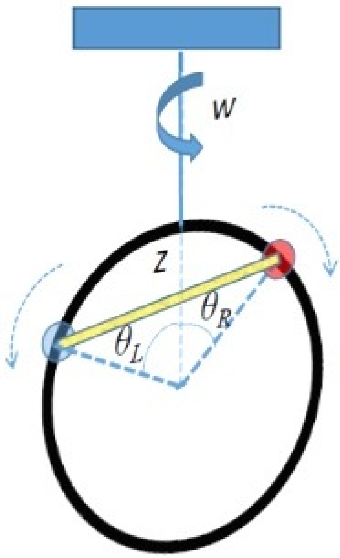


(6) Rewrite the reaction N of the ring on the right bead in the presence of the spring and determine the new angle $\tilde{\theta}$ for which N vanishes. Again, in this Part II, consider that the motion starts initially at rest from the top of the ring. (1 pt)

(7) Rewrite the total reaction $N_T(\theta)$. Determine the new angular interval where the tension of the thread T is smaller than Mg for $k = 0.2$ N/m. (0.5 pts)

Part III

Next, suppose that the entire system from Part II, consisting of the ring, the spring, and the two beads, rotates on a vertical plane about the z axis.



(8) Write the Lagrangian of this system as a function of the configuration variables $\theta_R, \dot{\theta}_R, \theta_L, \dot{\theta}_L$, and $\varphi, \dot{\varphi}$, the azimuthal angle of the rotation of the whole system around the z-axis. (1.5 pts)

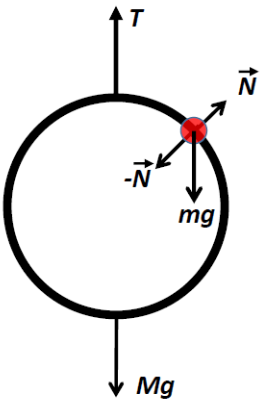
(9) Obtain the azimuthal angular momentum of the system J_φ. Determine the rotation velocity $\dot{\varphi}$ and its range of variation for the initial conditions $\theta_L(0) = \theta_R(0) = 0$. (1 pt)

(10) Write the Euler-Lagrange equations for θ_L and θ_R. What are the most general initial conditions that ensure that $\theta_L(t) = \theta_R(t)$? Then obtain $\dot{\theta}_R(t)$ and $N_R(t)$ in terms of $\theta_R(t)$, R, m, M,, and k for the case $\theta_L(0) = \theta_R(0) = 0$, and $\dot{\theta}_L(0) = \dot{\theta}_R(0) = 0$. (2 pts)

Solution

Part I

1) Draw the relevant forces in the problem, namely, the weight of each bead mg, the reaction N of the ring on the beads, and the tension in the thread T. Write Newton's second law separately for the tangential and normal components (only for the right bead) and obtain from the first the modulus of the bead's velocity v as a function of the angle $\theta \in [0, \pi]$.



Let us consider first the right bead's tangencial and normal components:

$$mR\frac{d^2\theta}{dt^2} = mg\sin\theta, \; m\frac{v^2}{R} = mg\cos\theta - N. \tag{1}$$

Now, $\frac{dv}{dt} = v\frac{dv}{Rd\theta} = g\sin\theta \;\rightarrow\; v = \sqrt{v_0^2 + 2gR\left(\cos\theta_0 - \cos\theta\right)}$

$$\frac{mv^2}{2} = \frac{mv_0^2}{2} + mgR\left(\cos\theta_0 - \cos\theta\right)$$
$$m\frac{v^2}{R} = mg\cos\theta - N$$

2) Write the reaction N of the ring on the right bead as a function of m, g, R and θ. Focus on the case where the motion starts initially at rest from the top of the ring and determine the angle $\tilde{\theta}$ for which N vanishes.

$$N(\theta) = mg\cos\theta + \frac{mv^2}{R} = \frac{mv_0^2}{R} + mg\left(3\cos\theta_0 - 2\cos\theta\right), \; \theta \in [0, \pi] \tag{2}$$

Under the initial conditions $\theta_{R0} = \theta_{L0} = 0$ and $\dot{\theta}_{R0} = \dot{\theta}_{L0} = 0$

$$N_y^R(\theta) = mg\;\sin\theta\left(3\cos\theta - 2\right), \qquad N_z^R(\theta) = mg\;\cos\theta\left(3\cos\theta - 2\right). \tag{3}$$

For the left bead: $N_y^L(\theta) = -N_y^R(\theta)$, $N_z^L(\theta) = N_z^R(\theta)$.

The angle $\tilde{\theta}$ for which the reaction N of the ring on each bead vanishes is given by:

$$N(\tilde{\theta}) = mg(3\cos\tilde{\theta} - 2) = 0 \;\;\rightarrow\; \cos\tilde{\theta} = 2/3 \;\rightarrow\; \tilde{\theta} \;\approx\; 48^0 \tag{4}$$

3) Write the total reaction $\vec{N}_T(\theta_L, \theta_R)$ of the ring on both beads for the case considered in 2). Compute its value for $\theta = 0$ and $\theta = \pi/2$.

$$N_{Ty}(\theta) \equiv 0 \;\text{ because of axial symmetry,} \quad N_{Tz}(\theta) = 2mg\;\cos\theta\left(3\cos\theta - 2\right) \tag{5}$$

$$N_{Tz}(\theta) = \ 2mg\cos\theta\ (3\cos\theta - 2),\ N_{Tz}(\theta = 0) = 2mg,\ \ N_{Tz}(\theta = \pi/2) = 0. \tag{6}$$

4) What are the most general initial conditions that ensure that the thread can remain vertical? Choose the simplest values for these conditions, and give - for them - the tension T on the thread.

The total force that the ring exerts on the thread is

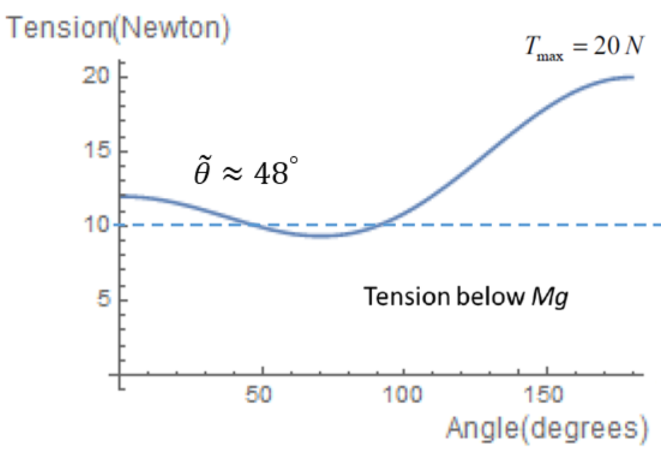
$$M\vec{g} - \vec{N}_R(\theta_R) - \vec{N}_L(\theta_L). \tag{7}$$

The horizontal component of this force must cancel out for the thread to remain vertical. For this it must be $N_{Ry}(\theta_R) + N_{Ly}(\theta_L) = 0\ \forall t$. With some hindsight it can be seen that this is a case that only requires axially symmetric (but otherwise arbitrary) initial conditions on both beads, giving $\theta_R = \theta_L,\ \dot{\theta}_R = \dot{\theta}_L,\ \forall t = 0$. Hence, the tension on the vertical thread is

$$T = Mg + N_{Rz}(\theta) + N_{Lz}(\theta)\ \Rightarrow\ T = Mg + N_T(\theta) = Mg + 2mg\cos\theta\ (3\cos\theta - 2), \tag{8}$$

the second equation corresponds to the simplest case where $\theta_{R0} = \theta_{L0} = 0$ and $\dot{\theta}_{R0} = \dot{\theta}_{L0} = 0$.

5) Determine the angular intervals where the tension T on the thread obtained above becomes smaller than Mg, the maximum value of T, and at what angles it it is reached. Make a plot of T vs θ showing those cases.



$\theta \in [\tilde{\theta}, \pi/2] \rightarrow T \leq M\,g = 10N$

thread tension in Newtons as a function of angle θ (without spring).
The tension T on the thread is smaller than Mg for

$$\cos\theta\ (3\cos\theta - 2) < 0$$

i. e., in the interval $\theta \in [\tilde{\theta}, \pi/2]$, where $\tilde{\theta} = \arccos 2/3 \approx 48^\circ$. The maximum tension

$$T = Mg + 10mg = 20N$$

corresponds to $\theta_0 = 180^0$ which is at the bottom of the ring.

Part II

6) Rewrite the reaction N of the ring on the right bead in the presence of the spring and determine the new angle $\tilde{\theta}$ for which N vanishes. Again, in this Part II, consider that the motion starts initially at rest from the top of the ring..

We begin assuming an axisymmetric fall; hence, the distance between the beads is $2x = 2R\text{sen}\theta$, so that the elastic force on the right bead is $F_E = -k(2R\text{sen}\theta)$.

Integrating the tangential component of the Newton equation gives

$$\frac{1}{2}m(v^2 - v_0^2) = +mgR\,(\cos\theta_0 - \cos\theta) + \frac{1}{4}k((2R\sin\theta_0)^2 - (2R\sin\theta)^2)), \tag{9}$$

i. e., energy conservation. The normal component is directly

$$m\frac{v^2}{R} = mg\cos\theta + 2kR\sin^2\theta - N, \tag{10}$$

from these equations

$$N = mg(3\cos\theta - 2cos\theta_0) + kR(4\sin^2\theta - \sin^2\theta_0). \tag{11}$$

For the rest of this part we will consider the case $\theta_0 = 0$ and $v_0 = 0$. The reaction of the ring vanishes for an angle $\tilde{\theta}$ given by

$$N(\tilde{\theta}') = 0 \;\rightarrow\; \cos\tilde{\theta}' = \frac{3mg \pm \sqrt{9m^2g^2 - 32mgkR + 64k^2R^2}}{8kR} \tag{12}$$

with a valid solution $\cos\tilde{\theta}' \simeq 0.455 \;\rightarrow\; \tilde{\theta}' \simeq 63^0$.

7) Obtain again an analytic expression for the total reaction $N_T(\theta)$. Determine the new angular interval where the tension of the thread T is smaller than Mg, where $k = 0.2$ N/m.

The interval is given by the condition $N_{Tz} = 2\cos(\theta)N(\theta) < 0$. There are two posibilities i) $\cos(\theta) > 0, N(\theta) < 0$, and ii) $\cos(\theta) < 0, N(\theta) > 0$. they are separated by $\cos(\tilde{\theta}') = 0$. But

$$\cos(\theta) > 0, N(\theta) > 0 \;\text{ for } 0 \le \theta < \tilde{\theta}',\; \text{and}\; \cos(\theta) < 0, N(\theta) < 0 \;\text{ for } \pi/2 < \theta \le \pi.$$

Finally, the requested interval is

$$\theta \in (\tilde{\theta}', \pi/2)$$

that starts at a θ value larger than the obtained for $k = 0$. This could be expected 'a priori' because the spring stores energy that adds to the gravitational energy. Therefore, the bead has to fall further than before to compensate it.

Part III

8) Write the Lagrangian of this system as a function of the three configuration variables θ_R, θ_L, and φ, the azimuthal angle of the rotation of the whole system around the z-axis.

Taking into account that for a ring rotating around any of its diagonals the inertial momentum is $MR^2/2$, we get

$$L(\varphi, \dot{\varphi}, \theta_L, \dot{\theta}_L, \theta_R, \dot{\theta}_R) = \frac{1}{4}MR^2\dot{\varphi}^2 + \sum_{i=R,L}\left(\frac{1}{2}mR^2\sin^2\theta_i\dot{\varphi}^2 + \frac{1}{2}mR^2\dot{\theta}_i^2\right) - \sum_{i=R,L} mgR\cos\theta_i - \frac{1}{2}k(2R)^2\sin^2\left[\frac{1}{2}(\theta_L+\theta_R)\right] \tag{13}$$

or,

$$L = \frac{1}{4}MR^2\dot{\varphi}^2 + \sum_{i=R,L}\left(\frac{1}{2}mR^2\sin^2\theta_i\dot{\varphi}^2 + \frac{1}{2}mR^2\dot{\theta}_i^2 - mgR\cos\theta_i\right) - kR^2[1-\cos(\theta_L+\theta_R)]. \tag{14}$$

9) Obtain the azimuthal angular momentum of the system J_φ. Determine the rotation velocity $\dot{\varphi}$ and its range of variation.

Using the Euler-Lagrange equations first for the azimuthal angle φ,

$$\frac{\partial L}{\partial \varphi} = 0,\ J_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \left[\frac{1}{2}MR^2 + mR^2(\sin^2\theta_L + \sin^2\theta_R)\right]\dot{\varphi},\ \dot{J}_\varphi = 0. \tag{15}$$

Since φ is an ignorable coordinate, J_φ remains constant.

$$J_\varphi = \left[\frac{1}{2}MR^2 + mR^2\ (\sin^2\theta_R + \sin^2\theta_L)\right]\dot{\varphi} \tag{16}$$

Let us take the initial conditions: $\theta_{R0} \neq \theta_{L0}$ and $\dot{\varphi}_0 = w_0$; then

$$\dot{\varphi} = \frac{1 + \frac{2m}{M}(\sin^2\theta_{R0} + \sin^2\theta_{L0})}{1 + \frac{2m}{M}(\sin^2\theta_R + \sin^2\theta_L)}\, w_0. \tag{17}$$

In the case $\theta_{R0} = \theta_{L0} = 0$,

$$\dot{\varphi} = \frac{w_0}{1 + \frac{2m}{M}(\sin^2\theta_R + \sin^2\theta_L)}, \tag{18}$$

and the variation range of the angular velocity $\dot{\varphi}$ is:

$$w_0/(1+4m/M)\ \leq \dot{\varphi}\ \leq w_0 \tag{19}$$

with the minimal value happening for $\theta_R = \theta_L = \pi/2$

10) Write the Euler -Lagrange equations for θ_L and θ_R. What are the most general initial conditions that ensure that $\theta_L(t) = \theta_R(t)$?. Obtain $\dot{\theta}_R(t)$ and $N_R(t)$ in terms of $\theta_R(t)$, R, m, M, and k for the case $\theta_L(0) = \theta_R(0) = 0$, and $\dot{\theta}_L(0) = \dot{\theta}_R(0) = 0$.

The E-L equations are

$$mR^2\ddot{\theta}_i = \frac{1}{2}mR^2 \sin 2\theta_i \dot{\varphi}^2 + mgR\sin\theta_i - kR^2 \sin(\theta_L + \theta_R),\ \text{for } i = L, R \tag{20}$$

It will help to use $\ddot{\theta}_i = \frac{1}{2}\frac{d\dot{\theta}_i^2}{d\theta}$ in what follows.

Note that the differential equations are identical for both angles θ_R and θ_L. Therefore, their solution is the same function of time $\theta(t)$ if both have identical initial conditions. Now Eq. (18) reduces to:

$$\dot{\varphi} = \frac{w_0}{1 + 4\frac{m}{M}\sin^2\theta}, \tag{21}$$

and the E-L eqs. read

$$m\ R\ \frac{d\dot{\theta}^2}{d\theta} = m\ R\ \sin 2\theta \left(\frac{w_0}{1 + 4\frac{m}{M}\sin^2\theta}\right)^2 + 2\ m\ g\sin\theta - 2\ k\ R\ \sin 2\theta \tag{22}$$

which can be integrated to give

$$m\ R\ \dot{\theta}^2 = m\ R\ w_0^2 \left(\frac{\sin^2\theta}{1 + 4\frac{m}{M}\sin^2\theta}\right) - 2\ m\ g\ (\cos\theta - 1) + k\ R\ (\cos 2\theta - 1). \tag{23}$$

Since the constraints were already used when writing the Lagrangian L Eq. (14), we have to resort to Newton's equation for the radial components

$$m\ R\ \dot{\theta}^2 = \ m\ g\ \cos\theta + k\ R\ (1 - \cos 2\theta) - \ N \tag{24}$$

to obtain N. Finally,

$$N = -m\ R\ w_0^2 \left(\frac{\sin^2\theta}{1 + 4\frac{m}{M}\sin^2\theta}\right) + m\ g\ (3\cos\theta - 2) - 2k\ R\ (\cos 2\theta - 1). \tag{25}$$