Estudiantes

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Real Sociedad Española de Física

PLANCKS 2020

Preliminares de PLANCKS 2020, España

Crossing a river: a classical problem with a twist

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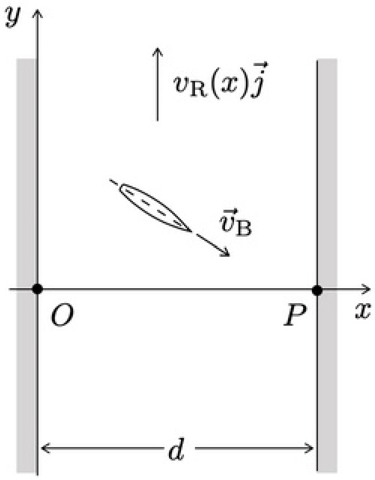
A river flows in the positive direction of the y axis (see figure). Assume that the velocity of the water depends on the x coordinate, i.e., $\vec{v}_\mathrm{R}(x) = v_\mathrm{R}(x)\vec{j}$.



We wish to go from the pier at the origin O (on the left margin) to the pier P, located just opposite (on the right margin), at a distance d from O. Our boat, B, always moves at the same speed, $v_\mathrm{B} = |\vec{v}_\mathrm{B}|$, relative to the water, so we can only change course with the rudder.

1. (1 point) Give an integral expression for the time it takes to go from O to P in terms of the y component of the velocity of our boat relative to the water.
2. (1 point) Likewise, give an integral expression for the condition that the y coordinate of P is zero.
3. (2 points) Formulate the problem of finding the trajectory $y(x)$ corresponding to the shortest passage (in time). Solve it when the velocity of the water is uniform, i.e., when $v_\mathrm{R}(x) = v_\mathrm{R}\,\vec{j}$. How long is the crossing?
4. (3 points) Find the trajectory $y(x)$ corresponding to the shortest passage (in time) if the speed of the water increases as the square root of the distance to the left margin, i.e., $v_\mathrm{R}(x) = v_\mathrm{B}\sqrt{x/d}$ (notice that the maximum speed of the water, on the right bank, coincides with that of the boat). Make a rough sketch of the trajectory.
5. (3 points) How long is the crossing in the case of question 4? How long would it take us if we went straight? Compare with the minimum crossing time.

Solution

1. Let $\vec{r} = x\vec{i} + y\vec{j}$ be the position of our boat relative to the banks of the river. Then,

$$v_\mathsf{B}^2 = \dot{x}^2 + (\dot{y} - v_\mathsf{R})^2.$$

Since $dt = dx/\dot{x}$ is the time that takes for our boat to travel a distance dx away from the left bank, we have

$$t = \int_0^x \frac{dx'}{\dot{x}(x')} = \int_0^x \frac{dx'}{\sqrt{v_\mathsf{B}^2 - [\dot{y}(x') - v_\mathsf{R}(x')]^2}} = \int_0^x \frac{dx'}{\sqrt{v_\mathsf{B}^2 - \xi^2(x')^2}}, \tag{1}$$

where

$$\xi(x) = \dot{y}(x) - v_\mathsf{R}(x) \tag{2}$$

is the y component of the velocity of our boat relative to the water. The crossing time is, then,

$$\boxed{T[\xi] = \int_0^d \frac{dx}{\sqrt{v_\mathsf{B}^2 - \xi^2(x)^2}}}. \tag{3}$$

Note it is a functional of $\xi(x)$.

2. Similarly,

$$y = \int_0^t \dot{y}(t')dt' = \int_0^t \frac{\dot{y}(t')}{\dot{x}(t')}[\dot{x}(t')dt'] = \int_0^x \frac{\xi(x') + v_\mathsf{R}(x')}{\sqrt{v_\mathsf{B}^2 - \xi^2(x')}}dx'. \tag{4}$$

Since $y(d) = 0$, we have

$$\boxed{0 = C[\xi] := \int_0^d \frac{\xi(x) + v_\mathsf{R}(x)}{\sqrt{v_\mathsf{B}^2 - \xi^2(x)}}dx}. \tag{5}$$

3. • The problem of finding the trajectory $y(x)$ corresponding to the shortest passage (in time) can be formulated as

$$\boxed{\min_\xi T[\xi], \quad \text{subject to} \quad C[\xi] = 0}.$$

The constrain $C[\xi] = 0$ can be implemented through the introduction of a Lagrange multiplier, λ. So, the problem becomes that of minimizing

$$S[\xi] := \int_0^d \frac{1 + \lambda(\xi + v_\mathsf{R})}{\sqrt{v_\mathsf{B}^2 - \xi^2}}dx := \int_0^d L(\xi(x), \lambda)dx.$$

The Euler-Lagrange equation is simply, $\partial L/\partial \xi = 0$, which gives

$$\frac{\lambda v_\mathsf{B}^2 + (1 + \lambda v_\mathsf{R})\xi}{(v_\mathsf{B}^2 - \xi^2)^{3/2}} = 0.$$

The solution can be written as

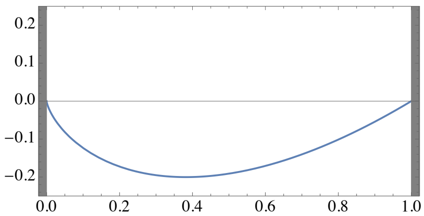
$$\boxed{\xi(x) = -\frac{\lambda v_\mathsf{B}^2}{1 + \lambda v_\mathsf{R}(x)}}. \tag{6}$$

To determine λ, we impose the constrain (5). Once λ is determined, we substitute in Eq. (6) to obtain an explicit expression for $\xi(x)$. Finally, by substituting $\xi(x)$ in (3) and (4) we obtain the duration of the crossing and the expression of the trajectory $y(x)$, respectively.

- If v_{R} is constant, Eq. (6) simply is
$$\xi(x) = -\frac{\lambda v_{\mathrm{B}}^2}{1+\lambda v_{\mathrm{R}}} = \xi = \text{constant}.$$
Substituting in (5), we obtain
$$\frac{\xi + v_{\mathrm{R}}}{\sqrt{v_{\mathrm{B}}^2 - \xi^2}} d = 0 \quad \Rightarrow \quad \xi = -v_{\mathrm{R}} \quad \Rightarrow \quad \dot{y} = 0 \quad \Rightarrow \quad \boxed{y = 0}.$$
Thus, the best strategy is to go straight from O to P and, from Eq. (3), the duration of the crossing is
$$\boxed{T = \frac{d}{\sqrt{v_{\mathrm{B}}^2 - v_{\mathrm{R}}^2}}}.$$

4. We next assume that
$$v_{\mathrm{R}} = v_{\mathrm{L}} \sqrt{\frac{x}{d}}. \tag{7}$$
Eq. (6) becomes
$$\xi(x) = -\frac{\lambda v_{\mathrm{B}}^2}{1+\lambda v_{\mathrm{B}}\sqrt{x/d}} = -\frac{\mu}{1+\mu\sqrt{u}} v_{\mathrm{B}}, \tag{8}$$
where we have defined the dimensionless variables $\mu = \lambda v_{\mathrm{B}}$ and $u = x/d$. Eq. (5) is
$$d \int_0^1 \frac{\sqrt{u} + (u-1)\mu}{\sqrt{(1+\mu\sqrt{u})^2 - \mu^2}} du = 0. \tag{9}$$
Let us compute the corresponding indefinite integral, which will be also used to obtain $y(x)$ after determining the value of the Lagrange multiplier μ (λ). The obvious change of variable that works for this integral is $\mu\sqrt{u} + 1 = \mu \cosh\phi$.

The indefinite integral we wish to compute can be written as
$$\int \left(\frac{2}{\mu^2} \cosh\phi + 2\sinh^2\phi \cosh\phi - \frac{2}{\mu} \cosh 2\phi \right) d\phi = \frac{2}{\mu^2} \sinh\phi + \frac{2}{3} \sinh^3\phi - \frac{1}{\mu} \sinh 2\phi.$$
Undoing the change of variables we obtain
$$\int \frac{\sqrt{u} + (u-1)\mu}{\sqrt{(1+\mu\sqrt{u})^2 - \mu^2}} du = \frac{2}{3\mu^3} \left(1 + \mu^2 u - \mu\sqrt{u} - \mu^2\right) \sqrt{(1+\mu\sqrt{u})^2 - \mu^2}. \tag{10}$$
Substituting the integration limits, the condition (9) [equivalent, (5)] is
$$\frac{2d}{\mu^3}(1-\mu) \left[\sqrt{1+2\mu} - (1+\mu)\sqrt{1-\mu}\right] = 0 \quad \Rightarrow \quad \mu = 1.$$
(One can easily check that the term in square brackets does not vanish for any real value of μ other than $\mu = 0$, which is not a solution.) Thus, Eq. (8) becomes
$$\xi = -\frac{1}{1+\sqrt{u}} v_{\mathrm{B}} = -\frac{v_{\mathrm{B}}}{1+\sqrt{x/d}}.$$
Substituting $\mu = 1$ in Eqs. (10) lead to
$$\boxed{y(x) = \frac{2d}{3} \left(\frac{x}{d} - \sqrt{\frac{x}{d}}\right) \sqrt{\left(1+\sqrt{\frac{x}{d}}\right)^2 - 1}}, \tag{11}$$
which gives the optimal trajectory of our boat. Figure 1 shows a plot of it.



Plot of the optimal trajectory. The values of the dimensionless coordinates $x/d := u$ and y/d are displayed on the corresponding axes.

5. • To compute the duration of the optimal passage, [Eq. (1)] we first compute the indefinite integral

$$\int \frac{1+\sqrt{u}}{\sqrt{(1+\sqrt{u})^2-1}}du = 2\int \cosh\phi(\cosh\phi-1)d\phi = \phi + (\cosh\phi - 2)\sinh\phi,$$

where we have used the same change of variables we used above. We obtain

$$T_{\text{opt.}} = \frac{d}{v_{\text{B}}}\int_0^1 \frac{1+\sqrt{u}}{\sqrt{(1+\sqrt{u})^2-1}}du = \frac{d}{v_{\text{B}}}\text{arccosh}\,2 \approx 1,317\times\frac{d}{v_{\text{B}}}.$$

• Let us turn to finding the duration of the passage had we crossed along a straight line from O to P. Recall that the speed of the water is given by Eq. (7) and that now $\dot{y} = 0$. From the definition of ξ, Eq. (2), we see that

$$\xi(x) = -v_{\text{R}}(x) = -v_{\text{B}}\sqrt{\frac{x}{d}} = -v_{\text{B}}\sqrt{u}. \quad (12)$$

Substituting in Eq. (3) we obtain that the duration of the crossing would be

$$T_{\text{str.}} = \frac{d}{v_{\text{B}}}\int_0^1 \frac{du}{\sqrt{1-u}} = -2\frac{d}{v_{\text{B}}}\sqrt{1-u}\Big|_0^1 = 2\frac{d}{v_{\text{B}}}.$$

• We see that $T_{\text{opt.}} < T_{\text{str.}}$. More precisely, the optimal crossing is 34% shorter in time than going straight from pier to pier.

Preliminares de PLANCKS 2020, España

Photon Gas

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The photon gas (or black body radiation) is a quantum mechanical system of photons in thermal equilibrium at temperature T with the walls of a cavity. This thermodynamic behavior of photons in blackbody radiation has played a crucial role in the development of physics in 20th century.

Consider a container of volume V, whose walls are maintained at temperature T. Suppose it has been emptied of matter by a vacuum pump. However, the container cannot be entirely empty because the walls radiate photons into the container. A portion of these photons scatter off the walls, with some of them being absorbed and new ones being emitted continuously. Thus, an apparently empty container actually is filled with a photon gas. When the average electromagnetic absorption and emission rates are equal, an equilibrium exists between the photon gas and the walls of the container. That is to say, electromagnetic radiation interacts with matter, and reaches a state of thermodynamics equilibrium at a definite temperature.

Using statistical thermodynamics arguments, the Helmholtz free energy for the photon gas in a container with wall temperature T and volume V can be expressed as:

$$F(T,V) = -\left(\frac{4\sigma}{c}\right)\frac{VT^4}{3}, \tag{1}$$

where V is the container volume, T is the wall temperature, σ is the Stefan-Boltzmann constant, and c is the speed of light.

1. (1 point) Find expressions for the entropy, S, and the internal energy, U, of the gas as a function of T and V.
2. (3 points) Next, consider a slow isothermal reversible volume change of the container. The photon gas undergoes a reversible expansion having initial and final volumes V and $2V$, respectively, whilst in equilibrium with a heat reservoir at temperature T. Find the work, W, performed by the gas, and the heat, Q, transferred between the gas and the reservoir. Both W and Q has to be written as a function of T and V.

In statistical mechanics the internal energy of the photon gas in a container of volume V and temperature T is

$$U(V,T) = \int_0^\infty d\omega\, n(\omega)\, E(\omega)$$

where $E(\omega)$ is the average energy per standing wave of frequency ω, and $n(\omega)d\omega$ is the number of standing waves in the interval between ω and $\omega + d\omega$.

3. (2 points) Compute $E(\omega)$. As starting point use that the probability for finding the photon gas with energy in the interval between E and $E+dE$ is $P(E) = \frac{1}{kT}e^{-E/kT}$ where k is the Boltzmann constant.
4. (2 points) Compute $n(\omega)d\omega$ in a cubic cavity of side L taking into account that the photon gas is composed of all the electromagnetic standing waves permitted in the cavity, i. e., $\sum_{\mathbf{k}}, \vec{A}_{\mathbf{k}} e^{i(\vec{k}\vec{r}-\omega t)}$, where the summation extends over all wave numbers producing stationary waves in the cavity, $\vec{k} = \frac{2\pi}{L}(n_x, n_y, n_z),\ n_i = 0, \pm 1, \pm 2, ...$
5. (2 points) Finally, compute[1] the internal energy of the photon gas confined in a cavity of volume V at Temperature T. Obtain the value of the Stefan-Boltzmann constant σ by matching the thermodynamical and statistical results.

[1] You may have to use one of the following integrals

$$\int_0^\infty dx \frac{x^{2n-1}}{e^x - 1} = \frac{(2\pi)^{2n} B_n}{4n}, n = 1, 2, 3...$$

where $B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, B_4 = \frac{1}{30}$.

Solutions

1. The differential form of the Helmholtz potential is given by

$$dF = -SdT - PdV. \tag{2}$$

Using Eqs (1) and (2), it then follows that

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = \left(\frac{4\sigma}{c}\right)\frac{4VT^3}{3}. \tag{3}$$

The Euler form of the Helmholtz potential is

$$F = U - TS \tag{4}$$

By combining Eqs. (1), (3), and (4), it can be derived that

$$U = F + TS = \left(\frac{4\sigma}{c}\right)VT^4. \tag{5}$$

2. The work performed by the gas is

$$W = -\int_V^{2V} PdV \tag{6}$$

As the temperature is constant, Eq. (2) gives $dF = -PdV$, thus

$$W = \int_V^{2V} dF = F(T, 2V) - F(T, V) = -\left(\frac{4\sigma}{c}\right)\frac{VT^4}{4}. \tag{7}$$

The heat transferred during the process is

$$Q = \int TdS = \int T\left(\frac{\partial S}{\partial V}\right)_T dV \tag{8}$$

Using Eq. (3), it can be obtained that

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{4\sigma}{c}\right)\frac{4T^3}{3}. \tag{9}$$

It then follows that

$$Q = \int_V^{2V}\left(\frac{4\sigma}{c}\right)\frac{4T^4}{3}dV = \left(\frac{4\sigma}{c}\right)\frac{4VT^4}{3}. \tag{10}$$

3. The average value of the energy will be

$$\bar{E} = \frac{\sum_n E_n P(E_n)}{\sum P(E_n)}$$

where the sum extends over all possible values of the energy.

The energy of the gas of photons of frequency ω will take the values $\hbar\omega, 2\hbar\omega, 3\hbar\omega, \dots$, according to the number of photons with this frequency present in the gas; then

$$E(\omega) = \frac{\sum_{n=0}^{\infty} n\,\hbar\omega\, e^{-n\hbar\omega/kT}}{\sum_{n=0}^{\infty} e^{-n\hbar\omega/kT}}$$

The denominator is simple to compute: $\sum_{n=0}^{\infty}(e^{-x})^n = (1-e^{-x})^{-1}$, where we introduced the notation $x = \hbar\omega/kT$.

Now, notice the following

$$E(\omega) = kT\,\frac{\sum_{n=0}^{\infty} n\,x\,e^{-nx}}{\sum_{n=0}^{\infty} e^{-nx}} = -kTx\frac{d}{dx}\ln\sum_{n=0}^{\infty}(e^{-x})^n.$$

Therefore,

$$E(\omega) = -\hbar\omega\,\frac{d}{dx}\ln(1-e^{-x})^{-1} = \hbar\omega\frac{1}{e^x-1} = \frac{\hbar\omega}{e^{\hbar\omega/kT}-1}.$$

4. The wave numbers of the possible standing waves have to satisfy

$$(k_x, k_y, k_z) = \frac{2\pi}{L}(n_x, n_y, n_z),\; n_i = 0, \pm1, \pm2, ...$$

Hence, the number Δn_x of possible values of the component k_x in the interval Δk_x is equal simply to the number of values of n_x in the corresponding interval, then

$$(\Delta n_x, \Delta n_y, \Delta n_z) = \frac{L}{2\pi}(\Delta k_x, \Delta k_y, \Delta k_z)$$

The number of possible oscillation modes for which the components of the wave vector lie in the cell of sides $\Delta k_x,\, \Delta k_y,\, \Delta k_z$ is the product of those numbers that is, $\Delta n_x \Delta n_y \Delta n_z = V/(2\pi)^3\Delta k_x \Delta k_y \Delta k_z$. In the same way, the number of modes for which the absolute value of k lies in the range between k and $k+dk$ is $n(k)dk = V/(2\pi)^3\,4\pi\,k^2\,dk$. Using that $\omega = kc$ and multiplying by 2 (since there are two independent polarizations for each mode), we obtain the number of quantum states of photons with frequencies between ω and $\omega + d\omega$:

$$n(\omega)d\omega = \frac{V}{\pi^2c^3}\,\omega^2\,d\omega$$

5. Collecting the above results, we obtain for the internal energy:

$$U(V,T) = \frac{V}{\pi^2c^3}\int_0^\infty d\omega\,\omega^2\,\frac{\hbar\omega}{e^{\hbar\omega/kT}-1} = VT^4\frac{k^4}{\pi^2c^3\hbar^3}\int_0^\infty dx\frac{x^3}{e^x-1} = \frac{\pi^2k^4}{15c^3\hbar^3}VT^4$$

where we used the footnote to get $\int_0^\infty dx\frac{x^3}{e^x-1} = \frac{(2\pi)^4}{8\cdot30}$.

Using thermodynamic considerations we got in question 1 that $U(V,T) = \frac{4\sigma}{c}VT^4$, therefore

$$\sigma = \frac{\pi^2k^4}{60c^2\hbar^3}.$$

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Confining charged particles in free space: The Penning Trap

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Confining charged particles in free space is a challenging task[1].

Circular paths of cyclotron frequency ω_c are achieved in $z = const$ planes by applying a homogeneous magnetic field $\vec{B} = (0, 0, B)$, but the motion in the z direction is not controlled. Nevertheless, this motion could be confined in principle by the restoring force of a harmonic electrostatic potential $V(z) \propto z^2$.

In free space the Laplace equation forces this potential to also depend on the x or y coordinates in somehow prescribed forms. Consider

$$V(x, y, z) = \frac{1}{2} V_0 \left(\frac{z}{d}\right)^2 + f(x, y)$$

where V_0 and d are arbitrary constants and f is a function of x and y to be determined.

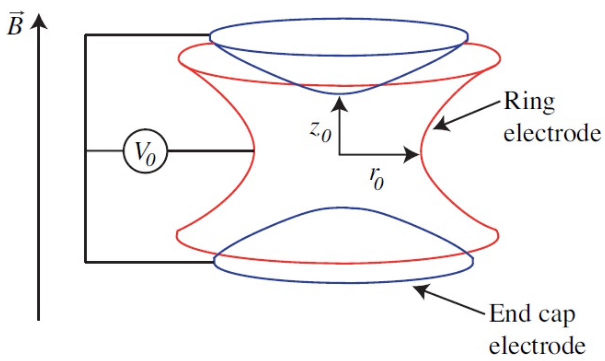
1. (1 point) Determine $f(x, y)$ such that a) the potential satisfies Laplace equation and b) it is rotationally symmetric around the z-axis and homogeneous. Determine the sign of V_0 to get a confining potential on the z-axis.

2. (1 point) Draw[2] equipotential surfaces in the plane xz for $V > 0$ and for $V < 0$. Indicate the values of the potential in these surfaces in terms of V_0, d and their minimal radial and axial distances to the origin of coordinates O (call them $\rho_0, z_0, ...$ etc. as appropriate). Also draw electric field lines in the four quadrants.

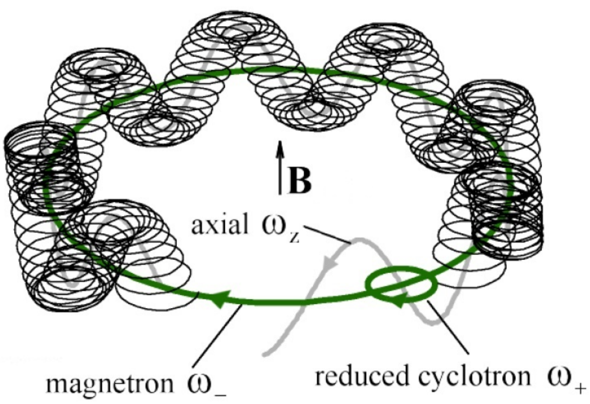
3. (1 point) Consider now two equipotential surfaces passing by the points $C(0, 0, z_0)$ and $R(x_0, 0, 0)$. Determine d and the ratio z_0/x_0 so that the potential difference between them is V_0 and the absolute value of the potential on them be equal, $|V_C| = |V_R|$.

We have seen that using an electrostatic potential it is not possible to generate potential minima in free space but only saddle points. Nevertheless it is possible to achieve three dimensional confinement by combining an electric field with an axial magnetic field.

This is the idea of the Penning trap that is achieved by the superposition of the homogeneous magnetic field[3] $\vec{B} = (0, 0, B)$ and the electric field $\vec{E}$ generated by the static potential $V(x, y, z)$ which was obtained in question 3 above. This potential is generated by simply applying a D. C. voltage V_0 between a conducting surface (the so called ring electrode) and the two conducting surfaces (end cap electrodes), as shown in the figure.



Penning trap setup



Particle trajectory

[1] These particles can be electrons, protons, ions, etc. Hereinafter call q and m to their charges and masses.
[2] A freehand sketch is enough, it not need be scaled, but include what you think is relevant to the question.
[3] Here we will assume that the magnetic field points in the positive direction of the z axis.

4. (0.5 points) Determine the equations describing the ring and the end cap surfaces.

The dynamics of the charged particle in the trap can be derived from the Lagrangian[4]

$$L(\vec{r}, \dot{\vec{r}}) = \frac{1}{2} m \dot{\vec{r}}^2 - q V(\vec{r}) + q \dot{\vec{r}} \cdot \vec{A}(\vec{r})$$

5. (1.5 points) Get the equations of motion of the charged particle in the trap in terms of the axial frequency along the z axis, ω_z, and the cyclotron frequency, ω_c, of the particle. Determine the frequencies of this motion, calling ω_+ (reduced cyclotron frequency) and ω_- (magnetron frequency) to the highest and the lowest of them respectively. Determine the trapping condition for the transverse motion in the x, y plane, i. e., the minimum value of the magnetic field such that the particle motion on the plane is bounded.

6. (2 points) Obtain the angular momentum $\vec{J}$ of and the time dependence of its z component J_z (consider only spinless particles).

7. (3 points) Show that the resulting transverse trajectory of the particle is the superposition of a rapid cyclotron motion and a slow magnetron motion. In general the transverse motion will be quasiperiodic. Under what conditions will the three-dimensional movement be periodic?. Determine the trajectory for initial conditions $\vec{r} = (\rho_0, 0, 0)$ and $\vec{v} = (0, 0, 0)$ at $t = 0$. Determine the bounds of the radial variable $\rho = \sqrt{x^2 + y^2}$. Finally, find J_z and the particle energy.

[4]For a homogeneous magnetic field the magnetic potential can be taken as $\vec{A} = \frac{1}{2}\vec{B} \times \vec{r}$.

SOLUTION

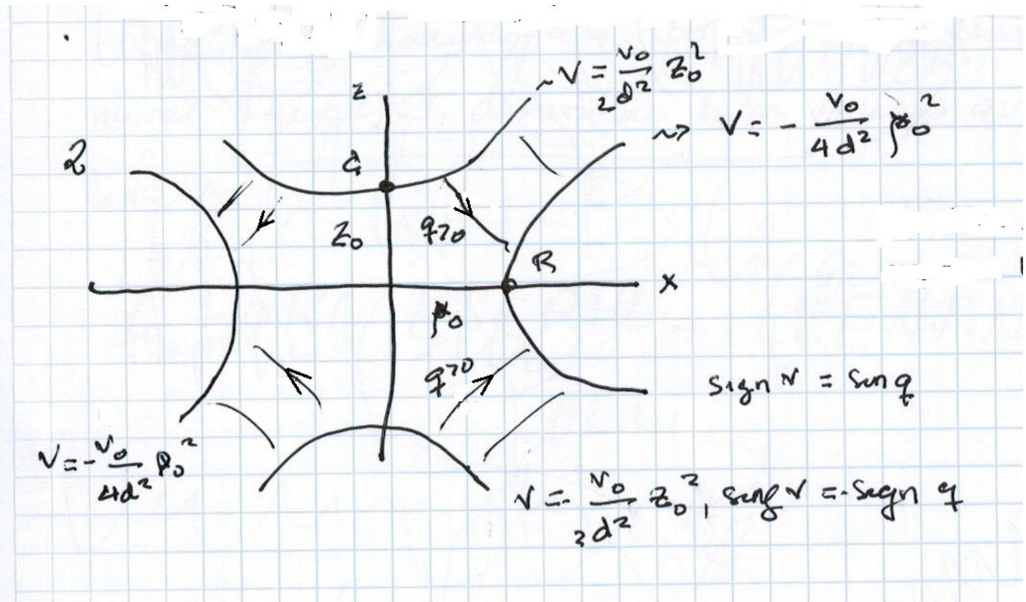
1. Determine $f(x, y)$

- f is rotationally symmetric $\rightarrow$ $f(x, y) = f(\rho)$ where $\rho = \sqrt{x^2 + y^2}$.
- f is homogeneous $\rightarrow$ $f(x, y) = \Phi \rho^n$ where Φ, n are constant to be determined.
- Laplace equation: $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{V_0}{d^2} = 0 \quad \rightarrow \quad n = 2 \text{ and } \Phi = -\frac{V_0}{4d^2}$.

Finally, $V(x, y, z) = \frac{V_0}{2d^2}\left(z^2 - \frac{1}{2}\rho^2\right)$ whith $\rho = \sqrt{x^2 + y^2}$.

The force on the z axis is $F_z = -(\frac{qV_0}{d^2})z$; it is a confining force if it is restoring $(\frac{qV_0}{d^2}) > 0 \rightarrow$ sign $V_0 =$ sign q.

2. Drawings



For $q < 0$, the electric field lines flow in the opposite direction to that shown in the drawing.

3.

- $V_C - V_R = \frac{V_0}{2d^2}(z_0^2 + \frac{1}{2}\rho_0^2) = V_0 \rightarrow d^2 = \frac{1}{2}(z_0^2 + \frac{1}{2}\rho_0^2)$
- $|V_C| = |V_R| \rightarrow z_0^2 = \rho_0^2/2 \rightarrow (\frac{z_0}{\rho_0}) = \frac{1}{\sqrt{2}}$

4.

- End cap surfaces $z^2 - \frac{1}{2}(x^2 + y^2) = z_0^2$
- Ring surface $z^2 - \frac{1}{2}(x^2 + y^2) = -\frac{1}{2}r_0^2$

5. From the Lagrangian $L(\vec{r}, \dot{\vec{r}}) = \frac{1}{2}\, m\, \dot{\vec{r}}^2 - q\, V(\vec{r}) - \frac{1}{2} q\, \dot{\vec{r}} \cdot \left(\vec{r} \times \vec{B}\right)$

- $\vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}} - \frac{1}{2}q\left(\vec{r} \times \vec{B}\right)$
- $\vec{K} = \frac{\partial L}{\partial \vec{r}} = -q\vec{\nabla}V + \frac{1}{2}q\left(\dot{\vec{r}} \times \vec{B}\right)$

Then, the Lagrange equations, $\dot{\vec{p}} = \vec{K}$, give

$$m\ddot{\vec{r}} - \frac{1}{2}q\left(\dot{\vec{r}} \times \vec{B}\right) = -q\,\vec{\nabla}V + \frac{1}{2}q\left(\dot{\vec{r}} \times \vec{B}\right),$$

written in components:

- $\ddot{x} = \frac{1}{2}\omega_z^2\,x + \epsilon\,\omega_c,\,\dot{y}$
- $\ddot{y} = \frac{1}{2}\omega_z^2\,y - \epsilon\,\omega_c\,\dot{x}$
- $\ddot{z} = -\omega_z^2\,z$

where ϵ is the sign of the particle's charge, $q = \epsilon|q|;$, $\omega_c = |q|B/m$ and $\omega_z = \sqrt{qV_0/md^2}$ are the cyclotron and axial frequencies respectively. The motions transversal and along the z axis remain decoupled. The equations for x and y can be rewriten as

$$\ddot{\vec{\rho}} + i\epsilon\omega_c\sigma_2\dot{\vec{\rho}} - \frac{1}{2}\omega_z^2\vec{\rho} = 0$$

where $\vec{\rho} = (x, y)^\top$ is projection of the vector position onto the plane xy and σ_2 is the Hermitian Pauli matrix. Instead, they can also be put in terms of the complex variable $\rho = x + iy$

$$\ddot{\rho} + i\epsilon\omega_c\dot{\rho} - \frac{1}{2}\omega_z^2\rho = 0$$

which will be used hereafter. Taking $\rho \propto e^{i\omega t}$ gives

$$\omega^2 + \epsilon\omega_c\omega + \frac{1}{2}\omega_z^2 = 0, \Rightarrow \omega = \epsilon\omega_\pm \;\text{ where }\; \omega_\pm = \frac{\epsilon}{2}\left(\omega_c \pm \sqrt{\omega_c^2 - 2\omega_z^2}\right)$$

In order that the motion be bounded both frequencies have to be real, leading to the trapping condition

$$\omega_c^2 - 2\omega_z^2 > 0 \quad \rightarrow \quad B^2 > \frac{2mV_0}{qd^2}$$

which gives the minimal magnetic field able to compensate the runaway effect of the radial electric field.

6. The angular momentum is $\vec{J} = \vec{r} \times \vec{p}$. Taking eq.(1) into account:

$$\vec{J} = m\,\vec{r} \times \dot{\vec{r}} + q\,\vec{r} \times \vec{A} = m\,\vec{r} \times \dot{\vec{r}} + \frac{1}{2}q\,\vec{r} \times \left(\vec{B} \times \vec{r}\right)$$

In components:

- $J_x = m\left((y\dot{z} - \dot{y}z) - \frac{\omega_c}{2}xz\right)$
- $J_y = m\left((z\dot{x} - \dot{z}x) - \frac{\omega_c}{2}xy\right)$
- $J_z = m\left((x\dot{y} - \dot{x}y) - \frac{\omega_c}{2}(x^2 + y^2)\right)$

The z component is constant, $\dot{J}_z = m\left((x\ddot{y} - \ddot{x}y) - \omega_c(x\dot{x} + y\dot{y})\right) = 0$ as can be seen after replacing in $\ddot{x}, \ddot{y}$ the values given by the Lagrange equations. From the initial conditions of the problem (given in question 7 below) we can get its value, $J_z = \frac{1}{2}m\omega_c\rho_0^2$.

7. The solution can be finally written as the superposition of the rapid cyclotron oscillation with frequency ω_+ and the slower magnetron oscillation with frequency ω_-.

$$\rho = \rho_+ e^{i\epsilon\omega_+ t} + \rho_- e^{i\epsilon\omega_- t}$$

where the constants $\rho_\pm$ will be determined by the initial conditions. The motion will be periodic when at some time t, after an integer number of periods T_z, T_+, and T_- have elapsed, the particle returns to its initial conditions. This requires that ω_-/ω_+ and ω_z/ω_+ are rational numbers.

The initial conditions $\vec{r} = (\rho_0, 0, z_0)\text{and } \vec{v} = (0,0,0)$ give

$$\rho_+ = -\frac{\omega_-}{\omega_+ - \omega_-}\rho_0, \text{ and } \rho_- = \frac{\omega_+}{\omega_+ - \omega_-}\rho_0$$

Finally, the path followed by the particle is:

- $x = \frac{\rho_0}{\omega_+ - \omega_-}\left(-\omega_- \cos\omega_+ t \ + \omega_+ \cos\omega_- t\right)$
- $y = \frac{\epsilon\rho_0}{\omega_+ - \omega_-}\left(-\omega_- \sin\omega_+ t \ + \omega_+ \sin\omega_- t\right)$
- $z = z_0 \cos\omega_z t$

The radial variable is

$$|\rho| = \frac{\rho_0}{\omega_+ - \omega_-}\sqrt{\omega_+^2 + \omega_-^2 - 2\omega_+\omega_- \cos(\omega_+ - \omega_-)t}$$

so that

$$\rho < \ |\rho| \ < \frac{\omega_+ + \omega_-}{\omega_+ - \omega_-}\rho_0$$

The particle energy is a constant; it can be obtained from the initial conditions:

$$E = \frac{1}{2}m\dot{r}^2 + qV(\vec{r}) = \frac{1}{2}mv^2 - \frac{1}{4}m\omega_z^2\rho_0^2$$

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Quantum Zeno effect

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By quantum Zeno effect we refer to the "freezing" of the observed dynamics of a quantum system when it is subjected to very frequent quantum measurements.

[*The first paragraphs are a brief introduction to quantum measurements - you can skip it if you are familiar with it*]

The measurement process is described in standard quantum physics by its own postulate. Imagine that you have a quantum system in a pure state $|\psi\rangle$. An ideal measuring apparatus is a device that measures an associated observable, O. The latter is an Hermitian operator with eigenstates ϕ_1, ϕ_2, ..., and eigenvalues o_1, o_2, ... We can express the quantum state as a linear superposition of observable eigenstates, $|\psi\rangle = \sum_n |c_n|^2 |\phi_n\rangle$. Then, according to the postulate, a quantum measurement is a probabilistic process whose possible outcomes are the eigenvalues of O, o_1, o_2, ..., which happen with probability $p = |c_1|^2$, $|c_2|^2$, ..., respectively.

As an example consider a two-level system. A complete basis spanning the Hilbert space is given by the two states,

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For example, $|1\rangle$ and $|2\rangle$ could be two atomic levels. We define the observable σ_z,

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which is also known as z-Pauli matrix. Imagine an atom and is initially in a linear superposition state,

$$|\psi\rangle = c_1|1\rangle + c_2|2\rangle,$$

and then we measure σ_z (this could be done, for example, by measuring the photoluminescence emitted by the atom, which may be level-dependent and, thus, be used to detect the atomic state). The measurement outcome will be 1 or -1 with probability $|c_1|^2$ or $|c_2|^2$, respectively.

[*OK, enough revision - this is the problem.*]

Consider now a two-level system subjected to the Hamiltonian,

$$H = \frac{\hbar\Omega}{2}\sigma_x, \tag{1}$$

where σ_x is the x-Pauli matrix

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

1. (2 points) The two level system is initially in the state $|\psi(0)\rangle = |1\rangle$. Calculate the evolution of the quantum state, $|\psi(t)\rangle$.
2. (4 points) Consider now that the system is initially in the state $|\psi(0)\rangle = |1\rangle$. You perform N measurements of σ_z separated by very short intervals of time, Δt, up to a total time $t = N\Delta t$. Δt is very short compared to the typical time scale of Hamiltonian evolution, $\Delta t\, \Omega \ll 1$. We define $P_1^{(N)}(t)$ as the probability that *all* the N measurement outcomes are $\sigma_z = 1$. This is the same as saying that the system remains in state $|1\rangle$ during the whole experiment. Calculate $P_1^{(N)}(t)$ and find out a time at which $P_1^{(N)}(t) = e^{-1}$, with e the base of the natural logarithm. Hint: you have to use the following limit,
$$e^{-x} = \lim_{n\to\infty}\left(1 - \frac{x}{n}\right)^n \; .$$

3. (1 point) Very briefly explain how the result in the previous part implies that the dynamics of the quantum system is slowed down by the measurement process.

4. (1 points) Consider now that the system is not subjected to any Hamiltonian dynamics, that is, $\Omega = 0$. However, the state $|1\rangle$ is unstable and it can decay to the level $|2\rangle$, for example, by radiative emission of photons. This means that if, the system is initially at state $|1\rangle$, the probability that it stays in $|1\rangle$ after a time t is $p_1(t) = e^{-\gamma t}$, with γ the decay rate.

   Imagine now that you monitor the system in the same way as in part (ii). That is, you measure the value of σ_z at very short time intervals Δt. What is the now the probability $P_1^{(N)}(t)$ that *all* measurement outcomes are $\sigma_z = +1$. Compare the result of (iv) with the one of part (ii) form the perspective of the slowing down of the system's dynamics.

5. (2 points) Is the zeno effect a general property, i. e., valid for arbitrary states in a two-level system with a Hamiltonian linear in the Pauli's matrices, like the one given in (1)? Check if this is the case.

1. SOLUTION. -

$$|\psi(t)\rangle = \cos(\Omega t/2)|1\rangle - i\sin(\Omega t/2)|2\rangle$$

(This can be obtained by explicitly solving the Schroedinger equation with $c_1(0) = 1$, $c_2(0) = 0$)

2. SOLUTION.- Everytime you measure, the probability of getting a $+1$ outcome is $\cos(\Omega\Delta t/2)^2$ (because we are assuming that the previous measurement outcome was also $+1$). Thus,

$$P_1^{(N)}(t) = \cos(\Omega\Delta t/2)^{2N} \approx \left(1 - \frac{1}{2}(\Omega\Delta t/2)^2\right)^{2N} = \left(1 - \frac{(\Omega^2\Delta t)t}{8N}\right)^{2N}$$

The limit $N \to \infty$ is

$$P_1^{(N)} \to e^{-(\Omega^2\Delta t/4)t}$$

An the time needed for that probability to get to e^{-1} is just the inverse of the Zeno decay rate $\Omega^2\Delta t/4$.

Variations of this problem could include questions like, what is the probability of measuring $+1$ at time t, irrespective of the outcome of previous measurements. This is a very typical problem in Markovian processes, but it involves using the binomial distribution and it turns into a mathematical problem. Another question could be: what is the average number of "quantum jumps" between 0 and t, or what is the probability distribution of the number of jumps (this is a Poissonian)

3. SOLUTION.- There is an effective decay rate that decreases with Δt, thus, it tends to zero as the time interval between measurements decreases. Actually, the typical evolution time without measurements is Ω^{-1}, which becomes longer by a factor $1/(\Omega\Delta t)$ if the system is being observed.

4. SOLUTION.- Same as before, but now

$$P_1^{(N)}(t) = \left(e^{-\gamma\Delta t}\right)^N = e^{-\gamma t}$$

So, no slowing down, just the same decay as when the system is not observed.

5. SOLUTION.- Any generic Hamiltonian $H = \frac{\hbar\Omega}{2}\hat{n}\cdot\vec{\sigma}$ can be put in the form (1) by means of a rotation, therefore it is enough to consider this Hamiltonian (1) with an arbitrary initial state of the form $|\psi(0)\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ with $|a|^2 + |b|^2 = 1$. The σ_x eigenvectors are $\lambda = \pm 1$ and its eigenstates

$$|\lambda\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

that we called before $|1\rangle, |2\rangle$ for $\lambda = 1, -1$ respectively. In this basis

$$|\psi(0)\rangle = \frac{(a+b)}{\sqrt{2}}|1\rangle + \frac{(a-b)}{\sqrt{2}}|2\rangle.$$

We can write

$$H = \sum_{\lambda=\pm 1} E_\lambda |\lambda\rangle\langle\lambda|, \Rightarrow e^{-i\frac{H}{\hbar}t} = \sum_{\lambda=\pm 1} e^{-i\frac{E_\lambda}{\hbar}t}|\lambda\rangle\langle\lambda|$$

with $E_\lambda = \lambda\hbar\Omega/2$, therefore

$$|\psi(t)\rangle = \frac{(a+b)}{\sqrt{2}}e^{-i\frac{\Omega t}{2}}|1\rangle + \frac{(a-b)}{\sqrt{2}}e^{+i\frac{\Omega t}{2}}|2\rangle.$$

It is straightforward to get

$$\langle\psi(0)|\psi(t)\rangle = \frac{|a+b|^2}{2}e^{-i\frac{\Omega t}{2}} + \frac{|a-b|^2}{2}e^{i\frac{\Omega t}{2}},$$

so the probability of finding the initial state after a time t is

$$P_{\psi(0)}(t) = |\langle\psi(0)|\psi(t)\rangle|^2 = \frac{1}{2}(1+R^2) + \frac{1}{2}(1-R^2)\cos\Omega t = \cos^2\frac{\Omega t}{2} + R^2\sin^2\frac{\Omega t}{2}$$

where we used that $|a|^2 + |b|^2 = 1$ and[1] $R := ab^* + a^*b$. For very small times $\Delta t \ll 1/\Omega$

$$P_{\psi(0)}(\Delta t) \approx 1 - \frac{1}{4}(1-R^2)(\Omega\Delta t)^2$$

Then, the probability of a sequence of N measurements spaced by very small time intervals Δt (so that $t = N\Delta t$), getting always the initial state is

$$P^N_{\psi(0)}(\Delta t) = \left(1 - \frac{(1-R^2)\Omega^2\Delta t}{4N}t\right)^N = e^{-\frac{(1-R^2)\Omega^2\Delta t}{4}t}$$

Finally, the result for the general case with a, b arbitrary, is the tiny decay rate $\frac{(1-R^2)\Omega^2\Delta t}{4} \leq \frac{\Omega^2\Delta t}{4} \ll \frac{\Omega}{2}$. For vanishing a or b, $R = 0$, recovering the previous result

[1] With all generality $a = e^{i\alpha}\cos\varphi, b = e^{i\beta}\sin\varphi, \Rightarrow R = \cos(\alpha-\beta)\sin 2\varphi, \; -1 \leq R \leq 1$. The cases $a = \pm b \Rightarrow R^2 = 1$ correspond to eigenstates of the Hamiltonian and there is no evolution