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Perihelion shift in Special Relativity

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The relativistic Lagrangian for a particle of mass m in motion in the central potential $V(r) = -k/r$ (with $k > 0$) is (in natural units $c = 1$).

$$L = -m\sqrt{1-\dot{\mathbf{r}}^2} - V(r) \qquad (r = |\mathbf{r}|).$$

1) Determine the quantitites $\mathbf{J}$ and E that are conserved as a result of the invariance of the Lagrangian under rotations and time translations (i.e., the relativistic angular momentum and energy respectively). Show that, as in the non relativistica case, the motion remains within a plane passing through the origin of coordinates. [**0.5 points**]

Solución:

$$\mathbf{J} = \mathbf{r} \times \frac{\partial L}{\partial \dot{\mathbf{r}}} = \frac{m\mathbf{r} \times \dot{\mathbf{r}}}{\sqrt{1-\dot{\mathbf{r}}^2}} \equiv T\mathbf{r} \times \dot{\mathbf{r}}\,, \qquad E = \dot{\mathbf{r}} \cdot \frac{\partial L}{\partial \dot{\mathbf{r}}} - L = \frac{m}{\sqrt{1-\dot{\mathbf{r}}^2}} - \frac{k}{r} \equiv T - \frac{k}{r}\,,$$

where $T = m/\sqrt{1-\dot{\mathbf{r}}^2}$ is the relativistic kinetic energy. Due to $\mathbf{J}$ and orthogonality between $\mathbf{r}$ and $\mathbf{J}$, the motion is in the plane orthogonal to $\mathbf{J}$ that goes through the origin.

2) Taking the plane of motion so that $\mathbf{J} = J\mathbf{e}_z$ with $J = |\mathbf{J}| > 0$, express the relativistic kinetic energy of the particle T as a function of J, r and $\dot{r}$. [**1 point**]

Solución: Be r y θ the polar coordinates in the plane of motion $z = 0$,

$$J = Tr^2\dot{\theta} \implies \dot{\mathbf{r}}^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 + \frac{J^2}{T^2r^2} \implies m^2 = T^2(1-\dot{\mathbf{r}}^2) = T^2(1-\dot{r}^2) - \frac{J^2}{r^2}$$
$$\implies T = \sqrt{\frac{m^2 + J^2/r^2}{1-\dot{r}^2}}\,.$$

3) Let $1/r = u(\theta)$ be the equation of the orbits. Using both, conservation of energy and relativistic angular momentum, find the first order differential equation satisfied by $u(\theta)$. (*Help:* according to subsection 2) $T^2 = A(r) + T^2\dot{r}^2$ for a certain function $A(r)$.) en [**2.5 points**]

Solución:

$$(E+ku)^2 = T^2 = m^2 + J^2u^2 + T^2\dot{r}^2 = m^2 + J^2u^2 + T^2\dot{\theta}^2 r'(\theta)^2 = m^2 + J^2u^2 + J^2u^4 r'(\theta)^2$$
$$= m^2 + J^2u^2 + J^2u'(\theta)^2 \implies u'(\theta)^2 = \frac{1}{J^2}\Big[(E+ku)^2 - m^2 - J^2u^2\Big]\,.$$

4) Let's assume from now on that (as it happens in the planetary motion) $k < J$. Using the equation in the previous section, prove that the relativistic energy E is positive. [**1 point**]

Solución: We can write the previous equation as follows

$$u'^2 = -\gamma^2u^2 + \frac{2kE}{J^2}u + \frac{E^2-m^2}{J^2} \equiv P(u)\,, \qquad \gamma^2 \equiv 1 - \frac{k^2}{J^2} > 0\,.$$

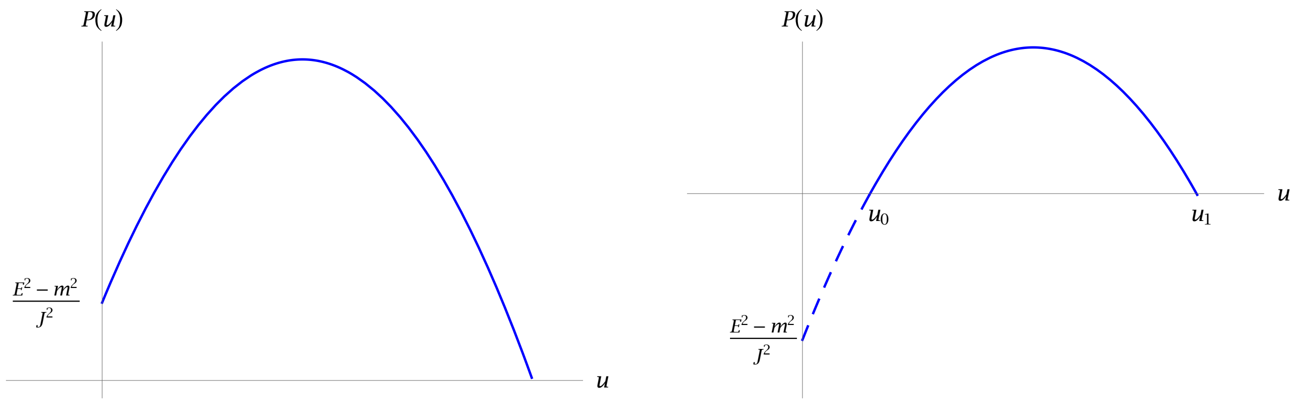

Figure 1: gráfica de $P(u)$ para $E \geq m$ (izda.) y $m\gamma \leq E < m$ (drcha.).

For $E \leq 0$, this equation implies that $E^2 \geq m^2$. In this case, the point $u = 0$, that is $r = \infty$, would be within the accesible region $P(u) \geq 0$. However, if the particle reaches the spatial infinity $E = T|_{r=\infty} > 0$, which is against the hypothesis that $E \leq 0$.

5) Determine for which values of E and J the particle describes a bounded orbit that does not reach the origin of coordinates. **[1 point]**

Solución: As seen in the previous question, the equation for the accesible region is $P(u) \geq 0$ and the energy E is positive. With $P(0) = (E^2 - m^2)/J^2$ and for $E \geq m$, the trajectories either fall in the origin, or they reach infinity, or both (cf. la Fig. 1 izda.). Hence, bound orbits with $0 < r_0 \leq r \leq r_1 < \infty$ should have an energy $E \in (0, m)$. Besides, the polynomial $P(u)$ must have two positive real roots. As in this case $P(0) < 0$ and $P'(0) = 2kE/J^2 > 0$, the necessary and sufficient condition for this to occur is that the discriminant of P is not negative, that is

$$\left(\frac{kE}{J^2}\right)^2 + \frac{\gamma^2}{J^2}(E^2 - m^2) \geq 0 \iff k^2E^2 + \gamma^2 J^2 (E^2 - m^2) \geq 0 \iff E \geq m\gamma\,.$$

(cf. Fig. 1 dcha.). Note that this last condition is consistent with the previous one ($0 < E < m$), since $\gamma < 1$. In short, the requested conditions are

$$m\gamma \leq E < m\,.$$

6) Find the equation of the bounded orbits in the previous section. (*Help*: derive the first-order equation for $u(\theta)$ and integrate the resulting second order differential equation.) **[2 points]**

Solución:

$$2u'u'' = u'P'(u) \implies u'' = \frac{1}{2}P'(u) = -\gamma^2 u + \frac{kE}{J^2} \iff u'' + \gamma^2 u = \frac{kE}{J^2} \implies u = \frac{kE}{\gamma^2 J^2} + A\cos(\gamma(\theta - \theta_0))\,,$$

where θ_0 is an integration constant and the constant A is easily calculated by substituting it into the first-order differential equation:

$$A = \frac{\sqrt{E^2 - m^2\gamma^2}}{\gamma J}$$

(Note that A can be taken non-negative without loss of generality, because if it were not it would suffice to change θ_0 for $\theta_0 + \pi/\gamma$).

7) A periapsis of a bounded orbit is a point of the orbit at minimum distance from the origin. The displacement of the periapsis of a bounded orbit is defined as $\Delta\theta = \delta\theta - 2\pi$, where $\delta\theta$ is the angle between two successive periapses. Calculate the displacement of the periapsis of the orbits of the previous section. en [**1 point**]

Solución: nótese que las órbitas del apartado anterior solo son cerradas si u es una función periódica de θ, es decir si γ es un múltiplo racional de 2π. Como $A \geq 0$, el ángulo θ en un periápside de la órbita está dado por

Note that the orbits of the previous section are only closed if u is a periodic function of θ, that is, if γ is a rational multiple of 2π. As $A \geq 0$, the angle θ in a periapsis of the orbit is given by

$$\theta = \theta_0 + \frac{2k\pi}{\gamma}, \qquad \text{con } k \in \mathbb{Z}\,.$$

Por tanto

$$\delta\theta = \frac{2\pi}{\gamma} \quad \Longrightarrow \quad \Delta\theta = 2\pi\left(\frac{1}{\gamma} - 1\right) > 0.$$

8) In planetary motion, $\gamma \simeq 1$ and closed orbits differ very little from the non-relativistic ellipses. Express approximately in this case the displacement of the periapsis as a function of the semimajor axis of the orbit a and its eccentricity e. [**1 point**]

Solución: As the closed orbits are approximately elliptical for $\gamma \simeq 1$, we can use as a very good approximation the relativistic formula relating J to a and e, that is

$$J^2 = mka(1-e^2)\,.$$

Substituting in the formula of the previous question, we get

$$\Delta\theta \simeq 2\pi(1-\gamma) = 2\pi\left(1 - \sqrt{1 - \frac{k^2}{J^2}}\right) \simeq \frac{\pi k^2}{J^2} = \frac{\pi k}{ma(1-e^2)} = \frac{\pi GM}{a(1-e^2)}\,,$$

where M is the Sun mass.

Comments.

- Restoring the speed of light we get

$$\Delta\theta = \frac{\pi GM}{c^2 a(1-e^2)}\,,$$

  This is exactly $1/6$ od the correct result obtained using General Relativty. This problem therefore demonstrates that Special Relativity is insufficient to explain the correct value of the perihelion displacement of the planets with an acceptable approximation.

- En el caso del planeta Mercurio se puede comprobar fácilmente que la aproximación utilizada ($k \ll J$) para obtener la fórmula de $\Delta\theta$ del último apartado está plenamente justificada. En efecto, en primer lugar la velocidad máxima de Mercurio es $v_{\max} = 58.98\ \mathrm{km} < 2 \cdot 10^{-4} c$, por lo que los efectos relativistas son pequeños y podemos calcular J utilizando la ecuación no relativista del apartado anterior. En tal caso (restaurando de nuevo la velocidad de la luz)

  In the case of the planet Mercury, it can be easily verified that the approximation used ($k \ll J$) to obtain the formula of $\Delta\theta$ in the last section is fully justified. In effect, Mercury's maximum speed is $v_{\max} = 58.98\ \mathrm{km} < 2 \cdot 10^{-4} c$, so the relativistic effects are small and we can calculate J using the non-relativistic equation of the previous section. In such a case (restoring the speed of light again)

$$\frac{k^2}{J^2 c^2} = \frac{R}{a(1-e^2)}\,,$$

  where $R = GM/c^2 = 1.477\ \mathrm{km}$ is the gravitational radius of the Sun. Using the values

$$a = 5.791 \cdot 10^7\ \mathrm{km}\,, \qquad e = 0.2056$$

  we get

$$\frac{k^2}{J^2 c^2} = 1.24 \cdot 10^{-8}.$$

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Refrigerators, efficiency and optimization

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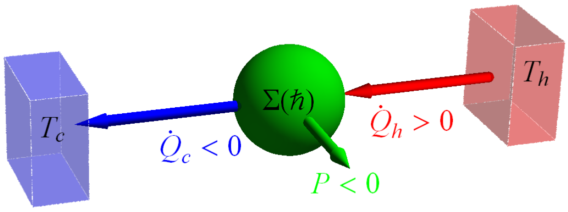

Figure 1: Representation of a refrigerator. Energy currents are indicated with the sign convention used.

Introduction and context

Consider a quantum system Σ in contact with two different thermal baths at temperatures T_c (cold) and T_h (hot) respectively, with $T_c < T_h$. When the whole system (machine) is operating as a refrigerator some power P is injected in Σ, some energy per unit time (energy current) $\dot{Q}_h$ is extracted from the system and delivered to the hot bath, and energy per unit time $\dot{Q}_c$ is extracted from the cold bath towards Σ. It is assumed that the whole system is operating at stationary conditions.

The energy per unit time that enters the system Σ is taken positive. In figure (1) the machine energy fluxes between baths and Σ are schematically represented. As Σ is a quantum system in contact with thermal reservoirs two important constants of the problem are the Planck's constant h and the Boltzmann's constant k_B. Notice that conservation of energy demands that

$$\dot{Q}_c + \dot{Q}_h + P = 0.$$

Three level maser

A simple choice for Σ is a three level system with Bohr frequencies ω_c, ω_h and $\omega = \omega_h - \omega_c$. A periodic driving with frequency ω is externally applied as power source. For instance, by applying a coherent laser field. As a matter of fact, Σ exchanges photons of energy $\hbar\omega_c$ and $\hbar\omega_h$ with the cold and hot baths respectively. From now on we shall set $\hbar = h/2\pi = k_B = 1$.

In addition, the Second Law of Thermodynamics states that the entropy production $\dot{S}$ (entropy produced per unit time) of the process satisfies

$$\dot{S} = -\frac{\dot{Q}_c}{T_c} - \frac{\dot{Q}_h}{T_h} \geq 0. \qquad (1)$$

For the three-level maser, in certain conditions that involve the high temperature limit, it is found that

$$\begin{aligned} \dot{Q}_c &= \omega_c I \\ \dot{Q}_h &= -\omega_h I \\ P &= -(\omega_c - \omega_h)I, \text{ with} \\ I &= \kappa \left(\frac{\omega_c}{\omega_h}\right)^{d-1} \left(\frac{\omega_h}{T_h} - \frac{\omega_c}{T_c}\right), \; d = 1,2,3 \quad \text{and } \kappa = \text{constant} > 0 \end{aligned} \qquad (2)$$

Suppose that all parameters are fixed except ω_c —for which it is assumed that a good experimental control is possible —and answer the following questions.

Cooling window

Show that the maximum frequency ω_c at which the machine can operate as a refrigerator is

$$\omega_{c,Max} = \omega_h \frac{T_c}{T_h}. \tag{3}$$

The set of frequencies within $(0, \omega_{c,Max})$ are named as the *cooling window* of the machine. [**1.5 points**]

Efficiency

Show that the efficiency of the refrigerator, defined as $\epsilon = \dot{Q}_c/P$, is such that $\epsilon \in [0, \epsilon_C]$, where

$$\epsilon_C = \frac{T_c}{T_h - T_c}$$ [**1.5 points**]

Maximum cooling power

At what frequency ω_c^* the extraction of energy per unit time from the cold bath is maximum? [**2 points**]

Efficiency at maximum cooling power

When the machine operates at frequency ω_c^*, what is the limit of the ratio

$$\frac{\epsilon(\omega_C^*)}{\epsilon_C} \tag{4}$$

when $\epsilon_C \to 0$? [**4 points**]

Tuning

What would happen if ω_c is tuned within the interval $(\omega_{c,Max}, \omega_h)$? [**1 point**]

Solutions

Cooling window

The machine operates as a refrigerator when $\dot{Q}_c > 0$ which is equivalent to $I > 0$. This means that

$$\left(\frac{\omega_h}{T_h} - \frac{\omega_c}{T_c}\right) > 0. \tag{5}$$

If all parameters are fixed, the previous condition implies that $\omega_c \leq \omega_h \frac{T_C}{T_h} = \omega_{c,Max}$.

Efficiency

The efficiency of this machine equals

$$\epsilon = \frac{\omega_c}{\omega_h - \omega_c}, \tag{6}$$

which is a monotonically increasing function of ω_c. Hence the maximum efficiency is reached for $\omega = \omega_{c,Max}$ frequency at which

$$\epsilon(\omega_{c,Max}) \equiv \epsilon_C = \frac{T_c}{T_h - T_c}, \tag{7}$$

Maximum cooling power

The energy per unit time extracted form the cold bath is $\dot{Q}_c(\omega_c)$ that reaches its maximum when the following conditions are satisfied,

$$\frac{\partial \dot{Q}_c}{\partial \omega_c} = 0 \quad \text{with} \quad \frac{\partial^2 \dot{Q}_c}{\partial \omega_c^2} < 0. \tag{8}$$

From the first condition in (8) it is obtained that for

$$\omega_c^* = \frac{d}{d+1}\omega_h \frac{T_c}{T_h}$$

, $\dot{Q}_c$ has an extreme. To confirm that such extreme is a maximum, we notice that for the range of frequencies $\omega_c \in [0, \omega_{c,Max}]$, $\dot{Q}_c$ is positive and that its only zeroes are located at the boundaries $\omega_c = 0$ (zero with multiplicity d) and $\omega_c = \omega_{c,Mac}$. Hence the extreme value of $\dot{Q}_c$ is a maximum.

The fact that the located extreme is a maximum can also be worked out explicitly evaluating the second derivative referred in (8).

Efficiency at maximum cooling power

The efficiency at ω_c^* is given by

$$\epsilon(\omega_c^*) = \frac{\alpha T_c}{T_h - \alpha T_c}, \quad \text{with} \quad \alpha = \frac{d}{d+1}. \tag{9}$$

As we are asked for the behavior of $\epsilon(\omega_c^*)$ for small ϵ_C. We can use the definition $\epsilon_C = T_c/(T_h - T_c)$ to get

$$T_c = T_h \frac{\epsilon_C}{1+\epsilon_C}, \tag{10}$$

that for small ϵ_C can be approximated by

$$T_c \approx T_h \epsilon_C (1 - \epsilon_C) + O(\epsilon_C^3) \tag{11}$$

By using this last expression in equation 9 the result follows, $i.e.$

$$\frac{\epsilon(\omega_c^*)}{\epsilon_C} \approx \frac{d}{d+1}, \quad \text{for small } \epsilon_C \tag{12}$$

Tuning

When $\omega_c \in (\omega_{c,Max}, \omega_h)$ it follows from equations (2) that $\dot{Q}_c < 0$, $\dot{Q}_h > 0$ and $P < 0$ which means that in this case the machine will operate as an engine, delivering photons to the laser field, energy that can be use to produced work.

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Motion of charged particles in electromagnetic fields

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Perpendicular, uniform, and constant electric and magnetic fields are applied on a particle of electric charge Q that is at rest at the origin of coordinates of an inertial system K, (assume that $\vec{E} = E\vec{j}$ and $\vec{B} = B\vec{k}$),

(a) Explain qualitatively what the particle motion will be. **(1 point)**

(b) Check the previous explanation by solving the equations of motion in the non-relativistic approximation in which the velocity of the particle is $v \ll c$. What conditions have to be met for this approach to be adequate? **(2 points)**

(c) Calculate the motion of the particle in the relativistic case. **(4 points)**
Help: Make a Lorentz transformation to an inertial system K' in which the electric field is zero. Determine the motion of the particle in that system and then transform back to the original system K. Assume that $E < cB$.

(d) Compare the previous relativistic result with that of the non-relativistic section (b). **(1 point)**

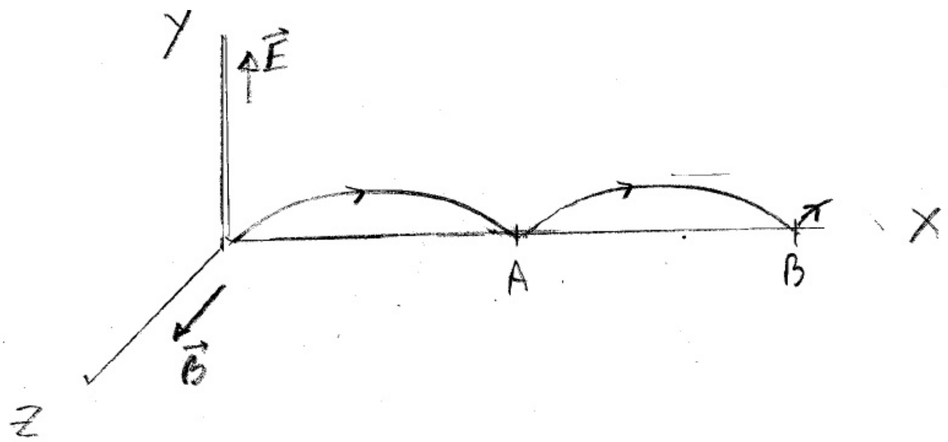
(e) Compare the relativistic results in K and K' obtained in section (c), discussing what it takes to repeat the motion according to each system, and also the orbit that describes the particle. **(2 points)**

Solution

(a) At rest $\vec{F}_E = QE\vec{j}$. For v along the Y axis there is an additional magnetic force $\vec{F}_B = Q(v\vec{j}) \times (B\vec{k}) = QvB\vec{i}$ pushing the particle in the direction of the X axis.

The faster it goes, the greater $\vec{F}_B$ is made and eventually the particle's trajectory curves back to the X axis. The charge now moves against $\vec{F}_E$, loses speed, $\vec{F}_B$ decreases and $\vec{F}_E$ takes control, leading the charge to rest at point A of the figure.

The process then begins again, taking the particle to point B, etc.



1.jpg

(b) $\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})$

- The electric force follows the Y axis.
- The magnetic force is perpendicular to B and therefore it is exerted in the XY plane.

- The particle is initially at rest. Hence $\vec{v}$ component along the Z axis will be zero, $\vec{v} = (v_x, v_y, 0)$.

$$\vec{F} = Q(E\vec{j} + Bv_y\vec{y} - Bv_x\vec{j}) = QBv_y\vec{y} + Q(E\vec{j} - Bv_x\vec{j})$$

$$\vec{F} = \frac{d\vec{p}}{dt}, \;\; \vec{p} = \gamma_v m\vec{v}, \;\; \gamma_v = (1 - v^2/c^2)^{-1/2} \underset{v \ll c}{\longrightarrow} 1$$

$$\vec{F} \approx m\dot{\vec{v}} \Rightarrow QBv_y = m\dot{v}_x, \; Q(E - Bv_x) = m\dot{v}_y$$

with $\omega_0 \equiv \frac{QB}{m}$ (that would be the cyclotron frequency that the particle would have in the absence of electric field), we have

$$\dot{v}_x = \omega_0 v_y, \; \dot{v}_y = \omega_0(\frac{E}{B} - v_x)$$

The solution of these differential equation is

$$x = \frac{C_1}{\omega_0}\sin\omega_0 t - \frac{C_2}{\omega_0}\cos\omega_0 t + \frac{E}{B}t + C_3 \qquad (1)$$

$$y = \frac{C_1}{\omega_0}\cos\omega_0 t + \frac{C_2}{\omega_0}\sin\omega_0 t + C_4 \qquad (2)$$

Initial conditions:

$$t = 0, \; v_x = v_y = 0 \Rightarrow C_1 = -\frac{E}{B}, \; C_2 = 0$$

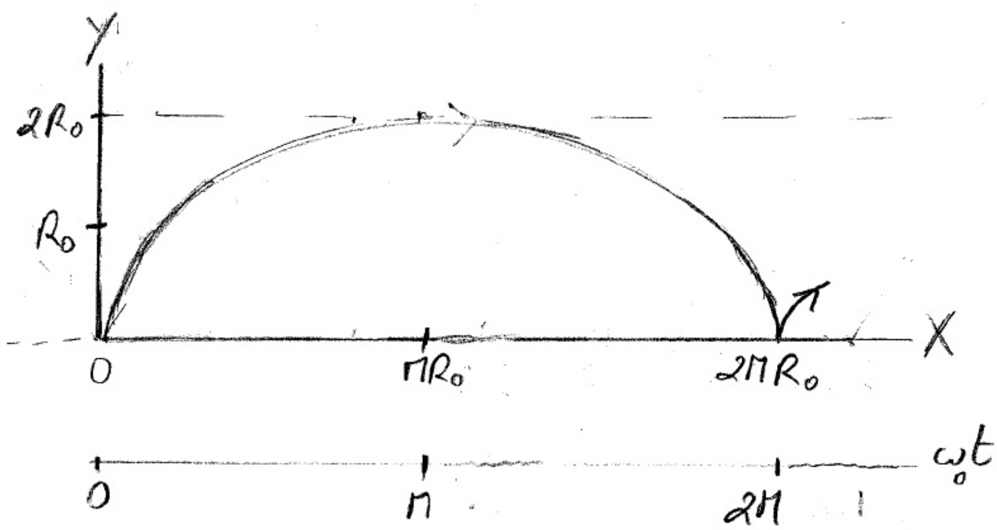
Then, $v_x = \frac{E}{B}(1 - \cos\omega_0 t), \;\; v_y = \frac{E}{B}\sin\omega_0 t \;\Rightarrow\; v \ll c$ if $\frac{E}{B} \ll c$, whereas the absolute magnitudes of E and B can be arbitrary.

At $t = 0, \; x = y = 0$, then $C_3 = 0, \; C_4 = \frac{E}{\omega_0 B}$. Finally

$$x = R_0(\omega_0 t - \sin\omega_0 t) \qquad (3)$$

$$y = R_0(1 - \cos\omega_0 t) \qquad (4)$$

where $\omega_0 \equiv \frac{QB}{m}$ and $R_0 \equiv \frac{E}{\omega_0 B} = \frac{mE}{QB^2}$. When $\omega_0 t = 2\pi n \;\Rightarrow\; x = R_0\omega_0 t = 2\pi n R_0, \; y = 0$.



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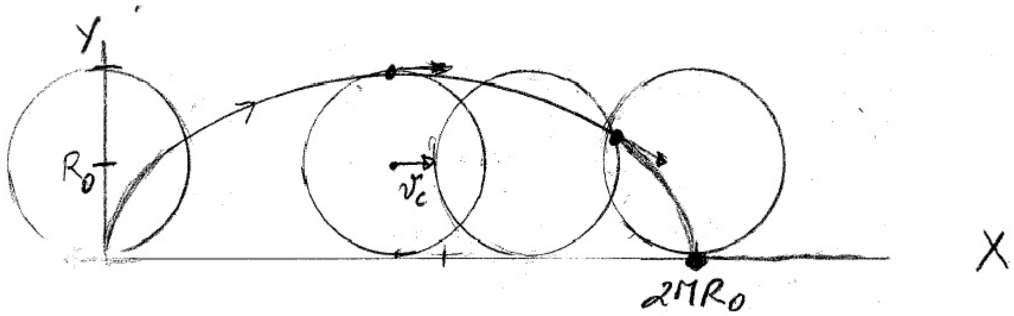
The previous result is the parametric equation of a cycloid $(x - R_0\omega_0 t)^2 + (y - R_0)^2 = R_0^2$ which is the equation of a circle of radius R_0 with center given by the coordinates $(R_0\omega_0 t, R_0, 0)$ traveling in the X direction with constant velocity $\vec{v}_c = (R_0\omega_0, 0, 0) = R_0\omega_0\vec{i} = \frac{E}{B}\vec{i}$. Again we see that $v \ll c \ \leftrightarrow ; \frac{E}{B} \ll c$. Then for any given instant T, for K frame the particle is at one of the points of a circle with center at $(R_0\omega_0 t, R_0, 0)$ and radius R_0.



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(Visit https://en.wikipedia.org/wiki/Cycloid)

As expected, the distance traveled along the X axis until the motion is repeated coincides with the length of the circumference.

(c) The relativistic case for which $v \ll c$ is not fulfilled is

$$QBv_y = m\frac{d}{dt}(\gamma_v v_x), \quad Q(E - Bv_x) = m\frac{d}{dt}(\gamma_v v_y)$$

which is very complicated to solve because $\gamma_v = (1 - v^2/c^2)^{-1/2}$, where $v^2 = v_x^2 + v_y^2$, can not be taken from the derivative.

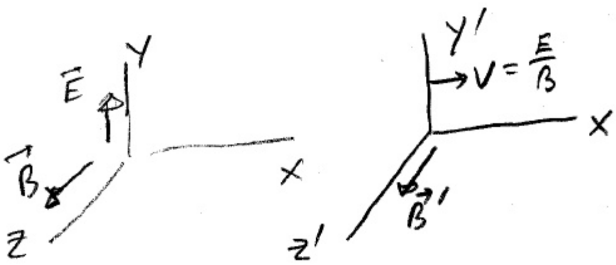
It is simpler to solve it by doing a Lorentz transformation that takes us to another inertial system K' in which there is only a magnetic field, so that the equation of motion is easy to solve.

Since $\vec{E} \cdot \vec{B}$ is a Lorentz scalar, $\vec{E}\ perp \vec{B}$ in K implies $\vec{E} \cdot \vec{B} = 0 = \vec{E}' \cdot \vec{B}'$ and we can find a system K' with a velocity V such that $\vec{E}' = 0,\ \vec{B}'\ ne 0$

$$\vec{B}_\parallel = 0 = \vec{B}'_\parallel \quad \vec{B}_\perp = \quad \gamma(\vec{B}' + \frac{\vec{\beta}}{c} \times \vec{E}')_\perp = \gamma\vec{B}' \tag{5}$$

$$\vec{E}_\parallel = 0 = \vec{E}'_\parallel \quad \vec{E}_\perp = \quad \gamma(\vec{E}' - c\vec{\beta} \times \vec{B}')_\perp = -\gamma\vec{V} \times \vec{B}'_\perp = -\vec{V} \times \vec{B}_\perp \tag{6}$$

Finally, $\vec{E}' = 0$ and $\vec{B}' = \frac{1}{\gamma}\vec{B}$. Also $\vec{E} = \vec{V} \times \vec{B}$. therefore $V = \frac{E}{B}$ which is always possible because $\frac{E}{B} < c$.



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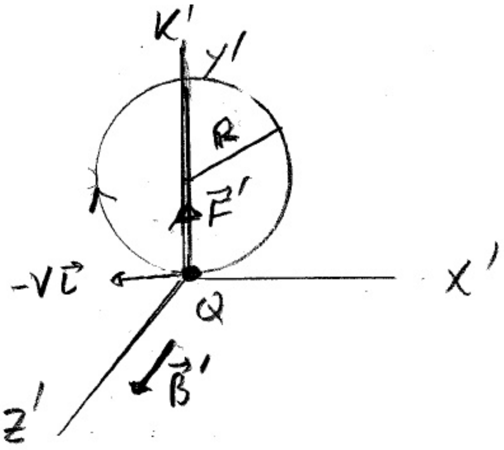
Since for $t = 0$ the particle is at rest in K, for $t' = 0$ it will have a velocity $-V\vec{i}$ in K'. The magnetic field $\vec{B}'$ does not work on the particle so that the modulus of its velocity remains constant and the equation of the movement is easy to solve, describing the circumference of radius R:

$$QvB' = m\gamma\frac{V^2}{R} \quad \Rightarrow \quad R = \frac{m\gamma V}{QB'} = \frac{m\gamma V^2}{QB}$$

with angular frequency $\omega' = \frac{V}{R} = \frac{QB}{m\gamma^2}$



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The trayectory is

$$\begin{aligned} x' &= -R\sin\omega t' & (7)\\ y' &= R(1-\cos\omega t') & (8)\\ z' &= 0 & (9) \end{aligned}$$

i.e., the circumference $x'^2 + (y'-R)^2 = R^2$ of radius R and centre $(0, R, 0)$.

By using Lorentz transformations the trajectory in K is:

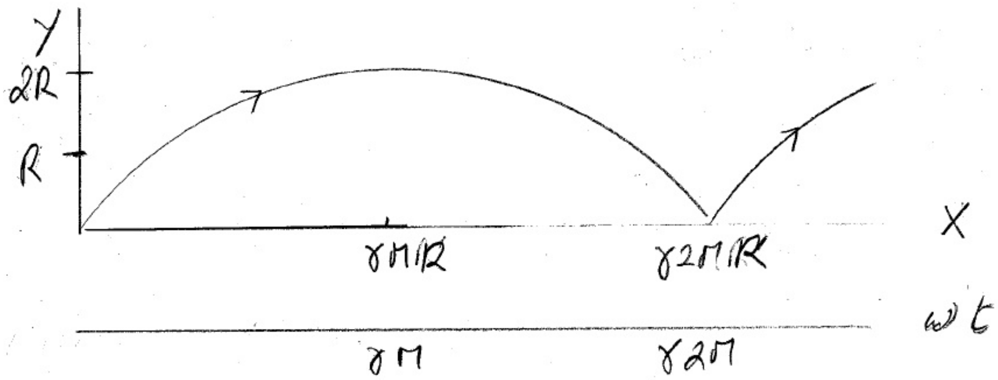
$$\begin{aligned} t' &= \gamma(t - \frac{V}{c^2}x) & (10)\\ z = z' &= 0 & (11)\\ y = y' &= R(1-\cos\omega t') & (12)\\ x' = \gamma(x - Vt) \to x &= Vt - \frac{R}{\gamma}\sin[\omega\gamma t'] & (13) \end{aligned}$$

Now, putting t' in terms of t and x,

$$\begin{aligned} x &= Vt - \frac{R}{\gamma}\sin[\omega\gamma(t - \frac{V}{c^2}x)] & (14)\\ y &= R - R\cos[\omega\gamma(t - \frac{V}{c^2}x)] & (15) \end{aligned}$$

with the following constants $V = E/B,\ \gamma = (1 - V^2/c^2)^{-1/2}$, $\omega = \frac{QB}{m\gamma^2}, R = V/\gamma$.

When $\omega\gamma(t - \frac{V}{c^2}x) = 2\pi n$, $y = 0$ and $x = Vt$. Hence $t = \gamma\frac{2\pi n}{\omega}$. Finally, $y = 0$ when $\omega = 2\pi n\gamma$ and $x = 2\pi n\gamma R$.



5.jpg

(d) This figure is similar to that of question (b) but with the horizontal axis stretched. In the limit $E/B \ll c,\ \gamma \to 1$ recovering the results of (b).

(e) Comparing K' with K. The time it takes for the motion to repeat itself in K' is, for example, what it takes for the particle to pass again through $x' = y' = 0$

$$\omega t' = 2\pi n \quad \to \quad t' = \frac{2\pi n}{\omega}$$

However, for K the time the particle takes to go again through $y = 0$ w as

$$\omega t = 2\pi n\gamma \quad \to \quad t = \frac{2\pi n}{\omega}\gamma$$

This result is no more than the effect of temporary dilation. We could have reached it using the Lorentz transformation

$$t = \gamma(t' - \frac{V}{c^2}x') \underset{x'=0}{=} \gamma t' \ \to \ t = \frac{2\pi n}{\omega}\gamma$$

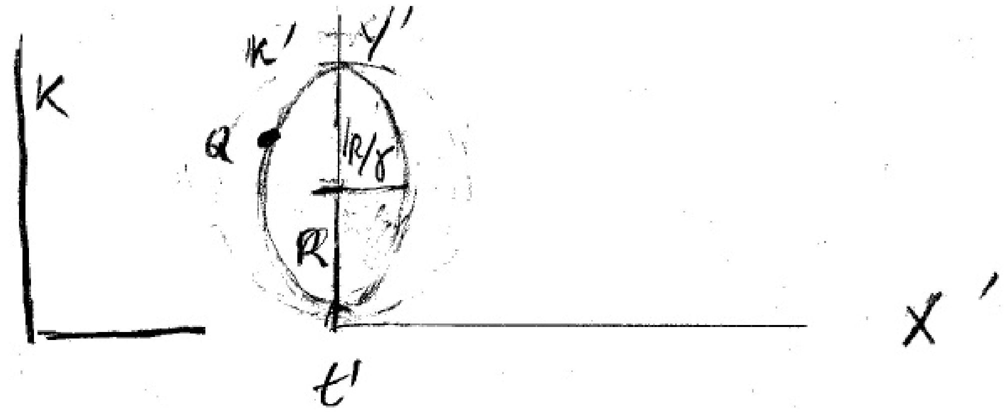
K observe that the motiont takes more time to run in K'. According to K everything goes more slowly in K'. A clock, such as the particle in K' submitted to B', goes more slowly than if it were at rest.

What K' interprets as a magnetic process, K interprets it as electric + magnetic. They do not coincide in the concrete quantities that they measure, but coincide in that the equations of motion are the same. In K the equations of the trajectory satisfy

$$\gamma^2(x - R\omega t)^2 + (y - R)^2 = R^2$$

$$\frac{(x - R\omega t)^2}{(\frac{R}{\gamma})^2} + \frac{(y - R)^2}{R^2} = 1$$

which is the equation of an ellipse with minor axis R/γ, whose center $(R\omega t, R, 0)$ travels with velocity $(R\omega, 0, 0) = (E/B, 0, 0)$. For K, at any given time t, the particle is in one of the points of this ellipse.



6.jpg

This result is nothing more than the effect of length contraction. We got in K' $x'^2 + (y' - R)^2 = R^2$. The Lorentz transformation is $x' = \gamma(x - Vt)$, $y' = y$, $z' = z$ wich carried to the previous equation gives

$$\frac{(x - Vt)^2}{(\frac{R}{\gamma})^2} + \frac{(y - R)^2}{R^2} = 1$$

This can also be understood as due to the relativistic kinematics that changes the force for different observers, so that the particle follows different trajectories.

Estudiantes R.S.E.F.

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Spin waves in a ferromagnetic chain

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Spin waves are collective excitations of magnetic materials and they play a key role in the description of solid-state systems. In essence, spin waves can be described in much the same way as phonons in a crystal or electrons in a tight-binding model.

We will consider here one of the simplest examples of a magnetic system, namely, a ferromagnetic Heisenberg chain. This is a one dimensional array of N spins coupled by an isotropic interaction. At each site, j, of the chain we define a vector of spin-1/2 operators,

$$\vec{S}_j = (S_j^x, S_j^y, S_j^z). \tag{1}$$

Our Hamiltonian takes the form,

$$H_\mathsf{F} = -g\sum_{j=1}^{N} \vec{S}_j \cdot \vec{S}_{j+1}, \quad g > 0, \tag{2}$$

where we assume periodic boundary conditions, $\vec{S}_{N+1} = \vec{S}_1$ and units such that $\hbar = 1$.

- Consider first a classical description in which each spin is just a vector of length 1/2. What is the set of classical minimum-energy spin configurations? What is their energy? **(1 point)** SOLUTION: A classical spin can be characterized by a vector $\vec{S}_j = \frac{1}{2}\hat{n}_{\phi_j,\theta_j}$. The energy is minimized by maximizing the scalar products $\vec{S}_j \cdot \vec{S}_{j+1} = 1/4$. Thus, the minimum energy configuration consists of N parallel spins with energy $E_\mathsf{F} = -(g/4)N$.
- Let us go back to the quantum case. Define ladder operators,

$$\begin{aligned} S_j^+ &= S_j^x + i\,S_j^y, \\ S_j^- &= S_j^x - i\,S_j^y. \end{aligned} \tag{3}$$

  Show that the magnetic chain can be re-written like

$$H_\mathsf{F} = -g\sum_{j=1}^{N} \frac{1}{2}(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+) - g\sum_{j=1}^{N} S_j^z S_{j+1}^z \quad \textbf{(1 point)}. \tag{4}$$

  SOLUTION: This is really just a simple algebraic step to help solving the following points. One has to express $S_j^x = (S_j^+ + S_j^-)/2$, $S_j^y = (S_j^+ + S_j^-)/(2i)$ and calculate the scalar product in the Heisenberg equation.

- We define the eigenstates of S_j^z, $|+\rangle_j$, $|-\rangle_j$, with eigenvalue $+1/2$, $-1/2$, respectively. Consider the following quantum state,

$$|\Psi_\mathsf{F}\rangle = |-\rangle_1 |-\rangle_2 \ldots |-\rangle_N, \tag{5}$$

  Show that $|\Psi_\mathsf{F}\rangle$ is an eigenstate of H_F and calculate its energy, E_F. **(1 point)**

  SOLUTION: We only need to use that $S_j^-|-\rangle_j + 0$, so that

$$S_j^-|\Psi_\mathsf{F}\rangle = 0.$$

  The only non-zero contribution to the energy is thus the $z-$coupling:

$$S_j^z S_{j+1}^z |\Psi_\mathsf{F}\rangle = \frac{-1}{4}.$$

  Since there are N coupling terms we get $E_0 = -(g/4)N$.

- Calculate the degeneracy of the energy E_F. **(3 points)**

  SOLUTION: This part requires some understanding of rotational symmetry. First of all we must note that H_F is invariant under global rotations and thus it commutes with the total angular momentum operators ($\vec{S} = \sum_j^N \vec{S}_j$).

The state $|\Psi_\mathsf{F}\rangle$ can be written in terms of the eigenstates of the total angular momentum $\vec{S}^2$, S^z. In particular, we have that $|\Psi_\mathsf{F}\rangle = |S, -S\rangle$, with $S = N/2$.

We can define global ladder operators $S^+ = S^x + iS^y$. Since $[H_\mathsf{F}, S^{\alpha=x,y,z}] = 0$, then $[H_\mathsf{F}, S^+] = 0$. This means that by acting any number of times with S^+ on $|\Psi_\mathsf{F}\rangle$ we generate an eigenstate of H_F with energy E_F. Thus the whole set of states, $|S, M\rangle$ with $S = N/2$ and $M = -S, -S+1, \dots, S$ have the energy E_F and degeneracy is $2S + 1 = N + 1$.

- Let us define a set of N orthonormal states, $|\Phi_n\rangle$, $n = 1, \dots N$, that describe a collective spin excitation (spin-wave). Each state n is defined by means of a vector $\phi_j^{(n)}$,

$$|\Phi_n\rangle = \sum_{j=1}^{N} \phi_j^{(n)} S_j^+ |\Psi_\mathsf{F}\rangle. \tag{6}$$

You can see in equation (**??**) that $\phi_j^{(n)}$ is the probability amplitude of having a spin excitation at site j in the Heisenberg chain.

Calculate the N orthonormal states, $|\Phi_n\rangle$ that are eigenstates of the Hamiltonian H_F and calculate their energy, E_n.**(3 points)**

SOLUTION: The chain is subjected to periodic boundary conditions, and thus, it has to be diagonalized by plane-waves

$$\phi_j^{(n)} = \frac{1}{\sqrt{N}} e^{-i\frac{2\pi n j}{N}}.$$

Such that the spin-wave states take the form

$$|\Phi_n\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{-i\frac{2\pi n j}{N}} S_j^+ |\Psi_\mathsf{F}\rangle.$$

To find the energy we apply the Hamiltonian H_F

$$H_\mathsf{F}|\Phi_n\rangle = \left(-g \sum_{j=1}^{N} \frac{1}{2}(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+) - g \sum_{j=1}^{N} S_j^z S_{j+1}^z \right) \frac{1}{\sqrt{N}} \sum_{p=1}^{N} e^{-i\frac{2\pi n p}{N}} S_p^+ |\Psi_\mathsf{F}\rangle.$$

There are many ways to solve this problem, which in essence is just a tight-binding single-particle Hamiltonian. To make things easier, let us define the set of states with a spin excited at site p,

$$|p\rangle = S_p^+ |\Psi_\mathsf{F}\rangle = |-\rangle_1 \dots |+\rangle_p \dots |-\rangle_N.$$

We can easily show that

$$\sum_{j=1}^{N} S_j^+ S_{j+1}^- |p\rangle = |p-1\rangle,$$
$$\sum_{j=1}^{N} S_j^- S_{j+1}^+ |p\rangle = |p+1\rangle,$$
$$\sum_{j=1}^{N} S_j^z S_{j+1}^z |p\rangle = \left(\frac{1}{4}(N-2) - \frac{1}{4} 2 \right) |p\rangle.$$

The z-interaction term is calculated by noticing that there are $N - 2$ bonds with aligned spins and 2 bonds with anti-aligned spins.

Putting all together,

$$H_{\mathsf{F}}\frac{1}{\sqrt{N}}\sum_{p=1}^{N}e^{-i\frac{2\pi np}{N}}|p\rangle = -\frac{g}{2}\frac{1}{\sqrt{N}}\sum_{p=1}^{N}e^{-i\frac{2\pi np}{N}}|p-1\rangle - \frac{g}{2}\frac{1}{\sqrt{N}}\sum_{p=1}^{N}e^{-i\frac{2\pi np}{N}}|p+1\rangle - g(\frac{N}{4}-1)|p\rangle$$
$$= (-g\cos(2\pi n/N) - g(N/4-1))\frac{1}{\sqrt{N}}\sum_{p=1}^{N}e^{-i\frac{2\pi np}{N}}|p\rangle.$$

The energy is thus

$$E_n = -g\frac{N}{4} + g(1-\cos(2\pi n/N)).$$

The excitation energy is

$$E_n - E_{\mathsf{F}} = g(1-\cos(2\pi n/N)).$$

- Imagine that you create a low-energy spin-wave excitation, for example, a wave-packet formed by a linear combination of states $|\Phi_n\rangle$. By low-energy we mean that the quantum state is formed by spin-waves with excitation energy much lower than the ferromagnetic coupling constant, g (that is, states such that $E_n - E_{\mathsf{F}} \ll g$) Such excitation could be considered like an effective particle. Assume that spins in the chain are separated by distance a.

  What is the effective mass of this particle? (Use your result for E_n and justify qualitatively your answer - this question does not require a long calculation). **(1 point)**

  SOLUTION: Since this is a low-energy excitation, we only need to take into account small n. The wave-vector of the spin-plane waves is $k = 2\pi n/(Na)$, and we can write the energy in the form,

$$E_n - E_{\mathsf{F}} = g(1-\cos(ka)) \approx g\frac{1}{2}(ka)^2 = \frac{k^2}{2m_{\mathsf{eff}}}.$$

  The latter is the dispersion relation in terms of an effective mass. Thus we find

$$\frac{1}{m_{\mathsf{eff}}} = ga^2.$$