MATH-EMATI-CAL

Methods

MIKEL ÁLVAREZ URKIOLA

josemm.senovilla@ehu.es

Kepasousa@ehu.es

RILEY-HOBSON-BENCE : "Mathematical methods for physics & engineering"

MATHEWS-WALKER : "Mathematical methods of physics"

BARLOW : "Statistics

EXACT EQUATIONS

$P(x,y)\,dx + Q(x,y)\,dy = 0 \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

SOLUTION $\rightarrow \int P\,dx + \int \left\{ Q - \int \frac{\partial P}{\partial y}\,dx \right\} dy = 0$

1^{st} special case: $P(x)\,dx + Q(x)\,dy = 0$

SOL. $\rightarrow y = -\int \frac{P(x)}{Q(x)}\,dx$

2^{nd} special case: $P(x)\,dx + Q(y)\,dy = 0$

SOL. $\rightarrow \int P(x)\,dx = -\int Q(y)\,dy$

INTEGRATING FACTORS $(\mu(x,y))$

LOST SOLS? $\Rightarrow \frac{1}{\mu} = 0$

FAKE SOLS.? $\Rightarrow \mu = 0$

$\mu = \mu(x)$

If $\frac{\partial}{\partial y}\left[\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\right] = 0 \Rightarrow \mu = C \cdot e^{\int \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dx}$

$\mu = \mu(y)$

If $\frac{\partial}{\partial x}\left[\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\right] = 0 \Rightarrow \mu = C \cdot e^{\int -\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dy}$

$\mu = \mu(h(x,y))$

$$\mu = C \cdot \exp\left[\int \frac{\overbrace{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}^{\text{Must be rewritable as a function of } h}}{Q\frac{\partial h}{\partial x} - P\cdot\frac{\partial h}{\partial y}}\,dh\right]$$

SEPARABLE EQUATIONS

$R(x)\,S(y)\,dx + U(x)\,V(y)\,dy = 0$

$\Downarrow \mu = \frac{1}{S(y)\,U(x)} \rightarrow$ Lost sols.? $\Rightarrow S(y) = 0$

$\frac{R(x)}{U(x)}\,dx + \frac{V(y)}{S(y)}\,dy = 0$

LINEAR EQUATIONS

$y' + A(x)\,y = B(x)$ $\begin{cases} B=0 \rightarrow \text{Homogeneous} \\ B\neq 0 \rightarrow \text{Inhomogeneous} \end{cases}$

SOL. $\rightarrow y = e^{-\int A\,dx} \cdot \int B \cdot e^{\int A\,dx}\,dx + e^{-\int A\,dx} \cdot C$

HOMOGENEOUS EQUATIONS

$\rightarrow P, Q$ are homogeneous functions (= order)

$\hookrightarrow f(ax, ay) = a^{r} \cdot f(x,y)$ ← order

$P\,dx + Q\,dy = 0 \rightarrow y' = -\frac{P}{Q} = f\left(\frac{y}{x}\right) \xrightarrow{u = \frac{y}{x}} \int \frac{du}{f(u) - u} = \int \frac{dx}{x} + C \rightarrow y = \ldots$

BERNOULLI'S EQUATION

$y' + A(x)\,y = B(x)\cdot y^n, \; n \neq 0, 1 \xrightarrow{u = y^{1-n}} \underbrace{u' + (1-n)A(x)\cdot u = B(x)\cdot(1-n)}_{\text{Linear inhomogeneous}} \rightarrow y = \ldots$

$y' = f\left(\frac{ax + by + c}{\alpha x + \beta y + \gamma}\right)$

$ax + by + c \rightarrow r_1$

$\alpha x + \beta y + \gamma \rightarrow r_2$

If $r_1 // r_2 \left(\frac{a}{\alpha} = \frac{b}{\beta}\right)$:

$u = ax + by + c$

or

$u = ax + by$

If $r_1 \times r_2 \rightarrow$ They cut each other at (x_0, y_0)

$\begin{cases} u = x - x_0 \\ v = y - y_0 \end{cases} \; y' = \frac{dy}{dx} = \frac{dv}{du} = v' = f\left(\frac{au + bv}{\alpha u + \beta v}\right) = f\left(\frac{a + b\frac{v}{u}}{\alpha + \beta\frac{v}{u}}\right) \rightarrow$

$\xrightarrow{z = \frac{v}{u}} z'u + z = f\left(\frac{a + bz}{\alpha + \beta z}\right)$

$v' = \frac{dv}{du}, \; z' = \frac{dz}{du}$

LOWERING THE ORDER

· Eqs. without dep. var.

$$F(x, y', y'', \dots, y^{(n)}) = 0$$

$$\boxed{u = y'}$$

· Eqs. without indep. var.

$$F(y, y', y'', \dots, y^{(n)}) = 0$$

$$\boxed{u = y'} \rightarrow \begin{cases} y'' = \dot{u} \cdot u \\ y''' = \ddot{u} \cdot u + \dot{u}^2 \cdot u \\ \dots \end{cases} \qquad \dot{u} = \frac{du}{dy}$$

New dep. var $\rightarrow$ "y" is the indep. var.

· Equidimensional-in-x ODEs

If $x \rightarrow ax$ $\begin{cases} y' \rightarrow a^{-1} y' \\ y'' \rightarrow a^{-2} y'' \\ \dots \end{cases}$ and the eq. doesn't change:

$$\boxed{t = \ln x} \rightarrow \begin{cases} x = e^t \\ y' = \dot{y}/x \\ y'' = \dfrac{\ddot{y} - \dot{y}}{x^2} \\ \dots \end{cases}$$

· Equidimensional-in-y ODEs

If $y \rightarrow ay$ $\begin{cases} y' \rightarrow ay' \\ y'' \rightarrow ay'' \\ \dots \end{cases}$ and the eq. remains the same:

$$\boxed{u = \frac{y'}{y}} \rightarrow \begin{cases} y' = uy \\ y'' = y(u' + u^2) \\ y''' = u^3 y + u'' y + 3uu'y \\ \dots \end{cases}$$

· Exact differentials

$$F(x, y, y', \dots, y^{(n)}) = 0 = \frac{d}{dx}\left[G(x, y, y', \dots, y^{(n-1)})\right]$$

We might as well work with $G(x, y, y', \dots, y^{(n-1)}) = C$ to solve the equation.

LINEAR ODEs

$\longrightarrow$ Sol.: $\boxed{y = y_{hom} + y_{part.\,inhom}}$

FOR THE HOMOGENEOUS PART:

- **Characteristic polynomial** $\rightarrow$ Only for linear hom. ODEs with ctant. coeff.

ROOTS
$$\begin{cases} K \text{ (multiplicity } m\text{)},\ K\in\mathbb{R} \rightarrow \text{Sols.: } e^{K},\ xe^{K},\ x^2e^{K},\ \ldots,\ x^m e^{K} \\ K_1 + K_2 i \ \text{(mult. } m\text{)} \rightarrow \text{Sols.: } e^{K_1x}\cos(K_2x),\ xe^{K_1x}\cos(K_2x),\ \ldots,\ x^m e^{K_1x}\cos(K_2x) \\ K_1 - K_2 i \ (\text{"}) \rightarrow \text{Sols.: } e^{K_1x}\text{sen}(K_2x),\ xe^{K_1x}\text{sen}(K_2x),\ \ldots \end{cases}$$

- **D'Alembert's method for 2^{nd} order hom. ODEs**

$y'' + a_1y' + a_2y = 0$, y_1 is a part. sol. of the eq.

$$\boxed{u = C_2 \cdot \frac{e^{-\int a_1 dx}}{y_1^2} \longrightarrow y = C_1y_1 + y_1\int u\,dx}$$

COMPLETE LINEAR ODEs

Remember: $Ly = b_1 + b_2 \rightarrow$ Sol.: $y = y_{hom} + y_{P1} + y_{P2}$

- **Variation of parameters method**

2ND ORDER

$y'' + a_1y' + a_2y = b$

$\hookrightarrow y_{hom} = C_1y_1 + C_2y_2$

$y_P = g(x)\,y_1 + h(x)\,y_2$

$$\begin{cases} g'y_1 + h'y_2 = 0 \\ g'y_1' + h'y_2' = b \end{cases}$$

3RD ORDER

$y''' + a_1y'' + a_2y' + a_3 = 0$

$\hookrightarrow y_{hom} = C_1y_1 + C_2y_2 + C_3y_3$

$y_P = f(x)\,y_1 + g(x)\,y_2 + h(x)\,y_3$

$$\begin{cases} f'y_1 + g'y_2 + h'y_3 = 0 \\ f'y_1' + g'y_2' + h'y_3' = 0 \\ f'y_1'' + g'y_2'' + h'y_3'' = b \end{cases}$$

· Undetermined coefficients

Given $\mathcal{L}t = g(t)$ (linear inhom. ODE with constant coeff.):

① Calculate the solution for the homogeneous part

↳ Characteristic polynomial

② Try to guess the particular sol. of the inhom. eq. based on the following criteria:

$g(t)$	$y_{PART}(t)$ guess
$ae^{\beta t}$	$Ae^{\beta t}$
$a\cos(\beta t)$ $b\sin(\beta t)$ $a\cos(\beta t)+b\sin(\beta t)$	$\to A\cos(\beta t)+B\sin(\beta t)$
n^{th} degree polyn.	$A_n t^n + A_{n-1}t^{n-1} + \dots + A_1 t + A_0$

③ Substitute in the equation to obtain the values of A, B, etc.

IMPORTANT!

$$* \ \mathcal{L}x = g(t) + h(t) \longrightarrow y = y_{hom} + \underbrace{y_{P1}}_{\text{corresponds to } \mathcal{L}t = g(t)} + \underbrace{y_{P2}}_{\text{'' '' } \mathcal{L}t = h(t)}$$

* If $\mathcal{L}x = g(t)\cdot h(t) \to$ Multiply the guessed solutions for both $g(t)$ & $h(t)$.

E.g.: • $g(t) = te^{4t} \to y_{PART} = (At+B)Ce^{4t} \to y_{PART} = (At+B)e^{4t}$

• $g(t) = 6t^2 - 7\sin(3t) + 9 \to y_{PART} = At^2 + Bt + C + D\cos(3t) + E\sin(3t)$

! * If any of the terms of the guessed particular sol. appears in the solution for the homogeneous part, multiply that term by t.

LAPLACE TRANSFORM

DEF: $f(t) \longrightarrow F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$f(t) \in F(\alpha) \Leftrightarrow |f(t)| < Me^{\alpha t},\ \forall t > t_0 \Leftrightarrow \exists\, \mathcal{L}[f(t)]\ \forall s > \alpha$

HEAVISIDE FUNCTION $\Rightarrow$ $\theta(t-a) = \begin{cases} 1, & t > a \\ 0, & t < a \end{cases}$

$\dot{\theta}(t-a) = \underset{\text{DIRAC'S DELTA}}{\underset{\Downarrow}{\delta(t-a)}} = \begin{cases} \infty, & t = a \\ 0, & t \neq a \end{cases}$

PROPERTIES

* $\mathcal{L}[e^{at} f(t)] = F(s-a)$
* $\mathcal{L}[\theta(t-a) f(t-a)] = e^{-as} F(s)$
* $\mathcal{L}[t^n f(t)] = (-1)^n \dfrac{d^n F(s)}{ds^n}$
* $\mathcal{L}[f(at)] = \dfrac{1}{a} F\left(\dfrac{s}{a}\right)$
* $\mathcal{L}\left[\dfrac{d^n f}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)$
* Linearity: $\mathcal{L}[af + bg] = a\mathcal{L}[f] + b\mathcal{L}[g$

USEFUL TRANSFORMS (Assuming $f(t) = \theta(t) f(t)$)

$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	$\dfrac{1}{s}$	$\sin(at)$	$\dfrac{a}{s^2+a^2}$
e^{at}	$\dfrac{1}{s-a}$	$\cos(at)$	$\dfrac{s}{s^2+a^2}$
t^n	$\dfrac{n!}{s^{n+1}}$	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$
$\theta(t-a)$	$\dfrac{e^{-as}}{s}$	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$

* $\mathcal{L}^{-1}\left[\dfrac{e^{-bs}}{s-a}\right] = \theta(t-b)\, e^{a(t-b)}$

HEAVISIDE FORMULA

$$\frac{P(s)}{Q(s)} = \frac{A}{(s-a)^n} + \frac{B}{(s-a)^{n-1}} + \dots + \frac{N}{s-a} + \frac{M}{s-b}$$

$$A = \lim_{s \to a} (s-a)^n \frac{P(s)}{Q(s)}$$

$$B = \lim_{s \to a} \frac{d}{ds}\left[(s-a)^n \frac{P(s)}{Q(s)}\right]$$

$\vdots$

$$N = \lim_{s \to a} \frac{d^n}{ds^n}\left[(s-a)^n \frac{P(s)}{Q(s)}\right]$$

$$M = \lim_{s \to b} (s-b) \frac{P(s)}{Q(s)}$$

DEF.: A function $f(x)$ is called ANALYTIC at $x=a$ if the Taylor expansion for $f(x)$ around $x=a$ has a positive radius of convergence and converges to $f(x)$.

Equations of the kind $y'' + P(x)\, y' + Q(x)\, y' = 0$ will be solved in this chapter.

- If $P(x)$ & $Q(x)$ are analytic at $x=x_0$, then $x=x_0$ is an ordinary point. Otherwise, it is a singular point.
  - If $p(x) \equiv \underbrace{(x-x_0)P(x)}_{\to\ 1^{st}\text{ order pole}}$ & $q(x) \equiv \underbrace{(x-x_0)^2 Q(x)}_{\to\ 2^{nd}\text{ order pole}}$ are analytic at $x=x_0$, then it's a regular singular point.
  - If not, $x=x_0$ is an irregular singular point.

From now on, we'll only solve the eqs. around $x=0$.

SERIES METHOD $\rightarrow$ $x=0$ MUST BE AN ORDINARY POINT

$$\boxed{y = \sum_{n=0}^{\infty} a_n x^n} \rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \rightarrow y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} n(n-1)\, a_n x^{n-2}$$

- Put everything together in a single series $\rightarrow \sum_{n=..}^{\infty} [\cdots\cdots]\, x^n = 0$
- Get the recurrence relation
- Use it to "guess" y. $\rightarrow$ First particular solution.
- With D'Alembert's method, obtain the 2nd part. sol.

FROBENIUS' METHOD

- Multiply the eq. by some factor of x so the lowest grade of the term corresponding to y'' is 2.

$$\boxed{y=\sum_{n=0}^{\infty} a_n x^{n+\lambda}} \rightarrow y'=\sum_{n=0}^{\infty}(n+\lambda)a_n x^{n+\lambda-1} \rightarrow y''=\sum_{n=0}^{\infty}(n+\lambda)(n+\lambda-1)a_n x^{n+\lambda-2}$$

- Do some changes of variables so every series ends with the same power of x, which must be the lowest posible one (usually "$n+\lambda$").
- You should get something like this (for example):

$$\sum_{n=0}^{\infty}[\cdots]x^{n+\lambda}+\sum_{n=1}^{\infty}[\cdots]x^{n+\lambda}+\sum_{n=2}^{\infty}[\cdots]x^{n+\lambda}=0$$

- Now, take the $n=0$ & $n=1$ terms out of the two first series:

$$\underbrace{\quad\quad}_{n=0 \text{ term}} x^{\lambda}+[\underbrace{\quad + \quad}_{n=1 \text{ terms}}]x^{\lambda+1}+\sum_{n=2}^{\infty}[\cdots+\cdots+\cdots]x^{n+\lambda}=0$$

Every term must be equal to 0!

Solve for λ

$\boxed{\text{Use the biggest solution.}}$

Might give some info. about a_1 or a_0.

Get recurrence relation

In principle, $a_0 \neq 0$ will be assumed.

- Using the data obtained in the last step, one should be able to get the first part. sol.
- With D'Alembert's method the 2nd part. sol is obtained as well.

Given the following system: $\begin{cases} \dot{x} = P(x,y) \\ \dot{y} = Q(x,y) \end{cases}$

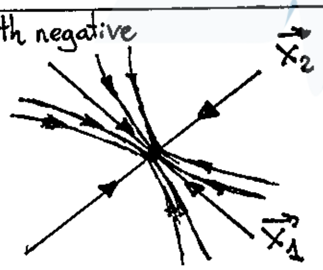
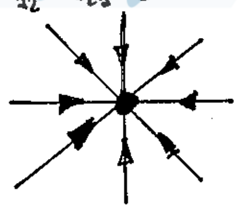
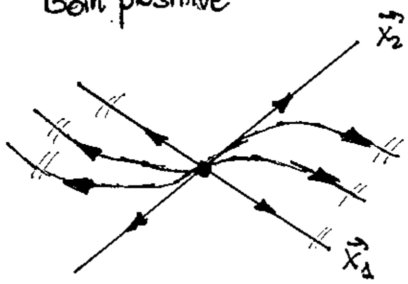
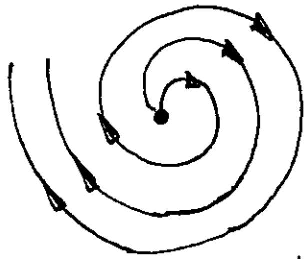
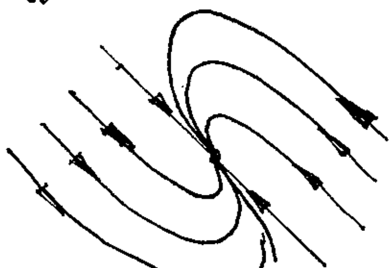
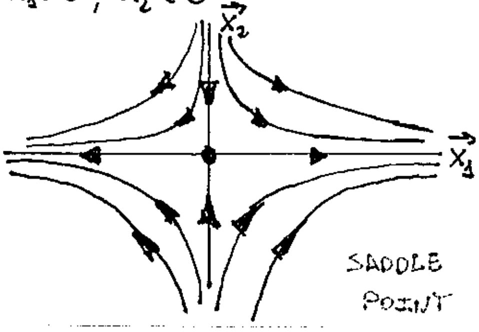
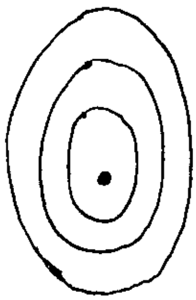
① Find out which are its fixed points.

↳ Let (x_0, y_0) be a fixed point. Then, $P(x_0,y_0) = Q(x_0,y_0) = 0$.

↳ $\begin{cases} \dot{x} = 0 \rightarrow \dots \\ \dot{y} = 0 \rightarrow \dots \end{cases}$ $\Rightarrow$ Sketch the results in a diagram to see it better.

② $$\mathbb{A} = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix} \rightarrow \mathbb{A}_{(x_0,y_0)} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

③ The eigenvalues ($\Rightarrow$ roots) & eigenvectors of $\mathbb{A}_{(x_0,y_0)}$ will determine the character of the fixed point.

DIFFERENT & REAL ROOTS $(K_1 > K_2)$	COMPLEX ROOTS $K_+ = \alpha + i\omega$, $K_- = \alpha - i\omega$	SAME, REAL ROOTS $(K_1 = K_2)$
Both negative  ASYMP. STABLE NODE	$\alpha < 0$  STABLE FOCUS	$a_{12} = a_{21} = 0$  STAR NODE
Both positive  UNSTABLE NODE	$\alpha > 0$  UNSTABLE FOCUS	$\lvert a_{12}\rvert + \lvert a_{21}\rvert \neq 0$  DEGENERATE NODE
$K_1 > 0$, $K_2 < 0$  SADDLE POINT	$\alpha = 0$  ⊛ CENTER	⊛ Watch out for symmetries! Because of the non-linear terms, the center might become a focus.

④ Look for particular solutions:

- Make $x \to x_0 \Rightarrow \dot{x} = 0 = P(x_0, y)$

  If one manages to get $x_0 = K$, then $x = K$ is a part. sol.

- Do the same for y

In the diagram, part. sols. are straight lines (with arrows) which help to sketch it

⑤ Look for symmetries $\begin{cases} x \to -x \\ y \to -y \\ t \to -t \end{cases}$

(Example on p. 102)

DEF.: Given $f, g \in C^n([a,b])$. The scalar product relative to the weight $\rho(x)$ is defined as

$$\langle f | g \rangle = \int_a^b \bar{f} \, g \, \rho \, dx$$

DEF.: L is a self-adjoint operator if $\langle Lf | g \rangle = \langle f | Lg \rangle$.

The "L" we'll work with is of this kind:

$$Ly = \frac{d}{dx}\left(P \frac{dy}{dx}\right) + Qy = a_0 y'' + a_1 y' + a_2 \rightarrow \rho = \frac{1}{a_0} \exp\left(\int \frac{a_1}{a_0} dx\right)$$

Self-adjoint if: $\Big[P \cdot W[\bar{f}, g] \Big]_a^b = 0$

S-L PROBLEM

$Ly + \lambda y = 0$ ⊕ Boundary conds. → Must be homogeneous.

The λs which allow for solutions are EIGENVALUES; each "y" corresponding to a λ is called an EIGENFUNCTION.

"L" IS SELF-ADJOINT
$\begin{cases} \text{* All eigenvalues are real.} \\ \text{* Eigenfunctions corresponding to different eigenvalues are orthogonal.} \end{cases}$

Might be useful:

* $y'' + \lambda y = 0$ ⊕ $y(0) = y(a) = 0 \rightarrow y_n = \sin\left(\frac{n\pi x}{a}\right)$, $n = 1, 2, 3, \dots$

* $y'' + \lambda y = 0$ ⊕ $\begin{cases} y(0) = y(2\pi) \\ y'(0) = y'(2\pi) \end{cases} \rightarrow y_n = A\sin(nx) + B\cos(nx)$, $n \in \mathbb{Z}$

Given $f(x)$ with period T, the Fourier series of f is defined as:

$$\circledast \quad f(x) = \sum_{n=-\infty}^{\infty} c_K(f) \cdot e^{iK\omega x} \quad \text{pro } \omega = \frac{2\pi}{T}, \quad x \in (0,T)$$

$$c_K(f) = \langle e^{iK\omega x} | f \rangle$$

$$\circledast \quad f(x) = a_0 + \sum_{K=1}^{\infty} a_K(f) \cos(K\omega x) + \sum_{K=1}^{\infty} b_K(f) \sin(K\omega x), \quad x \in (0,T)$$

$$a_0 = \int_0^T f(x) \frac{dx}{T}$$

$$a_K = 2 \cdot \int_0^T f(x) \cos(K\omega x) \frac{dx}{T}$$

$$b_K = 2 \cdot \int_0^T f(x) \sin(K\omega x) \frac{dx}{T}$$

If f is even $\rightarrow b_K = 0$, $K = 1, 2, 3, \dots$

If f is odd $\rightarrow a_0 = a_K = 0$, $K = 1, 2, 3, \dots$

If f has a jump at $\tilde{x}$, the FOURIER SERIES converges to $\dfrac{f(\tilde{x}^+) + f(\tilde{x}^-)}{2}$.

Given $Ly + \mu y = f(x) \oplus$ HOM. B.C. on "a" and "b":

μ: ctant.

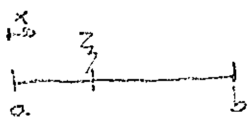
HOMOGENEOUS PROBLEM (with $\mu \to \lambda$)	INHOM. PROBLEM (with μ)
No solution ↳ Only trivial sol. ↳ μ NOT an eigenvalue	$y(x) = \sum_{n=0}^{\infty} \frac{\langle y_n \vert f \rangle}{\mu - \lambda_n} y_n(x)$
Solution ↳ μ IS an eigenvalue ↳ λ_p	• $\langle y_p \vert f \rangle \neq 0 \to$ No solution • $\langle y_p \vert f \rangle = 0 \to y(x) = \sum_{\substack{n=0 \\ n \neq p}}^{\infty} \frac{\langle y_n \vert f \rangle}{\mu - \lambda_n} y_n(x) + \underbrace{C_p}_{\text{arbitrary}} y_p(x)$

GREEN'S METHOD: Assuming $\mu \neq \lambda_n, \forall n$:

$$\boxed{y(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi}$$

$$\circledast \; G(x,\xi) = \rho(\xi) \sum_n \frac{y_n(x) \cdot \overline{y_n}(\xi)}{\mu - \lambda_n}$$

$$\circledast \; \overset{(A)}{LG + \mu G = \delta(x-\xi)} \oplus \text{HOM. B.C. on "a" and "b"} \oplus \text{Continuity everywhere.}$$

- Fix $\xi \Rightarrow$ 

- Compute the prob. for $x < \xi$ and $x > \xi \to$ Apply B.C. to get rid of some ctants.

- Apply continuity $\to G(\xi^-, \xi) = G(\xi^+, \xi) \to$ Only one ctant. left.

- Put everything together using Heaviside θs.

- For the last constant: (A) - Compute derivatives
  - Eliminate δs
  - Using eq.(A), get the ctant.

  (B) - Use: $\begin{cases} G = g_1 \theta(\xi - x) + g_2 \theta(x - \xi) \\ \frac{1}{a_0(\xi)} = (g_2' - g_1')(\xi) \end{cases}$
  - Get the constant.

$\left.\begin{array}{l}\text{PDE}\\ \text{I.C.} \to u|_{\Gamma} \overset{=}{} E(x,y)\end{array}\right\}$

- If Γ is a characteristic curve of the PDE, it will not have a valid solution.
- Sol. will exist in every point of every characteristic Γ crosses.

1st ORDER PDEs

$$A(x,y)\,u_x + B(x,y)\,u_y + C(x,y)\,u = D(x,y)$$

CHARACTERISTICS : $\dfrac{dx}{A} = \dfrac{dy}{B} \Rightarrow F(x,y) = c_1$

$$\left.\begin{array}{l}\eta = F(x,y)\\ \text{Choose } \xi\end{array}\right\} (x,y) \longrightarrow (\eta,\xi) \Rightarrow \underline{\text{New PDE}} \Rightarrow \text{Solve it}$$

↳ Choose wisely...

2nd ORDER PDEs

$$A(x,y)\,u_{xx} + B(x,y)\,u_{xy} + C(x,y)\,u_{yy} = F(x,y,u,u_x,u_y)$$

CHARACTERISTICS : $A\left(\dfrac{dy}{dx}\right)^2 - B\,\dfrac{dy}{dx} + C = 0$

$$B^2 - 4AC \begin{cases} > 0 \to \text{Hyperbolic} \to 2 \text{ real sols.} & \begin{cases}\xi = \varphi(x,y)\\ \eta = \psi(x,y)\end{cases}\\[2ex] = 0 \to \text{Parabolic} \to 1 \text{ real sol} & \begin{cases}\xi = \varphi(x,y)\\ \eta = x\end{cases}\\[2ex] < 0 \to \text{Elliptic} \to 2 \text{ imaginary sols. (complex conj). } (\varphi, \bar{\varphi}) & \begin{cases}\xi = \mathrm{Re}(\varphi)\\ \eta = \mathrm{Im}(\varphi)\end{cases}\end{cases}$$

PARTIAL DIFFERENTIAL EQUATIONS: SEPARATION OF VARIABLES

LAPLACIAN OPERATOR

$$\nabla^2 f = \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{1}{r^2 \sin\theta}\left\{\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 f}{\partial \varphi^2}\right\}$$

LAPLACE EQ.

$\nabla^2 \phi = 0 \rightarrow$

- spherical coord. $\rightarrow \phi = \sum_{l=0}^{\infty}\left(a_l r^l + \frac{b_l}{r^{l+1}}\right)\sum_{m=-l}^{m=l} C_{lm} Y_{lm}(\theta,\varphi)$
- Cylindrical coord. $\rightarrow \phi = \sum_{m,\varkappa}\left(A_m e^{\sqrt{\varkappa} z} + B_m e^{-\sqrt{\varkappa} z}\right)\cdot\left(C_m e^{im\varphi} + D_m e^{-im\varphi}\right)\cdot J_m(\sqrt{\varkappa}\rho)$

$-\infty < m < \infty$

$C_m \cos(m\varphi) + D_m(\operatorname{sen}(m\varphi))$

POISSON EQ.

$$\nabla^2 \phi = -C\rho(r)\ ;\ C = \begin{cases} 1/\varepsilon_0 \\ -4\pi G \end{cases}$$

HELMHOLTZ EQ.

$$\nabla^2 \phi + a\phi = 0,\quad a>0$$

HEAT EQ.

$$\nabla^2 \phi = \frac{1}{a^2}\frac{\partial \phi}{\partial t}$$

WAVE EQ.

$$\nabla^2 \phi = \frac{1}{V^2}\frac{\partial^2 \phi}{\partial t^2}$$

$$R'' + \frac{1}{\rho}R' + \left(\varkappa - \frac{m^2}{\rho^2}\right)R = 0$$

SCHRÖDINGER EQ.

$$\frac{\hbar}{2m}\nabla^2 \phi + V(\vec r)\phi = i\hbar \frac{\partial \phi}{\partial t}$$

SEPARATION OF VARIABLES

$$\phi = X(x)Y(y)Z(z) \xrightarrow[\text{EQ}\cdot\frac{1}{\phi}]{} \underbrace{\quad}_{(x)}^{-l} + \underbrace{\quad}_{(y)}^{m} + \underbrace{\quad}_{(z)}^{n} = 0 \rightarrow$$

The solution will be a linear comb. of every possible solution $\rightarrow \phi = \sum_{l,m,n} XYZ\, A_{lmn}$

SPECIAL FUNCTIONS

LEGENDRE POLYNOMIALS

$$\frac{1}{\sin\theta}\cdot\frac{1}{Z}\frac{d}{d\theta}\left(\sin\theta \frac{dZ}{d\theta}\right) + Q - \frac{m^2}{\sin^2\theta} = 0$$

$Z = P_l^m(\cos\theta)$

$x = \cos\theta$, $y(x) = Z(\theta)$, $x \in [-1,1]$

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(Q - \frac{m^2}{1-x^2}\right)y = 0$$

$Q = l(l+1)$

$m=0$

$$y(x) = P_l(x) = \frac{1}{2^l}\sum_{j=0}^{2j\le l}\frac{(-1)^j}{j!}\frac{[2(l-j)]!}{(l-2j)!\,(l-j)!}x^{l-2j}$$

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2-1)$
$P_3(x) = \frac{1}{2}(5x^3-3x)$

MAIN PROPERTIES:

- Orthogonality $\rightarrow \int_{-1}^{1} P_l(x)P_n(x)\,dx = 0,\ l\neq n$
- $\int_{-1}^{1} P_l^2(x)\,dx = \frac{2}{2l+1}$
- $\{P_l(x)\}_{l\in\mathbb{N}}$ is a complete set on $L^2_{[-1,1]}$

$0 < |m| \le l$

$$y(x) = P_l^m(x) = \frac{1}{2^l\, l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}\left[(x^2-1)^l\right]$$

$P_1^1(x) = (1-x^2)^{1/2} = \sin\theta$
$P_2^1(x) = 3x(1-x^2)^{1/2} = 3\sin\theta\cos\theta$
$P_2^2(x) = 3(1-x^2) = 3\sin^2\theta$
$P_3^1(x) = \frac{3}{2}(5x^2-1)(1-x^2)^{1/2} = \frac{3}{2}(5\cos^2\theta-1)\sin\theta$
$P_3^2(x) = 15x(1-x^2) = 15\cos\theta\sin^2\theta$
$P_3^3(x) = 15(1-x^2)^{3/2} = 15\sin^3\theta$

MAIN PROPERTIES:

- $P_l^{-m}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x)$
- $P_l^0(x) = P_l(x)$
- $P_l^m(-x) = (-1)^{l+m}P_l^m(x)$
- $P_l^m(\pm 1) = 0$
- Two scalar products
  - $\rho = 1$
    - ORTHOG. $\rightarrow \int_{-1}^{1} P_l^m(x)P_{l'}^m(x) = 0,\ l\neq l'$
    - NORM $\rightarrow \int_{-1}^{1}(P_l^m(x))^2\,dx = \frac{(l+m)!}{(l-m)!}\cdot\frac{2}{2l+1}$
  - $\rho = \frac{1}{1-x^2}$
    - ORTHOG. $\rightarrow \int_{-1}^{1} P_l^m\cdot P_{l'}^m\frac{dx}{1-x^2} = 0$
    - NORM $\rightarrow \int_{-1}^{1}(P_l^m(x))^2\frac{dx}{1-x^2} = \frac{(l+m)!}{(l-m)!\,m}$

SPHERICAL HARMONICS

$$Y_{l,m}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\cdot\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\,e^{im\varphi}$$

* Angular part of the Laplace eq. solution.
* If the prob. doesn't have φ dependence, then

$$Y_{l,m}(\theta) = P_l^m(\cos\theta)$$

BESSEL FUNCTIONS

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \left(\varkappa - \frac{m^2}{\rho^2}\right)R = 0$$

$x = \sqrt{\varkappa}\rho$, $y(x) = R(\rho)$; weight: ρ

$$x^2y'' + xy' + (x^2 - m^2)y = 0$$

BESSEL EQUATION

$\lambda^2 - m^2 = 0$ $\begin{cases}\lambda_1 = m \\ \lambda_2 = -m\end{cases}$

1^{st} solution: $y_1(x) = J_m(x) = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!\,\Gamma(l+m+1)}\left(\frac{x}{2}\right)^{2l+m}$

2^{nd} solution

- $\lambda_1 - \lambda_2 \notin \mathbb{N} \rightarrow y_2(x) = J_{-m}(x)$
- $\lambda_1 - \lambda_2 \in \mathbb{N}$, $2m$ even, $m>0$ $\rightarrow y_2(x) = N_m(x)$ — Irregular at axis!

$\lambda_1 - \lambda_2 \in \mathbb{N}$, $2m$ odd $\rightarrow m = k + 1/2$, $k\in\mathbb{N} \rightarrow \begin{cases} J_{k+1/2} = y_1 \\ J_{-(k+1/2)} = y_2\end{cases}$

PROPERTIES OF J_m:

- $J_m' = \frac{1}{2}(J_{m-1} - J_{m+1})$
- $(x^m J_m)' = x^m J_{m-1}$
- $(x^{-m}J_m)' = -x^{-m}J_{m+1}$
- $1 = J_0^2 + 2\sum_{m=1}^{\infty}J_m^2$
- NORM: $\int_0^a J_m^2\left(\frac{\alpha_n^{(m)}}{a}\rho\right)\rho\,d\rho = \frac{a^2}{2}J_{m+1}^2(\alpha_n^{(m)})$
- MUTUAL ORTHOG.: $\int_a^b J_m(\alpha x)J_m(\beta x)\,x\,dx = 0$

$$x^2y'' + xy' - (x^2+m^2)y = 0$$

$$y(x) = I_m(x) = i^{-m}\cdot J_m(ix)$$

~ PROBABILITY ~

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

BAYES THEOREM

$$P(A|B) = P(B|A) \cdot \frac{P(A)}{P(B)}$$

DECOMPOSITION THEOREM

$$P(A) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B})$$

If $\{B_i\}$ is a complete and mutually exclusive set of events:

$$P(A) = \sum_{i=1}^{N} P(A|B_i)P(B_i)$$

...r variations

$\leftarrow k \rightarrow$ | | ... | | |

...-1 $\quad n-k+2 \quad n-k+1$

$$\frac{...!}{(...k)!} \equiv V_k^n$$

Permutations

Ways of arranging n objects : $n!$

Combinations

$\hookrightarrow k!$ diff. ways of arranging

$$\frac{V_k^n}{k!} = \; : \; C_k^n = \frac{n!}{(n-k)!\,k!}$$

...OMIAL DISTRIBUTION

...e repeat an experiment n times $\begin{cases} \text{Success} \to p \\ \text{Failure} \to 1-p \end{cases}$

...ob. of k successes: $P(k) = \dfrac{n!}{(n-k)!\,k!}\, p^k (1-p)^{n-k}$

$* \; \langle k \rangle = np$

$* \; \sigma^2 = np(1-p)$

POISSON DISTRIBUTION

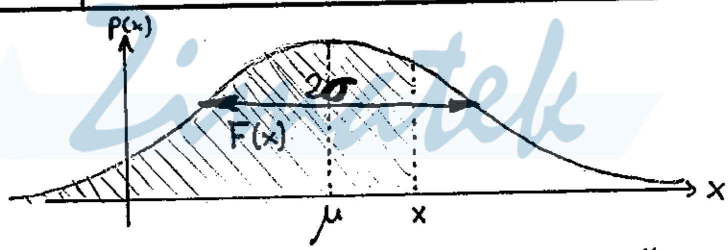
- We repeat an experiment n times $\begin{cases} \text{Success} \to p \\ \text{Failure} \to 1-p \end{cases}$
- $p \ll 1 \Rightarrow \langle k \rangle = np = \lambda$ $\quad \hookrightarrow \gg 1$ $\quad \hookrightarrow$ Continuum (over time)
- Prob. of k successes: $P(k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$

$* \; \langle k \rangle = \lambda$

$* \; \sigma^2 = \lambda$

...USSIAN DISTRIBUTION

$$p(x)dx = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad x \in (-\infty, \infty)$$



$* \; \langle x \rangle = \mu$

$* \; \sigma^2 = \langle (x - \langle x \rangle)^2 \rangle$

$$F(x) = P_r(X < x) = \int_{I_1}^{x} p(x)dx = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} \exp\left[-\frac{(t-\mu)^2}{2\sigma^2}\right] dt$$

STD. GAUSSIAN: $p(z)\,dz = \dfrac{1}{\sqrt{2\pi}} \exp\left[-\dfrac{z^2}{2}\right]$ $\quad (\mu = 0, \sigma = 1)$

$$\Phi(z) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left[-\frac{t^2}{2}\right] dt \to F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$\boxed{\text{Binomial } (n,p)}$

$n \to \infty$, p finite $\to$ $\boxed{\text{Gaussian}}$

$n \to \infty$, $p \to 0$ // $\lambda = np$, finite $\to$ $\boxed{\text{Poisson } (\lambda)}$

$\boxed{\text{Gaussian}} \xleftarrow{\lambda \gg 1} \boxed{\text{Poisson } (\lambda)}$

$\hookrightarrow$ BINOMIAL $\begin{cases} \mu = np \\ \sigma^2 = np(1-p) \end{cases}$

$\hookrightarrow$ POISSON $\begin{cases} \mu = \lambda \\ \sigma^2 = \lambda \end{cases}$

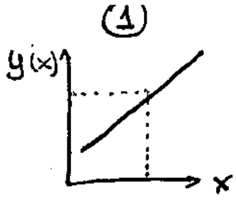
CENTRAL LIMIT THEOREM

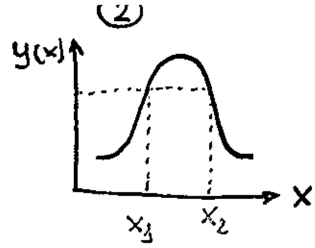
- $\{x_1, \dots, x_i, \dots, x_n\}$ independent random vars.

$\quad \hookrightarrow P(x_i) \begin{cases} \mu_i \\ \sigma_i^2 \end{cases}$

- $y \equiv \dfrac{1}{n} \displaystyle\sum_i x_i \Rightarrow \begin{cases} \bullet \; \langle y \rangle = \dfrac{1}{n} \sum_i \mu_i \\ \bullet \; \sigma_y^2 = \dfrac{1}{n^2} \sum_i \sigma_i^2 \\ \bullet \; n \to \infty \Rightarrow P(y) \to \text{Gaussian} \end{cases}$

FUNCTIONS OF RANDOM VARIABLES





$$\left.\begin{array}{l} x, P_x(x) \\ y = y(x) \end{array}\right\} \xrightarrow{?} P_y(y)$$

- Discrete case $\rightarrow \underline{X} = \{x_1, \ldots, x_n\}$

$$P_{\underline{x}}(\underline{x}) = \begin{cases} P(x_i) & , \ x = x_i \\ 0 & , \ \text{otherwise} \end{cases}$$

1 $\nearrow$ $P_y(y) = \begin{cases} P_x(x(y)) & , \ y = y_i \\ 0 & , \ \text{otherwise} \end{cases}$

2 $\searrow$ $P_y(y) = \begin{cases} \sum_j P_x(x_j) & , \ x_j = x(y) \\ 0 & , \ \text{otherwise} \end{cases}$

- Continuous case $\rightarrow x \in [I_1, I_2]$

1 $P_y(y)dy = P_x(x(y)) \left| \dfrac{dx}{dy} \right|$

2 $P_y(y) = \sum_i P_x(x_i(y)) \left| \dfrac{dx}{dy} \right|_{x_i(y)}$

$z = f(x,y)$ $\quad x \sim P_x(x) <^{\mu_x}_{\sigma_x^2}$ $\quad y \sim \ldots$

$\langle z \rangle = f(\mu_x, \mu_y)$

$$\sigma_z^2 = \left(\frac{\partial f}{\partial x}\bigg|_{\mu_x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\bigg|_{\mu_y}\right)^2 \sigma_y^2 + 2\left(\frac{\partial f}{\partial x}\bigg|_{\mu_x} \cdot \frac{\partial f}{\partial y}\bigg|_{\mu_y}\right) COV[x,y]$$

$COV[x,y] = 0 \Leftrightarrow x, y$ indep.

$COV[x,y] = \langle (x - \langle x \rangle)(y - \langle y \rangle) \rangle =$
$= \langle xy \rangle - \langle x \rangle \langle y \rangle$

~ STATISTICS ~

Sample $\Rightarrow \vec{X} = \{x_1, \ldots, x_N\} \rightarrow$ results of N experiments (DATA)

Population $\Rightarrow$ Basically, the distrib. which describes prob. of getting data for the sampl. $\rightarrow P(\vec{x}|\vec{a}) \rightarrow \mu, \sigma, \lambda, \ldots$

STUFF WE CAN OBTAIN FROM THE SAMPLE

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \qquad V_{xy} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})$$

$$S^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2 \qquad r_{xy} = \frac{V_{xy}}{S_x S_y}$$

ESTIMATORS

ESTIMATORS $\rightarrow$ When we don't know a certain parameter of the distrib. from which x is measured, just estimate it!

- Mean $(\mu) \rightarrow \bar{x}$
- Variance
  - If μ is known $\rightarrow \hat{\sigma}^2 = \dfrac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$
  - If μ is unknown $\rightarrow \hat{\sigma}^2 = \dfrac{N}{N-1} S^2$

ERRORS ON ESTIMATORS

$\bar{x} = \bar{x}_{obs} \pm \dfrac{\sigma}{\sqrt{N}}$ $\quad$ Might have to estimate it.

- If estim. is unbiased $\rightarrow$ Error $\equiv \sigma_{\hat{a}}$
- If estim. is biased $\rightarrow$ Error $\equiv \sigma_{\hat{a}}^2 + b^2(\hat{a})$

BE CAREFUL! Distinguish these 3 concepts:

- Sample Variance S^2
- Population variance σ^2 (or $\hat{\sigma}^2$, if estimated)
- Variance associated to $\bar{x}$ $\left(\sigma_{\bar{x}}^2 = \dfrac{\sigma^2}{N}\right)$

CONFIDENCE LIMITS & CONFIDENCE LEVELS

CONFIDENCE LIMITS & CONFIDENCE LEVELS $\Rightarrow$ What's the probability of finding the real value of parameter "a" between some two limits a_- & a_+, from the estimations we've made?

$$P(a_- \leq a \leq a_+) = 1 - \alpha - \beta \quad (\alpha = \beta \Leftrightarrow \text{Central confidence limits})$$

$$P(\hat{a}_{obs} < a_\alpha(a_+)) = \alpha = \int_{-\infty}^{\hat{a}_{obs}} d\hat{a} \, \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(\hat{a} - a_+)^2}{2\sigma^2}\right] = \Phi\left(\frac{\hat{a}_{obs} - a_+}{\sigma}\right)$$

$$P(\hat{a}_{obs} > a_\beta(a_-)) = \beta = \int_{\hat{a}_{obs}}^{\infty} d\hat{a} \, \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(\hat{a} - a_+)^2}{2\sigma^2}\right] = 1 - \Phi\left(\frac{\hat{a}_{obs} - a_-}{\sigma}\right)$$

1.1 DIFFERENTIAL EQS. (DEF. & TYPES)

- Ordinary equation (algebraic) ⇒ No derivatives (e.g.: $x^2+y^2=1$)

VS.

Diferential equations ⇒ With derivatives (e.g.: $x+yy'=0$)

- Notation:

$$y' = \frac{dy}{dx} \quad , \quad y'' = \frac{d^2y}{dx^2}$$

Typically, $y = y(x)$

y: dep. variable; x: indep. variable

O.D.E (ordinary differential equations)

Single variable

E.g.: $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$

P.D.E. (partial differential equations)

Several variables

E.g.: Heat equation (depends on position & time)

ORDER OF THE EQUATION ⇒ Highest derivative grade of the eq.

E.g.: 2nd order ← pendulum, Newton's 2nd law, Schrödinger   3rd order → Jerk (amount of variation in acceleration)

- Linearity → Power of derivatives means no linearity

E.g.: $(y')^2 + y = 0$ → Non-linear

$y' + xy = 0$ → Linear

$y' + x^3 y = 0$ → Linear

$y' + y^2 = 0$ → Linear

- General form of a linear equation

$$a_0(x)\,y^{(n)} + a_1(x)\cdot y^{(n-1)} + \dots + a_{n-1}(x)\cdot y' + a_n(x)\cdot y = b(x)$$

arbitrary functions

1.2. TYPES OF SOLUTIONS

General expression for any diff. eq. of order n:

$$F(x, y, y'', \dots, y^{(n)}) = 0$$

- Explicit solutions ⇒ $y = f(x)$
  ↳ the dependent variable is isolated.

How can a solution be checked?

It must satisfy $F(x, y, y'', \dots, y^{(n)}) = 0$.

$\forall x \in I$ satisfying , then $y = f(x)$ is an explicit solution with validity range I.

E.g.: $y = \sqrt{1 - x^2}$ is an exp. solution of O.D.E. $x + yy' = 0$.

Demonstration

$$y=\sqrt{1-x^2} \rightarrow y'=f'=-\frac{x}{\sqrt{1-x^2}}$$

$$x+y\cdot y'=x+\cancel{\sqrt{1-x^2}}\cdot\left(-\frac{x}{\cancel{\sqrt{1-x^2}}}\right)=x-x=0$$

Validity range $\Rightarrow y=\sqrt{1-x^2} \Rightarrow x\in(-1,1)$

- Implicit solution

$g(x,y)=0 \Rightarrow$ "y" is not isolated!

Must satisfy $F(x,y,y',\dots,y^{(n)})=0 \quad \forall x\in I$

We need to find $y', y'', \dots$ thus, we'll differentiate the eq. as a whole.

E.g.:

$$[\sin(\omega x)=0]'=\frac{\partial}{\partial x}[\sin(\omega x)=0]\overset{\omega x=u}{=\!=}\frac{\partial}{\partial x}[\sin u=0]$$

$$\frac{\partial}{\partial x}[\sin u]=\cos u\cdot u'=\cos u\cdot\frac{du}{dx}=(\cos\omega x)\cdot\omega$$

$$\text{CHAIN RULE} \Rightarrow \frac{\partial}{\partial x}[g(x,y(x))]=\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y}\cdot\frac{\partial y}{\partial x}$$

E.g.:

$$\sin(\omega x+y)=0 \overset{\omega x+y=u}{=\!=} \sin u=0$$

$$[\sin u]'=\cos u\cdot u'=\cos u\cdot\frac{\partial}{\partial x}(\omega x+\overset{y(x)}{y})=$$

$$=\cos u\cdot(\omega+y')=\cos(\omega x+y)\cdot(\omega+y')$$

We have deduced the following:

$$g(x,y)=0 \Rightarrow \frac{\partial g}{\partial x}+\frac{\partial g}{\partial y}\cdot\frac{\partial y}{\partial x}=0=\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y}\cdot y'$$

$$y'=-\frac{\partial g/\partial x}{\partial g/\partial y}=-\frac{g_x}{g_y}$$

$$F\left(x,\ y,\ \left(-\frac{\partial g/\partial x}{\partial g/\partial y}\right),\ \dots\right)$$

↳ expression for y', as demanded by $g(x,y)=0$

E.g.: Show that $x^2+y^2=1$ is an implicit solution for $x+y\cdot y'=0$

$$g(x,y)=x^2+y^2-1=0$$

$$y'=-\frac{\partial g/\partial x}{\partial g/\partial y}=-\frac{2\cancel{x}}{2y}=-\frac{x}{y}$$

Substitute it on the diff. eq.:

$$x+y\cdot y'=x+\cancel{y}\cdot\left(-\frac{x}{\cancel{y}}\right)=0$$

* Parametric solutions

$$\begin{cases} x=f(t) \\ y=g(t) \end{cases} \Rightarrow \text{"}t\text{" is an auxiliary variable which makes } y=y(t(x))$$

In order to check if it is a solution, we'll calculate $y', y'', \dots$

$$y'=\frac{dy}{dx}=\frac{\partial y}{\partial t}\cdot\frac{\partial t}{\partial x}=\frac{\partial y}{\partial t}\cdot\left(\frac{1}{\frac{\partial x}{\partial t}}\right)\overset{\text{for short}}{=}\frac{g'}{f'}$$

E.g. Show that $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$ is a parametric sol. for $x + y \cdot y' = 0$

$$y' = \frac{g'}{f'} = \frac{\cos t}{(-\sin t)} \to x + y \cdot y' = \cos t + \sin t \cdot \frac{\cos t}{(-\sin t)} = 0$$

E.g.: Prove that $y^3 - 3xy = 2C$ is the general solution
$\hookrightarrow g(x,y) = y^3 - 3xy - 2C$
for $y + (x - y^2) y' = 0$

$$y' = -\frac{\partial g / \partial x}{\partial g / \partial y} = -\frac{-3y}{3y^2 - 3x} = \frac{y}{y^2 - x} = \frac{y}{-(x - y^2)}$$

$$y + (x - y^2) y' = y + \cancel{(x - y^2)} \cdot \frac{y}{-\cancel{(x - y^2)}} = 0$$

1.3 EXISTENCE OF SOLUTION

Typically, equations have several solutions.

Families of solutions are expressions that bring together different solutions by using free constants. They might appear in explicit, implicit, parametric sols.

E.g.: Check that the following expressions are families of solutions for $x + y \cdot y' = 0$

a) $y = \sqrt{c^2 - x^2}$ b) $y = -\sqrt{c^2 - x^2}$ c) $x^2 + y^2 = c^2$ d) $\begin{cases} x = c \cdot \cos t \\ y = c \cdot \sin t \end{cases}$

* General solution: a family of solutions with as many parameters as the order of the equation.

n-th order O.D.E. $\Rightarrow$ general sol.: family with n constants,

E.g.:

Some solutions for $y+(x-y^2)y'=0$

$$\begin{cases} y^3-3xy=2 \rightarrow \text{NOT a g.s.} \\ y^3-3xy=2C \rightarrow \text{general solution} \\ y^3-3xy+A=B \rightarrow \text{general solution} \\ \quad (A-B=C) \end{cases}$$

↳ one ct.

* Particular solutions: obtained by giving values to the constants of a solution.

* Singular solutions: the ones not included in the general solution (no method, intuition, trial & error)

E.g.:

Prove that $y=C(x-C)$ is the general solution for

$(y')^2-xy'+y=0$

Among the following sols., which are particular and which singular?

a) $y=0$ PART.

b) $y=x-1$ PART.

c) $y=\dfrac{x^2}{4}$

1.4. UNIQUENESS OF THE SOLUTION

Equations generally have several solutions. But physics requires more precision: additional conditions have to be imposed, i.e. fix the constants in the families of solutions.

We can formulate an initial value problem:

1. Choose a value of the indep. variable $x=x_0$.
2. Choose the values of the dep. variable and its $(n-1)$ derivatives at x_0 $(y_0=y(x_0),\ y'_0=y'(x_0)\ldots)$

E.g.: Let's see that the initial value problem (ODE + condition(s))

$$\begin{cases} (y')^2 - xy' + y = 0 \\ y(0) = 1 \end{cases}$$

admits $y = \pm x - 1$ as a solution.

We've already seen that $y = cx^2 - c$ is the gen. sol. of that O.D.E..

Cond. of the problem: $y(0) = 1$, which is telling us that $x_0 = 0$

$$y(x_0) = y(0) = 1 = y_0$$

For our gen. sol.:

$$y(0) = c \cdot 0^2 - c^2 = -1 \therefore \to c = \pm 1 \to y = \pm x - 1 \quad \checkmark$$

E.g.: Prove that the problem $\begin{cases} (y')^2 - xy' + y = 0 \\ y(0) = 0 \end{cases}$ admits two real solutions, but no sol. at all when $y(0) = 1$ is demanded.

$$\text{Gen. sol.} \to y = cx^2 - c^2 \begin{cases} y(0) = 0 = c0^2 - c^2 \to c = 0 \quad \checkmark \\ y(0) = 1 = c \cdot 0^2 - c^2 \to \text{No real. sol.} \end{cases}$$

$$\text{Sing. sol.} \to y = \frac{x^2}{4} \quad \checkmark$$

$$\hookrightarrow y(0) = 0 = \frac{0^2}{4}$$

Let's consider a more general approach with $y(\overset{x_0}{0}) = 1$:

$$(y'(x_0))^2 - \underbrace{\cancel{x_0 \cdot y'(x_0)}}_{0} + \underbrace{y(x_0)}_{1} = 0 \to (y'(x_0))^2 + 1 = 0 \Rightarrow \text{Impossible}$$

Fixing initial values does not guarantee either existence or uniqueness.

Let's try to 'build a unique solution around x_0 by doing a Taylor series:

$$y = y(x_0) + y'(x_0)(x-x_0) + \dots + \frac{1}{n!} y^{(n)}(x_0)(x-x_0)^n$$

The first coefficient in the series is the initial value $y(x_0)=y_0$.
Next coefficients will be calculated by differentiation

$$y' = f(x,y) = f(x, y(x))$$

$$y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot f$$

The Taylor series can be used to build the unique solution if the series itself exists and converges.

PEANO'S PROBLEM: continuity of $f(x,y)$ guarantees existence but not uniqueness.

E.g.: Prove that $y=0$ and $y = \begin{cases} 0 & x \leq 0 \\ \left(\frac{2}{3}x\right)^{3/2} & x \geq 0 \end{cases}$

are two different solutions of the initial value problem: $\begin{cases} y' = y^{1/3} & \text{(ODE)} \\ y(0)=0 & \text{(condition)} \end{cases}$

Discuss the reason of the non-uniqueness.

$y=0$ case

Are both $\begin{matrix} y' = y^{1/3} \\ y=0 \end{matrix}$ satisfied?

$$y=0 \longrightarrow y'=0$$

$$\underset{\overset{\|}{0}}{y'} = \underset{\overset{\|}{0}}{y^{1/3}}$$

Second case

$$y = \begin{cases} 0 & x \leq 0 \\ \left(\frac{2}{3}x\right)^{3/2} & x \geq 0 \end{cases} \longrightarrow y' = \begin{cases} 0 & x \leq 0 \\ \cancel{\frac{3}{2}} \cdot \left(\frac{2}{3}x\right)^{1/2} \cdot \cancel{\frac{2}{3}} \end{cases}$$

$$y^{1/3} = \begin{cases} 0 & x \leq 0 \\ \left[\left(\frac{2}{3}x\right)^{\frac{3}{2}}\right]^{\frac{1}{3}} = \left(\frac{2}{3}x\right)^{1/2} & x \geq 0 \end{cases}$$

$$\underline{y' = y^{1/3} = f(x,y)}$$
$$y(0) = 0$$

Lack of uniqueness due to the Taylor series.

$$y = y(x_0) + f(x_0,y_0)(x-x_0) + \left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right](x-x_0)^2 + \dots =$$

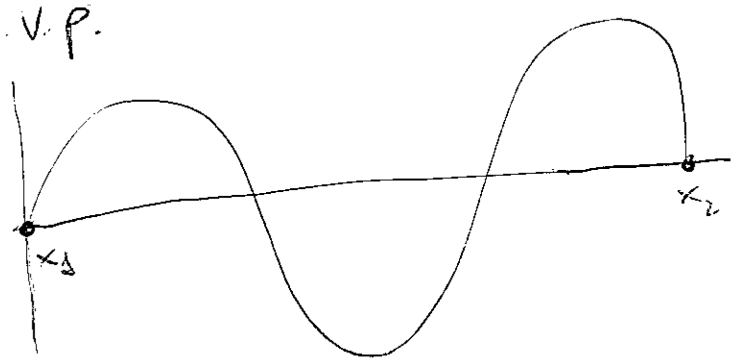
$$= 0 + 0\,(x-x_0) + \left[0 + \left[\frac{\partial}{\partial y}\left(y^{1/3}\right)\right] f\right]\Bigg|_{y_0=0} (x-x_0)^2 =$$

$$= 0 + 0 + \Big[0 + \underbrace{y^{-2/3}}_{\infty!!}\Big]_{y=0} + \dots$$

An equation may admit infinite solutions (family).

Physics demands fixing the free constants $\begin{cases}\text{initial value problem} \\ \text{boundary value problem}\end{cases}$

b.v.p.



$y(x_1)=0$

$y(x_2)=0$

In boundary value problems, we fix values of the dependable variable and/or its derivatives at different values of the indep. variable.

E.g.: $y(x_1), y(x_2), \dots, y'(x_1), \dots, y''(x_1), y''(x_2), \dots$

E.g.: In convenient dimensionless variables, the harmonic oscilator cas be written as $\underline{y''+y=0}$.

Typically, $\ddot{x}+\omega^2 x=0$. We have absorbed ω in t and then have renamed $t \to x$ and $x \to y$.

i) Check that the gen. sol. is $y = A \cdot \cos t + B \cdot \operatorname{sen} t$

ii) Show that the following boundary problems have 1, 0 & ∞ sols. respectively:

a) $y(0)=1$, $y\left(\frac{\pi}{2}\right)=2$

b) $y(0)=1$, $y(\pi)=2$

c) $y(0)=0$, $y(\pi)=0$

i) $y = A\cdot\cos t + B\cdot\sin t$

$y'' = -A\cdot\cos t - B\cdot\sin t \equiv -y$

$y'' + y = (-y) + y = 0$ ✓

ii) a) $y(0)=1,\ y(\pi/2)=2$

$$\begin{cases} y(0) = A\cdot\cos 0 + \cancel{B\cdot\sin 0} = 1 \to \boxed{A=1} \\ y(\pi/2) = \cancel{A\cdot\cos \pi/2} + B\cdot\sin \pi/2 = 2 \to \boxed{B=2} \end{cases}$$

b) $y(0)=1,\ y(\pi)=2$

$$\begin{cases} y(0) = A\cdot\cos 0 + \cancel{B\cdot\sin 0} = 1 \to A=1 \\ y(\pi) = A\cdot\cos \pi + \cancel{B\cdot\sin \pi} = 2 \to A=2 \end{cases} \to \text{Not possible}$$

c) $y(0)=0,\ y(\pi)=0$

$$\begin{cases} y(0) = A\cdot\cos 0 + \cancel{B\cdot\sin 0} = 0 \to A=0 \\ y(\pi) = A\cdot\cos\pi + \cancel{B\cdot\sin\pi} = 0 \to -A=0 \end{cases} \to \underline{A=0} \quad (B=\lambda)$$

PROBLEMS: 1 - 7

E.g.: Check that $y = \dfrac{1-C\cdot e^{2x}}{1+C\cdot e^{2x}}$ is the gen. sol. for $y' = y^2 - 1$. Find by inspection two easy solutions.

This question makes sense since we have 1st order ODE and an expression with 1 free parameter.

We should try expressions like $y=a$, $y=ax+b$, $y=ax^n$, $y=ax^2+b$

For example: $y=\pm 1$

2.1. GEOMETRIC MEANING

A curve in the (x,y) plane is described by a finite equation $\varphi(x,y)=0$. In order to obtain a family of curves, a free constant should be introduced in the eq.: $\varphi(x,y,c)=0$.

E.g.: $x^2+y^2-c^2=0 \Rightarrow$ circumferences of different radii.

So, why should we consider curves in ODE lectures? There is a close relationship among them:

$$\left.\begin{array}{l}\varphi(x,y,c)=0\\ \quad + \\ \left[\varphi(x,y,c)=0\right]'\end{array}\right\} \longrightarrow \text{can be combined as } \underline{F(x,y,y')=0} \text{ (c is eliminated)}$$

Remember that $\left[\varphi(x,y,c)\right]' = \dfrac{\partial\varphi}{\partial x} + \dfrac{\partial\varphi}{\partial y}\cdot\underbrace{\boxed{\dfrac{\partial y}{\partial x}}}_{y'} = 0$

Ex. 2.1.

Differentiate $x^2+y^2-c^2=0$ to show that its associated ODE is $x+yy'=0$

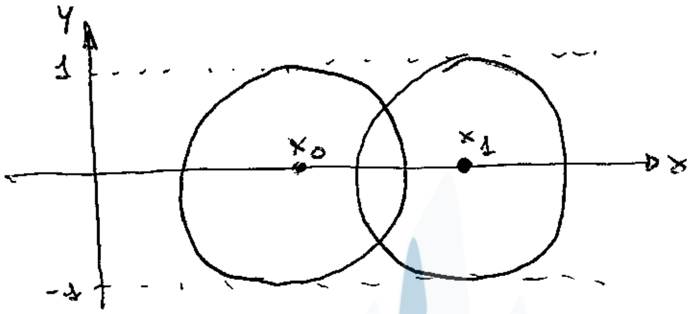
$$\left[x^2+y^2-c^2=0\right]' \Rightarrow 2x+2yy'-0=0 \rightarrow x+yy'=0$$

If $\varphi(x,y,c)=0$ gives $F(x,y,y')=0$ by differentiation (and some extra algebra), then the former is an integral curve of the latter.

<u>CAREFUL!</u> Eliminating c can add/remove solutions!

Ex. 2.2.

Find the ODE for the unit circumferences with their center along the "x" axis. Discuss possible singular sols.



Unit circumference with center in $(0,0)$: $x^2+y^2-1=0$

In order to move along the "x" axis, we shift "x":

$$\left[(x-c)^2+y^2-1=0\right]' \rightarrow 2(x-c)+2yy'=0 \rightarrow$$

$\hookrightarrow (x-c)^2=1-y^2$

$$\rightarrow (x-c)^2+(y\cdot y')^2=0 \rightarrow (1-y^2)+(y\cdot y')^2=0$$

We can find the sing. sol. by inspection:

$$y=a \rightarrow y'=0$$

$$1-a^2+0=0 \rightarrow a=\pm 1=y$$

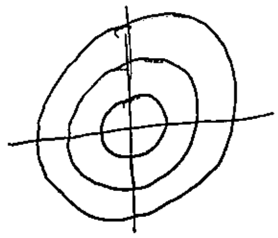
There's no way to fix the constant c to obtain $y=\pm 1$.

Therefore, $y=\pm 1$ are singular sols.

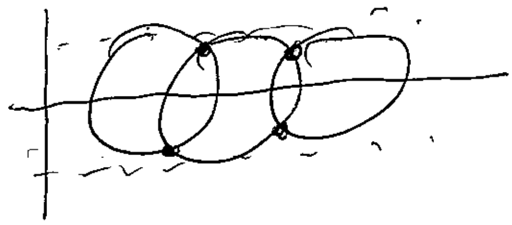
• Special families of curves: CONGRUENCES

DEF: families of curves that don't cross each other (well defined derivative at each point.

E.g.: $x^2+y^2=c^2$ ✓



This is NOT a congruence:



As each curve is associated with a single value c, each point has one asso. value of c. As a consequence, $\varphi(x,y,c)=0$ can be used to solve for c at any given point.

If a family of curves is a congruence, $\varphi(x,y,c)=0$ can be rewritten as $u(x,y)=c$.

We'll calculate the ODE of a congruence:

$$[u(x,y)=c]' \Rightarrow u'(x,y)=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x}=\boxed{\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot y'=0}$$

* Symetric form:

$$dx\cdot[u']=\frac{\partial u}{\partial x}\cdot dx+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\cancel{\partial x}}\cdot\cancel{dx}=\boxed{\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}\cdot dy=0}$$

NOTATION: $P=\frac{\partial u}{\partial x}$, $Q=\frac{\partial u}{\partial y}$

* Normal form: $y'=f(x,y)$

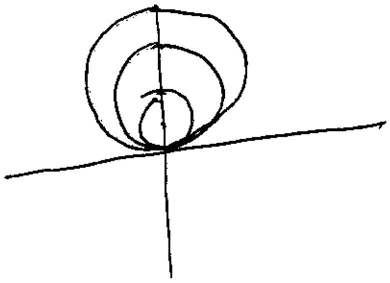
$$P\cdot dx+Q\cdot dy=0$$

$$Q \cdot dy = -P \cdot dx \rightarrow \frac{dy}{dx} = -\frac{P}{Q} = y' = f(x,y)$$

Ex. 2.4.

Show that the ODE of the circumferences of arbitrary radius centered at the "y" axis and touch the "x" axis is

$$y' = \frac{2xy}{x^2 - y^2}$$



$$x^2 + (y-c)^2 - c^2 = 0$$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} \cdot dy = P \cdot dx + Q \cdot dy$$

↳ The equation of a congruence (in symmetric form) is an exact differential.

SCHWARZ'S THEOREM

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{d^2 u}{\partial x \partial y} = \frac{d^2 u}{\partial y \partial x}$$

Example: integration of an exact equation

Our example:

$$\underset{P}{x} \cdot dx + \underset{Q}{y} \cdot dy = 0$$

Is it exact? $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ $\left\{ \begin{array}{l} \frac{\partial P}{\partial y} = 0 \\ \frac{\partial Q}{\partial x} = 0 \end{array} \right.$ $\rightarrow$ YES

By definition, $\Rightarrow$ constant with respect to "x"

$$P = \frac{\partial u}{\partial x} \Rightarrow u = \underbrace{\int P \cdot dx + h(y)}_{*}$$

By definiton again,

$$Q = \frac{\partial u}{\partial y} = y \Rightarrow u = \int x \cdot dx + h(y) = \frac{x^2}{2} + h(y)$$

but from the previous work I've found:

$$Q = \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\underbrace{\left[\frac{x^2}{2} + h(y)\right]}_{*} = x + h'(y)$$

For compatibility:

$$h'(y) = y \rightarrow h(y) = \int y \cdot dy + C = \frac{y^2}{2} + C$$

Our result is:

$$u = \frac{x^2}{2} + \frac{y^2}{2} + C$$

$$[u]' = x + y \cdot y' = 0 \rightarrow x + y \cdot \frac{dy}{dx} = 0$$

Ex. 2.5

Solve the O.D.E.:

$$(x + y + 1)\,dx + (x - y^2 + 3)\,dy = 0$$

Is it exact?

$$\frac{\partial P}{\partial y} = 1 \quad , \quad \frac{\partial Q}{\partial x} = 1 \longrightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \rightarrow \text{It is}$$

$$U = \int P \cdot dx + h(y) = \int (x+y+1)\,dx + h(y) = \frac{x^2}{2} + xy + x + h(y) + C$$

$$h'(y) = 3 - y^2 \rightarrow h(y) = 3y - \frac{y^3}{3}$$

$$\cancel{U = \frac{x^2}{2} + xy + x + xy - \frac{y^3}{3} + 3y + C = \frac{x^2}{2} - \frac{y^3}{3} + 2xy + x + 3y + C}$$

1^{st} special case:

Equations without dependent variable

- $P(x)\,dx + Q(x)\,dy = 0$

$$\frac{dy}{dx} = -\frac{P(x)}{Q(x)} \Rightarrow y = -\int \frac{P(x)}{Q(x)}\,dx$$

E.g.: $y' = -x \rightarrow y = -x^2 + C$

2^{nd} special case:

Equations with separated variables

$$P(x)\,dx + Q(y)\,dy = 0$$

"x" & "y" "do not mix"

Inmediately exact $\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 0 \Rightarrow U = \int P \cdot dx + \int Q \cdot dy$

1^{st} special case example:

$$y^2\left((y')^2 + 1\right) = 0$$

Think boldly, take "x" as the dependent variable

$$x \equiv x(y) \Rightarrow \frac{\partial x}{\partial y} = \dot{x} \rightarrow y' = \frac{1}{\dot{x}} \Rightarrow y^2\left(\frac{1}{(\dot{x})^2} + 1\right) = 1$$

2^{nd} special case example:

$$(1+y)\cdot e^y\cdot y' = 2x \xrightarrow{sol.} u = y\cdot e^y - x^2 + C$$

GENERAL PROCEDURE OF INTEGRATION OF AN EXACT EQ.

1) Check that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

2) $u = \int P\cdot dx + h(y)$

3) $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\int P\cdot dx + h(y)\right] = \int \frac{\partial P}{\partial y}\cdot dx + h'(y) = Q$

4) $h = \int \left\{ Q(y) - \int \frac{\partial P}{\partial y}\cdot dx \right\}\cdot dy$

So, we've got 2 different ways to obtain the same result:

WAY 1

$$u = \int P\cdot dx + h(y)$$

$$h(y) = \int \left\{ Q - \int \frac{\partial P}{\partial y}\cdot dx \right\} dy$$

WAY 2

$$u = \int Q\cdot dy + h(x)$$

$$h(x) = \int \left\{ P - \int \frac{\partial Q}{\partial x}\, dy \right\} dx$$

2.4. INTEGRATING FACTOR

An example first:

$\frac{x}{y}dx + dy = 0$ is not exact $\left(\frac{\partial Q}{\partial x} = 0, \frac{\partial P}{\partial y} \neq 0\right)$, but if we multiply it with y, it becomes exact.

Doing sthng. like that is sometimes possible.

If a non-exact eq. $Pdx + Qdy = 0$ becomes exact when multiplied by $\mu(x,y)$, then this function is an integrating factor for that eq. Generally, the solutions of $\mu(Pdx + Qdy) = 0$ will also be sols. for $Pdx + Qdy = 0$.

1st exception

Let's consider $x \cdot y \cdot dx + y^2 dy = 0$. It has a singular solution $y=0$.

On the other hand, it accepts the integrating factor $\mu = \frac{1}{y}$:

$\frac{1}{y}(xydx + y^2dy) = x \cdot dx + y \cdot dy = 0.$

$y=0$ is no longer a solution, so we've lost one root of $\frac{1}{\mu}$, and in general one has to see this happens with these roots.

SUMMARY: Sols. which give $\frac{1}{\mu} = 0$ may get lost.

2nd exception

- The integration factor can introduce fake solutions.
- $\mu(x,y)=0$ can describe solutions of the new eq. that were not solutions of the original one.

Ex. 2.8

Show that $\mu = \frac{1}{xy^2}$ is an integrating factor for

$$\underbrace{(xy+y^2)}_{P}dx \underbrace{-x^2}_{Q} dy = 0$$

Get the gen. sol. Is there any sing. solution? Are there lost solutions?

$$\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q)$$

$$\frac{\partial}{\partial y}\left[\frac{1}{xy^2}(xy+y^2)\right] = \frac{\partial}{\partial x}\left[-\frac{1}{xy^2}x^2\right] \Rightarrow -\frac{1}{y^2} = -\frac{1}{y^2} \checkmark$$

$$U = \int(\mu P)dx + \int\left\{\mu Q - \int\frac{\partial}{\partial y}(\mu P)dx\right\}dy =$$

$$= \int\left(\frac{1}{x}+\frac{1}{y}\right)dx + \int\left(-\frac{x}{y^2} - \int\frac{\partial}{\partial y}\left(\frac{1}{x}+\frac{1}{y}\right)dx\right)dy =$$

$$= \ln x + \frac{x}{y} + \int\left\{-\frac{x}{y^2} + \int\frac{1}{y^2}dx\right\}dy =$$

$$= \ln x + \frac{x}{y} + \int\left(-\frac{x}{y^2} + \frac{x}{y^2}\right)dy = \ln x + \frac{x}{y} + C = 0$$

Lost solutions?

$$\frac{1}{\mu} = xy^2 = 0 \Rightarrow y = 0$$

Added sols?

$\mu = 0 \to y = \infty \to$ Not continuous, not admissible as a sol., not added.

$$\frac{x}{y} + \ln x + C = 0 \xrightarrow{y=0} \frac{x}{0} + \ln x + C = 0$$

ok, but we'll rewrite the eq.

if $C = \infty$

$$x + y \cdot \ln(x) + Cy = 0 \Rightarrow y(\ln(x) + C) = -x$$

$$y = \frac{-x}{\log(x) + C} \underset{C = \frac{1}{D}}{=\!=} -\frac{Dx}{D \cdot \log(x) + 1} \to y = 0 \text{ corresponds to } D = 0$$

$y=0$ has not been lost, as it is a part case of the gen sol!!

Anticipating ex. 2.a.

Integrate

$$-(x-4)dx - \frac{y^2-3}{y^4}dy = 0 \Rightarrow -\frac{x^2}{2} + 4x + \frac{1}{y} - \frac{1}{y^3} + C = 0$$

2.5 SEPARABLE EQUATIONS

- In their symmetric form, the dep. var. and the ind. var. can be grouped in factors:

$$R(x)\,S(y)\,dx + U(x)V(y)\,dy = 0$$

- Always admit the int. factor $\mu = \dfrac{1}{S(y)\,U(x)}$

- The new eq. is separable:

$$\frac{R(x)}{U(x)}\,dx + \frac{V(y)}{S(y)}\,dy = 0$$

Lost sols. (may be):

$\mu = 0 \longrightarrow \boxed{S(y)=0}$ → they don't tell us anything

$\cancel{U(x)=0}$ ↳ Doesn't give any info.

Example:

$x(1+y)y' = y \rightarrow -y\,dx + x(1+y)\,dy = 0$

$R(x) = -1 \qquad U(x) = x \qquad \mu = \dfrac{1}{S\cdot U} = \dfrac{1}{xy}$

$S(y) = y \qquad V(y) = 1+y$

Gen. sol. $\rightarrow |y|\cdot e^{y} = C|x|$

$S(y) = y \overset{?}{=} 0$

$y=0$ may be lost.

$|y|e^{y} = C\cdot|x| \xrightarrow{\forall x} 0\cdot e^{0} = C|x| \rightarrow$ The sol. is recovered for $C=0$, so the sol is particular, not singular, so it wasn't lost.

Ex. 2.9

Solve the eq.: $\underbrace{(x-4)}_{R}\underbrace{y^4}_{S}dx \underbrace{-x^3}_{U}\cdot\underbrace{(y^2-3)}_{V}dy=0$

$\mu=-\frac{1}{x^3y^4}$ - - - $\Rightarrow$ Gen. sol.: $\frac{1}{x}+\frac{2}{x^2}-\frac{1}{y}+\frac{1}{y^3}=C=\frac{1}{D}$

($y=0$ is not lost, as it's recovered when $D=0$).

2.6 SPECIAL INTEGRATING FACTORS

Int. factors generally satisfy:

Exact eqs. $\rightarrow \frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$

Exact thanks to "μ" $\rightarrow \frac{\partial(\mu P)}{\partial y}=\frac{\partial(\mu Q)}{\partial x} \Rightarrow \frac{\partial \mu}{\partial y}P+\mu\cdot\frac{\partial P}{\partial y}=\frac{\partial \mu}{\partial x}Q+\mu\frac{\partial Q}{\partial x} \Rightarrow$

$\Rightarrow \frac{\partial \mu}{\partial y}P-\frac{\partial \mu}{\partial x}Q=\mu\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) \xrightarrow[\text{by } \mu]{\text{Dividing}}$

$\Rightarrow \frac{\partial(\ln \mu)}{\partial y}P-\frac{\partial(\ln \mu)}{\partial x}Q=\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}$

* 1^{st} case: $\mu=\mu(x)$
  ↳ Does not depend on "y"

$\frac{\partial(\ln \mu)}{\partial x}=\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$ is the condition to be satisfied
↳ a function of x only

but then,

$\frac{d}{dy}\left[\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)\right]=0$, which is the final condition

and then, $\mu=C\cdot e^{\int\left[\frac{1}{Q}\left(\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}\right)\right]dx}$

E.g.:

$$\underbrace{(2x^2+y)}_{P}\,dx + (\underbrace{x^2y-x}_{Q})\,dy = 0$$

$$\frac{\partial}{\partial y}\left(\frac{1}{\underbrace{x^2y-x}_{Q}}\Big(\underbrace{1}_{\frac{\partial P}{\partial y}} - \underbrace{(2xy-1)}_{\frac{\partial Q}{\partial x}}\Big)\right) = \frac{\partial}{\partial y}\left(\frac{1}{x(xy-1)}(2-2xy)\right) = \frac{\partial}{\partial y}\left(-\frac{2}{x}\right) = 0$$

$$\mu = C\cdot \exp\int -\frac{2}{x}\,dx = C\cdot x^{-2}$$

2nd case: $\mu = \mu(y)$

$$\frac{\partial(\ln\mu)}{\partial y} = -\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$\frac{\partial}{\partial x}\left[-\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\right] = 0 \longrightarrow \text{Must satisfy}$$

$$\mu = C\cdot e^{\int -\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dy}$$

Ex. 2.11:
Discuss whether eq. $(3xy+y^2)+(x^2+xy)y'=0$ admits $\mu=\mu(y)$.

EZIN DA.

3rd case: $\mu = \mu(h(x,y))$

Dependence on "x" and "y" through an auxiliary function and so,

If $\mu = \mu(h(x,y))$

$$\frac{\partial\mu}{\partial y} = \frac{\partial\mu}{\partial h}\cdot\frac{\partial h}{\partial y} = \mu'\cdot\frac{\partial h}{\partial y}$$

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q\frac{\partial h}{\partial x} - P\frac{\partial h}{\partial y}}$$ must be rewrittable as a function of h.

$$\frac{\partial \mu}{\partial x} = \mu' \cdot \frac{\partial h}{\partial x}$$

Ex. 2.12

Solve $(3xy + y^2)dx + (3xy + x^2)dy = 0$ using an int. factor of the form : $\mu(x+y)$

$$\frac{\partial P}{\partial y} = 3x + 2y$$

$$\frac{\partial Q}{\partial x} = 3y + 2x$$

$$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}(x+y) = 1$$

$$\frac{\partial h}{\partial y} = \frac{\partial}{\partial y}(x+y) = 1$$

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q\frac{\partial h}{\partial x} - P\cdot\frac{\partial h}{\partial y}} = \frac{3x + 2y - 3y - 2x}{(3xy + x^2) - (3xy + y^2)} = \frac{x-y}{x^2 - y^2} = \frac{1}{x+y} = \frac{1}{h}$$ ✓

$$\mu = C \cdot \exp \int \frac{1}{h}\, dh = C \exp \ln h = C \cdot h \rightarrow \text{set } C = 1$$

New eq. $\mu(Pdx + Qdy) = 0$

H.W. $\rightarrow$ $\boxed{u = x^3y + 2x^2y^2 + y^3x + C}$

2.7. LINEAR EQUATIONS

1st order linear equations look like this:

$$y' + A(x)\cdot y = B(x)$$

Case 1: $B=0 \Rightarrow$ Homogeneous (Gauss is vacuum $\vec{\nabla}\cdot\vec{E}=0$)

Case 2: $B\neq 0 \Rightarrow$ Inhomogeneous (" in presence of charge $\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0}$)

Linear eqs. always accept the following integrating factor:

$$\mu(x) = e^{\int A(x)dx}$$

So, we've got $y' + A(x)\cdot y = B(\cancel{x})$ and we multiply it by μ:

$$e^{\int A(x)dx}\cdot y' + A(x)\cdot e^{\int A(x)\cdot dx}\cdot y = B(x)\cdot e^{\int A(x)dx} \qquad \underline{\left(e^{\int f dx}\right)' = f\cdot e^{\int f\cdot dx}}$$

$$\left[y\cdot e^{\int A\cdot dx}\right]' = B(x)\cdot e^{\int A\cdot dx}$$

$$\int \left[y\cdot e^{\int A\cdot dx}\right]' dx = \int B(x)\cdot e^{\int A\cdot dx} dx$$

↳ we want to know this so we get rid of the "'".

$$y\cdot e^{\int A\cdot dx} = \int B(x)\cdot e^{\int A\cdot dx} dx + C$$

$$y = e^{-\int A\cdot dx}\cdot\left[\int B(x)\cdot e^{\int A\cdot dx}\cdot dx + C\right]$$

1st order eq. → Sol with a free constant and with a term that is 0 if $B=0$.

$$y = \underbrace{e^{-\int A\cdot dx}\cdot\int B\cdot e^{\int A\cdot dx}\cdot dx}_{\text{PART. SOL. OF INHOMOGENEOUS SOL.}} + \underbrace{e^{-\int A\cdot dx}\cdot C}_{\text{SOL. OF HOMOGENEOUS SOL.}}$$

general solution of a first order linear dif. eq. = gen. sol. of the homogen. eq. + particular sol. of the non-homogen. sol.

$$y = e^{-\int A dx} \cdot C + e^{-\int A dx} \cdot \int B \cdot e^{\int A dx} dx$$

Ex. 2.13

Solve $xy' + (1+x)y = e^x$

· Rearrange:

$$y' + \frac{1+x}{x} y = \frac{e^x}{x}$$ (Apply a $\mu(x)$, which doesn't add nor extract solutions)

$A(x) = \frac{1+x}{x}$

$B(x) = \frac{e^x}{x}$

$$y = e^{-\int A dx} \left[\int B \cdot e^{\int A \cdot dx} \cdot dx + C \right]$$

$$e^{-\int A dx} = e^{-\int \frac{1+x}{x} dx} = (\dots) = \frac{1}{x} \cdot e^{-x}$$ (the integration const. is not compulsory).

$$\int B \cdot e^{\int A \cdot dx} \cdot dx = \int \frac{e^x}{\cancel{x}} \cdot \cancel{x} \cdot e^x \cdot dx = \int e^{2x} \cdot dx = \frac{e^{2x}}{2} \quad (\;\text{''}\;)$$

$$y = \frac{e^{-x}}{x} \left(\frac{e^{2x}}{2} + C \right) = \frac{e^x}{2x} + \frac{C \cdot e^{-x}}{x}$$

2.8. TRANSFORMATION METHODS

Sometimes a change of variable helps.

2.9. HOMOGENEOUS EQUATIONS

• Def.: An homogeneous function of order r satisfies:

$$f(ax, ay) = a^r \cdot f(x,y)$$

If the P & Q of the symmetric form of an equation are homogeneous func. of the same order (r), then

$$P\,dx + Q\,dy = 0$$

is an homogeneous eq. of order r.

It can be proved that:

$$P\,dx + Q\,dy = 0 \text{ is homogeneous} \iff -\frac{P}{Q} = f\left(\frac{y}{x}\right)$$

• Useful change of variables for homogeneous eqs.:

$$u = \frac{y}{x} \Rightarrow y = ux \Rightarrow y' = (ux)' = u'x + u$$

and now,

$$P\,dx + Q\,dy = 0 \; ; \; y' = -\frac{P}{Q} = f\left(\frac{y}{x}\right)$$

• Do the change now:

$$y' = u'x + u = -\frac{P}{Q} = f\left(\frac{y}{x}\right) = f(u)$$

rearrange:

$$u'x + u = f(u)$$

$$\frac{du}{dx}x + u = f(u) \Rightarrow x\frac{du}{dx} = f(u) - u \Rightarrow \int \frac{du}{f(u)-u} = \int \frac{dx}{x} + C$$

The eq. has made the equation separable.

Ex. 2.16

Solve $(\sqrt{x^2+y^2}+y)\,dx - x\,dy = 0$

Homogeneous of order 1 by inspection

Don't believe it's second order!

$x \to ax$

$y \to ay$ $\Rightarrow \sqrt{a^2x^2+a^2y^2} + ay\,\cancel{dy} - ax\,dx = a(\sqrt{x^2+y^2}+y\,dy - x\,dx) = 0$

$y' = f\left(\frac{y}{x}\right)$ f homogeneous

$$y' = +\frac{\sqrt{x^2+y^2}+y}{x} = \sqrt{\frac{x^2}{x^2}+\frac{y^2}{x^2}} + \frac{y}{x} = \sqrt{1+\left(\frac{y}{x}\right)^2} + \frac{y}{x} = f\left(\frac{y}{x}\right)$$

$f(u) = \sqrt{1+u^2} + u$

$u = \frac{y}{x}$

The general formula:

$$\int \frac{du}{\sqrt{1+u^2}+\cancel{u-u}} = \int \frac{dx}{x} \Rightarrow \ln(u+\sqrt{1+u^2}) = \ln x + \ln C$$

We write the constant that way to get rid of logarithms

$$\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = Cx$$

2.11 EQUATIONS OF TYPE $y' = f\left(\frac{ax+by+c}{\alpha x+\beta y+\gamma}\right)$

• Two cases depending on the geometrical relation among the two straight lines:

$$ax+by+c=0$$
$$\alpha x+\beta y+\gamma=0$$

1st case: Parallel lines

$$\frac{a}{\alpha}=\frac{b}{\beta}=k$$

We can write

$$y' = f\left(\frac{ax+by+c}{k(ax+by)+\gamma}\right)$$

Change of variables $\Rightarrow$ either $u=ax+by+c$ or $u=ax+by$

Ex. 2.18

Solve $y'=\dfrac{x-y}{x-y-1}$

$$\begin{matrix} a=1 & b=-1 \\ \alpha=1 & \beta=-1 \end{matrix} \rightarrow u=x-y \Rightarrow (\ldots) \Rightarrow (x-y)^2+2y^2=2C$$

2nd case: Non-parallel lines

$$\frac{a}{\alpha}\neq\frac{b}{\beta}$$

• Let us suppose that our curves meet at (x_0,y_0), then

as $\left.\begin{matrix} ax+by+c=0 \\ \alpha x+\beta y+\gamma=0 \end{matrix}\right\} \Rightarrow \begin{matrix} ax_0+by_0+c=0 \\ \alpha x_0+\beta y_0+\gamma=0 \end{matrix}$

$$ax+by+c=0 = ax+by+c-(\underbrace{ax_0+by_0+c}_{=0})=0$$

$$\begin{cases} a(x-x_0)+b(y-y_0)=0 \\ \alpha(x-x_0)+\beta(y-y_0)=0 \end{cases}$$

These results suggest the changes:

$$\left.\begin{aligned} u&=x-x_0 \\ v&=y-y_0 \end{aligned}\right\} \Rightarrow \begin{aligned} ax+by+c&=au+bv \\ \alpha x+\beta y+\gamma&=\alpha u+\beta v \end{aligned}$$

So we propose:

$$\left.\begin{aligned} u&=x-x_0 \\ v&=y-y_0 \end{aligned}\right\} \Rightarrow \begin{aligned} & u'=(x-x_0)'=\frac{du}{dx}=1 \Rightarrow du=dx \\ & v'=(y-y_0)'=\frac{dy}{dx}=\frac{dv}{dx} \Rightarrow \frac{dy}{dx}=\frac{dv}{dx}=\frac{dv}{du} \end{aligned}$$

Remember our original expression:

$$y'=\frac{dy}{dx}=\frac{dv}{du}=f\left(\frac{ax+by+c}{\alpha x+\beta y+\gamma}\right)=f\left(\frac{au+bv}{\alpha u+\beta v}\right)$$

Ex. 2.19:

Solve $y'=\dfrac{x-y+1}{x+y-3}$

Where do they meet? $\to$ $x-y+1=x+y-3=0 \to (x,y)=(1,2)$

Change of vars: $u=x-1 \to x=u+1$
$v=y-2 \to y=v+2$

$$y'=\frac{dv}{du}=\frac{x-y+1}{x+y-3}=\frac{(u+1)-(v+2)+1}{(u+1)+(v+2)-3}=\frac{u-v}{u+v}$$

Let us now make momentarily denote a differential with respect to "u":

$$v'=\frac{u-v}{u+v}$$

In all $y'=f\left(\frac{ax+by+c}{\alpha x+\beta y+\gamma}\right)$ solutions the change $\left\{\begin{matrix} u=x-x_0 \\ v=y-y_0 \end{matrix}\right\}$ because it gives us an equation of the type $\frac{dv}{du}=f\left(\frac{au+bv}{\alpha u+\beta v}\right)$

We have to do now a change that corresponds to a homogeneous equation.

$z=\frac{v}{u} \rightarrow v=zu \rightarrow v'=z'u+z$ (because $'\equiv\frac{d}{du}$) and this makes the equation separable.

2.12 BERNOULLI'S EQUATION

These have the form

$$y'+A(x)y=B(x)\cdot y^n \quad , n\neq 0,1$$

$n=0 \rightarrow$ lin. inhomogeneous

$n=1 \rightarrow$ lin. homogeneous

* Convenient change of variables:

$$u=y^{1-n}$$

$$(y'?) \quad \frac{du}{dx}=u'=(y^{1-n})'=(1-n)\cdot y^{1-n-1}\cdot y'$$

so, $\boxed{u'=(1-n)y^{-n}y'} \Rightarrow y'=\frac{u'\cdot y^n}{(1-n)}$

Substituting,

$$y'+A(x)y=B(x)\cdot y^n$$

$$\frac{u'\cdot y^n}{1-n}+A(x)y=B(x)\cdot y^n$$

Dividing by y^n and multiplying $(1-n)$ on both sides:

$$u' + A(x)(1-n)\cdot y^{1-n} = B(x)(1-n)$$

$$\boxed{u' + (1-n)A(x)u = B(x)(1-n)} \Rightarrow \text{Lin. inhomogeneous.}$$

Ex. 2.20:

Solve $y' - y\cdot\cos x = \frac{1}{2}(\sin 2x)\cdot y^2$

Bernoulli with $n=2$:

$$u = y^{1-n} = y^{1-2} = y^{-1}$$

Identify A, B and apply formula:

$$A = -\cos x$$

$$B = \frac{1}{2}(\sin 2x)$$

$$u' + (1-n)A(x)u = B(x)(1-n)$$

$$u' + \cos x\cdot u = (1-2)\cdot\frac{\sin 2x}{2} = -\frac{\cancel{2}\cdot\sin x\cdot\cos x}{\cancel{2}}$$

$$u' + \cos x\cdot u = -\sin x\cdot\cos x$$

Which is a linear inhomogeneous eq.:

$$u = e^{-\int A\,dx}\left[C + \int B\cdot e^{\int A\,dx}\,dx\right] \quad (u' + Au = B)$$

$$u = e^{-\int\cos x\,dx}\cdot\left[C + \int(-\sin x\cdot\cos x)\,e^{\int\cos x\,dx}\,dx\right] =$$

$$= e^{-\sin x}\cdot\left[C - \int\sin x\cdot\cos x\cdot e^{\sin x}\right]$$

↳ Change of vars. → $\sin x = z$

$$(\ldots) \rightarrow y = \left(C\cdot e^{-\sin x} + 1 - \sin x\right)^{-1}$$

3.1. GEOMETRIC MEANING

- A generalization of the 1st order case
- Consider a flat family of curves with free parameters $(c_1, \dots, c_n)$
- The finite eq. $\varphi(x, y, c_1, \dots, c_n) = 0$ and its derivatives can be combined to give the corresponding $F(x, y, y', \dots, y^{(n)}) = 0$

$$\varphi(x, y, c_1, \dots, c_n) = 0$$

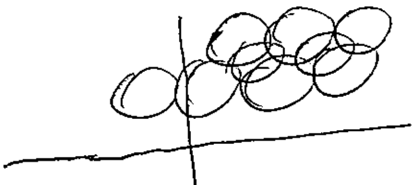
$$\frac{d}{dx}\left\{ \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} \cdot y' = 0 \right.$$

$$\frac{d^2}{dx^2}\left\{ \frac{\partial^2 \varphi}{\partial x^2} + 2 \cdot \frac{\partial^2 \varphi}{\partial x \partial y} y' + \frac{\partial^2 \varphi}{\partial y^2} (y')^2 + \frac{\partial \varphi}{\partial y} y'' = 0 \right.$$

- Each of the curves $\varphi(x, y, c_1, \dots, c_n) = 0$ is a solution to the corresponding ode, and it's also called an integral curve.

Ex. 3.1.

Which is the ODE of the unit circumferences?



$$\varphi(x, y, c_1, c_2) = \boxed{(x-a)^2 + (y-b)^2 - 1 = 0} \quad (1)$$

$$\frac{d}{dx}\left[\varphi(x, y, c_1, c_2) = 0\right] \Rightarrow$$

$$\Rightarrow \boxed{\cancel{2}(x-a) + \cancel{2}(y-b) y' = 0} \; (2) \Rightarrow \frac{d^2}{dx^2}[\varphi] = \frac{d}{dx}[\varphi']$$

$$1 + y'y' + (y-b)y'' = 0 \rightarrow \boxed{1 + (y')^2 + (y-b) \cdot y'' = 0} \quad (3)$$

$(2) \rightarrow (x-a)^2 = (y-b)^2 \cdot (y')^2$

$(1) \rightarrow (x-a)^2 = 1 - (y-b)^2$

$$0 = (y-b)^2 (y')^2 - 1 + (y-b)^2$$

$$\boxed{(y'^2+1)(y-b)^2 = 1} \quad (4)$$

Initially I had a finite eq. with 2 free parameters, so the final ode must be of order 2; therefore, at some time I'll have to use the expression with the second derivative.

Let's use it now:

$$(3) \rightarrow (y'')^2 (y-b)^2 = (1+yy')^2 \xrightarrow[(\cdots)]{(4)} (y'')^2 = (y'^2+1)^3$$

3.2. EXISTENCE AND UNIQUENESS THEOREM

- For an ode written in normal form:

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$$

If $f, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial y^2}, \dots, \frac{\partial^{(n-1)} f}{\partial f^{n-1}}$ are continuous and the ode has initial conditions $y(x_0) = y_0$, $y'(x_0) = y'_0, \dots, y^{(n-1)}(x_0) = y_0^{(n-1)}$, then the solution exists and is unique.

3.3. EQUIVALENCE AMONG EQUATIONS & SYSTEMS

· One can always lower the order of an ode by adding variables and equations.

· To do so, we define new vars:

$$y_1 \equiv y,\ y_2 \equiv y',\ \dots,\ y_n \equiv y^{(n-1)}$$

· This transforms the ode $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$ into this system:

$$y_1' = y_2$$
$$y_2' = y_3$$
$$\vdots$$
$$y_n' = f(x, y_1, \dots, y_n)$$

Ex. 3.2:

Write the eq. of a forced oscilator as a system:

· The original eq. is:

$$\ddot{x} + \omega^2 x = \frac{F(x)}{m}$$

Let the dep. var. and its n-1 ders. be the new variables:

2nd order equation $\longrightarrow$ 2 equation system

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned} \quad \Rightarrow \quad \begin{cases} \dot{x}_2 + \omega \cdot x_1 = \dfrac{F(x_1)}{m} \\ \dot{x}_1 = x_2 \end{cases}$$

3.4. LOWERING THE ORDER

* Equations without dep. variable

- They look like:

$$F(x, y', y'', \dots, y^{(n)}) = 0$$

- Defining $u = y'$ we get $u' = y''$, $u'' = y'''$, $\dots$, $u^{(n-1)} = y^{(n)}$

- And so, we get:

$$F(x, u, u', u'', \dots, u^{(n-1)}) = 0 \rightarrow \text{The order has been lowered.}$$

- The sol. (of the new eq) is:

$$\tilde{\varphi}(x, u, c_1, \dots, c_{n-1}) = 0$$

Which is, in fact, a 1^{st} order eq., since the change of vars. can be undone $(u = y')$.

Solving this intermidiate eq. gives us the gen. sol. to the original eq.: $\varphi(x, y, c_1, \dots, c_n) = 0$

Besides, every single sol. of the intermediate eq. will lead to a single sol. of the original one.

If the derivatives $y', \dots, y^{(m-1)}$ are missing, we can set $u = y^{(m)}$ so to lower the order of the eq. to $n-m$.

Example:

Consider $(y'')^2 = 240\, x^2 y'$ (y is missing!)

Let $u = y'$ $(u' = y'')$:

$$u' = \sqrt{(u')^2} = \sqrt{240 x^2 y'} = x\sqrt{240 u}$$

$$\frac{du}{\sqrt{u}} = \sqrt{240}\, x \cdot dx \Rightarrow \sqrt{u} = \sqrt{15}\,(x^2 + c_1) \Rightarrow u = 15\,(x^2 + c_1)^2 \Rightarrow$$

$$\Rightarrow y' = 15\,(x^2 + c_1)^2 \rightarrow y = 3x^5 + 5c_1 x^3 + 15c_1 x + 15c_2$$

By inspection, we can see that $y = a$ satisfies the original eq.
↳ particular sol.

Ex. 3.3

A puntual particle is falling down a vertical stright line due to gravity. Let friction be proportional to the velocity. Find $v(t)$ and prove there is a limiting velocity.

$$m\ddot{z} = -mg - k\dot{z}$$

dep. var. → z
ind. var. → t

$$\left\{ u = \dot{z}\,,\ \dot{u} = \ddot{z} \right.$$

$$m\dot{u} = -mg - ku \rightarrow \frac{m \cdot du}{-(mg + ku)} dt \Rightarrow u + \frac{m}{k} g = e^{-\frac{kt}{m} + c_1}$$

$$u = \dot{z} = v(t)$$

$$\lim_{t \to \infty} u = \lim_{t \to \infty} v = \lim_{t \to \infty} -\frac{m}{k} g + e^{-\frac{kt}{m} + c_1} = -\frac{m}{k} g$$

* Autonomous equations

- Equations in which the independent variable is absent:
- If $\varphi(x, y) = 0$ is a solution, then $\varphi(x - x_0, y) = 0$ is another sol. too $\forall x_0$
- Therefore, x_0 is one of the constants of the general solution and it will look like

$$\varphi(x - x_0, y, c_1, \ldots, c_{n-1}) = 0$$

· One can use $\boxed{u=y'}$ to lower the order and at the same time make $\boxed{y}$ our new independent variable, thus:

New notation: $\mathring{u} = \frac{du}{dy}$, $\mathring{} \equiv \frac{d}{dy}$

$$y' = u$$
$$y'' = \frac{du}{dy} \cdot y' = \mathring{u} \cdot y' = \mathring{u} \cdot u$$
$$y''' = \frac{d^2u}{dy^2} u + \left(\frac{du}{dy}\right)^2 u = \ddot{u} \cdot u + \mathring{u}^2 \cdot u$$

Substituting in the original equation, we obtain a new one of order $n-1$:

$$F(y, u, u', \dots, u^{(n-1)})$$

The gen. sol. to this equation will be like this:

$$\tilde{\varphi}(y, u, c_1, \dots, c_{n-1}) = 0$$

Undoing the change $u=y'$ we get a 1^{st} order eq. whose sol. eventually gives

$$\varphi(x-x_0, c_1, \dots, c_n) = 0$$

Ex. 3.4:

Solve $y'' = (2y+1)y'$

$\left(\text{Sol.} \rightarrow \frac{2}{\sqrt{4C_1-1}} \operatorname{arctg}\left(\frac{1+2y}{\sqrt{4C_1-1}}\right) = x + 2C_2\right)$

Autonomous eq. (x is not there).

$$y' = u \rightarrow u' = \frac{du}{dy} \cdot \frac{dy}{dx} = \mathring{u} y' = \mathring{u} u = y''$$
$$y'' = \mathring{u} u$$

$$\mathring{u} \cancel{u} = (2y+1) \cdot \cancel{u} \rightarrow \mathring{u} = \frac{du}{dy} = (2y+1) \xrightarrow{\text{Separable}} (\dots) \rightarrow u = y^2 + y + C_1$$

$$y' = y^2 + y + C_1 \xrightarrow[\text{(from book)}]{\text{Separable}} \frac{2}{\sqrt{4C_1-1}} \operatorname{arctg}\left(\frac{1+2y}{\sqrt{4C_1-1}}\right) = x + C_2$$

Careful! The simplification made us lose perhaps the $u=0$ solutions,

$u=0=y' \to y=C_3$, which is a singular solution.

* Equidimensional-in-x differential equations

These are invariant under the change $x \to ax$, and so,

$$F(ax, y, a^{-1}y', a^{-2}y'', \dots, a^{-n}y^{(n)}) = F(x,y,y',\dots,y^{(n)}) = 0$$

Each prime carries an a^{-1} !!

$x \to ax$

$$y' = \frac{dy}{dx} \to \frac{dy}{d(ax)} = \frac{1}{a}\cdot\frac{dy}{dx} = \frac{1}{a}y'$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \longrightarrow \frac{d}{d(ax)}\left(\frac{d}{d(ax)}y\right) = \frac{y''}{a^2}$$

These eqs. are transformed into autonomous ones by

$$\boxed{t \to \ln x}$$

Thus,

$$x = e^t$$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\cdot\frac{1}{x} = \frac{\dot{y}}{x}$$

$$y'' = \frac{d}{dx}\left(\frac{\dot{y}}{x}\right) = \frac{d}{dt}\left(\frac{\dot{y}}{x}\right)\cdot\frac{dt}{dx} = \frac{\ddot{y}x - \dot{x}\dot{y}}{x^2}\cdot\frac{dt}{dx} = \left[\frac{\ddot{y}x}{x^2} - \frac{\dot{x}\dot{y}}{x^2}\right]\frac{1}{x} = \frac{\ddot{y}-\dot{y}}{x^2}$$

$\dot{x} = x = e^t$

Ex. 3.5:

Solve $xy'' = yy'$

$x \to ax \Rightarrow \cancel{a}x\cdot\cancel{a^{-2}}\cdot y'' = y\cdot\cancel{a^{-1}}\cdot y' \to$ Remains the same

Changes: $y' = \frac{\dot{y}}{x}$, $y'' = \frac{\ddot{y}-\dot{y}}{x^2}$

$$\cancel{x}\left(\frac{\ddot{y}-\dot{y}}{\cancel{x^2}}\right)=y\cdot\frac{\dot{y}}{\cancel{x}} \rightarrow \ddot{y}-\dot{y}=y\dot{y} \rightarrow \boxed{\ddot{y}=\dot{y}(1+y)}$$ autonomous

Almost like the eq. $y''=(1+2y)y'$ studied in the previous example.

If we do new $y \rightarrow 2\,\text{old}\,y$ & $t \rightarrow x$ we recover it.

$$\text{Sol.} \rightarrow \frac{2}{\sqrt{4C_1-1}}\cdot \operatorname{arctg}\left(\frac{1+y}{\sqrt{4C_1-1}}\right)=t+C_2$$

As the eq. is not linear we cannot guarantee that every sol. will be included in the general one.

$y=C_3$ is a sing. sol.

* Equidimensional-in-y differential equations

These are invariant under the scaling $y \rightarrow ay$, and so

$$F(x,ay,ay',ay'',\dots,ay^{(n)})=F(x,y,y',\dots,y^{(n)})=0$$ Every "y" gives an "a"

They become autonomous by the change:

$$u=\frac{y'}{y}$$

$y'=uy$

$y''=u'y+uy'=u'y+u^2y=y(u'+u^2)$

$y'''=y'(u'+u^2)+y(u''+2uu')=uy(u'+u^2)+y(u''+2uu')=$

$=u\cdot u'\cdot y+u^3y+u''\cdot y+2u\cdot u'\cdot y=u^3y+u''y+3u\cdot u'\cdot y$

Ex. 3.6.

Solve $y\cdot y''=(y')^2$

$y \rightarrow ay \Rightarrow (ay)\cdot(ay'')=(ay')^2$ ✓

$$y[y(u'+u^2)] = (yu)^2 \rightarrow u' + \cancel{u^2} = \cancel{u^2} \rightarrow u' = 0 \rightarrow u = C_1$$

$$u = \frac{y'}{y} \rightarrow y = C_2 e^{C_1 x}$$

In this case, $y =$ ctant. is not a sing. sol. as we get it for $C_1 = 0$.

*Equations which are exact differentials

They satisfy:

$$F(x, y, y', \dots, y^{(n)}) = \frac{d}{dx}\left(G(x, y, y', \dots, y^{(n-1)})\right)$$

then, the quadrature

$$G(x, y, y', \dots, y^{(n-1)}) = C$$

gives us a first integral.

Ex. 2.7:

Solve $yy'' + (y')^2 = 0$

$$\frac{d}{dx}[\underbrace{yy'}_{G}] = \underbrace{yy'' + (y')^2}_{F}$$

So $\boxed{yy' = C_1}$ → first integral.

$$yy' = C_1 \rightarrow y \cdot \frac{dy}{dx} = C_1 \xrightarrow{\text{(separable)}} y^2 = 2C_1 x + C_2$$

$y =$ ctn. was a sol, to be checked, but we get it when $C_1 = 0$, so its a particular solution.

3.5. LINEAR DEPENDENCE OF EQUATIONS

The sols. of a linear homogeneous differential eq. form a vector space.

↳ The structure is induced by the usual addition and multiplication.

↳ The space is a subspace of the infinite dimensional of regular function.

- Notation: "y_k" denotes a certain solution

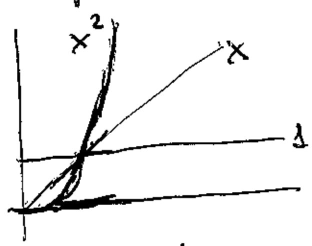
  * A linear combination will be another solution : $y = \sum_{k=1}^{n} c_k y_k$

The solution y_k will be linearly independent if

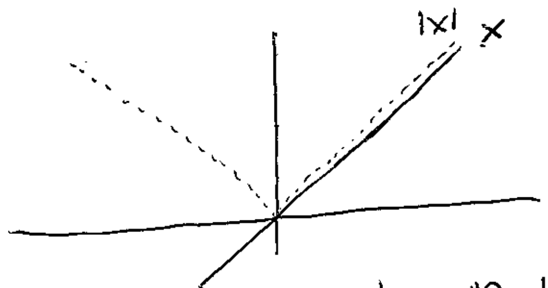
$$\sum_{k=1}^{n} c_k y_k = 0 \; (\forall x \in I) \text{ iff } c_1 = c_2 = \dots = c_k = 0$$

E.g.:

Linear independence graphically:



Linearly independent in $\mathbb{R}$ by visual inspection



Linearly independent in $\mathbb{R}$ but linearly dependent in either $(-\infty, 0)$ or $(0, \infty)$.

E.g.:

The x^n functions are linearly independent!

$$\{1, x, x^2, x^3, \dots, x^n\}$$
$$\quad y_1 \quad y_2 \quad y_3 \quad y_4 \qquad\quad y_{n+1}$$

A lin combination of them is a polinomial, and it only vanishes at its roots (they do not cover $\mathbb{R}$), so the only way a linear combination of them is identically 0 is that all coefficients are 0.

E.g.:

$\{x, |x|\}$

$c_1 y_1 + c_2 y_2 = 0$?

$$c_1 x + c_2 |x| = 0 \xrightarrow[\text{in } x=1;2]{\text{evaluate}} \begin{cases} c_1 + c_2 = 0 \\ c_1 \cdot 2 + c_2 \cdot 2 = 0 \end{cases} \Rightarrow c_1 = -c_2$$

There is a solution for $[0,\infty) \longrightarrow$ linearly dependent in $[0,\infty)$

$$c_1 x + c_2 |x| = 0 \xrightarrow[\text{in } x=1;-1]{\text{evaluate}} \begin{cases} c_1 \cdot 1 + c_2 \cdot 1 = 0 \\ -c_1 + c_2 = 0 \end{cases} \Rightarrow c_1 = c_2 = 0$$

There's going to be linear independence in any domain that includes $\{0\}$.

<u>WRONSKIAN</u>: A more formal way to discuss linear independence.

- If $\sum c_k y_k = 0$, then $(\sum c_k y_k)' = 0$ and $(\sum c_k y_k)'' = 0$ etc.

This allows to construct this system of algebraic equations:

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0$$
$$c_1 y_1' + c_2 y_2' + \dots + c_n y_n' = 0$$
$$\vdots$$
$$c_1 y_1^{(n-1)} + \dots + c_n y_n^{(n-1)} = 0$$

The c_n are the unknowns and we want to calculate them to check for linear indep.

If the det. of the coefficients of the system is equal to 0, then the system will be linearly dependent and so will be $y_1, y_2, \dots, y_n$.

Example: Are $y_1 = 2$ and $y_2 = e^x$ linearly indep.?

$$W[c_1, c_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 2 & e^x \\ 0 & e^x \end{vmatrix} = 2e^x \neq 0 \Rightarrow \text{Independent.}$$

Are $\{2, e^x, 2+e^x\}$ lin. dep.?

$$\begin{vmatrix} 2 & e^x & e^x+2 \\ 0 & e^x & e^x \\ 0 & e^x & e^x \end{vmatrix} = 0 \Longrightarrow \boxed{y_3 = y_1 + y_2}$$

Ex. Feb. 03:

Discuss whether $\{x-2, x^3-3, 6x^3-3x-6\}$ are linearly independent functions:

$$\begin{vmatrix} x-2 & x^3-3 & 6x^3-3x-6 \\ 1 & 3x^2 & 18x^2-3 \\ 0 & 6x & 36x \end{vmatrix} = (\dots) = 0$$

or

$$c_1(x-2) + c_2(x^3-3) + c_3(6x^3-3x-6) = 0 \quad \begin{cases} x=0 & -2c_1 - 6c_3 = 0 \\ x=2 & 6c_2 + 36c_3 = 0 \\ x=-1 & \cancel{-3c_1 - 9c_3} = 0 \end{cases}$$

The sols. are non-trivial so they're linearly dependent functions.

<u>FACT</u>: Given two functions $f(x)$ and $g(x)$ that are differentiable on some interval I:

1) If $W[f,g](x_0) \neq 0$ for some $x_0 \in I$, then $f(x)$ and $g(x)$ are lin. indep. $\forall x_0 \in I$.

2) If $f(x)$ and $g(x)$ are lin. dependent on I, then $W[f,g](x) = 0 \ \forall x \in I$.

3.6. LINEAR DIFFERENTIAL EQUATIONS

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_{n-1}(x)y' + a_n(x)y = b(x)$$

* Usually $a_0 = 1$ (for theoretical results)
* $a_1, \dots, a_n, b$ will be continuous in I.
* For faster notation: $a_i \equiv a_i(x)$ ↳ Don't forget 'x' dependence!!
* Operators (def):

$$D \equiv \frac{d}{dx}, \quad D^2 \equiv \frac{d^2}{dx^2}, \dots, D^n \equiv \frac{d^n}{dx^n}$$

* $L(x) = D^n + a_1 D^{n-1} + \dots + a_n D^0$

$$(Lf)(x) = f^{(n)} + a_1(x) f^{(n-1)} + \dots + a_{n-1} f' + a_n f = b(x)$$

or even shorter, $Lf = b$

* L is a linear operator:

$$L(c_1 f_1 + c_2 f_2) = c_1 L f_1 + c_2 L f_2$$

3.7. LINEAR HOMOGENEOUS EQUATIONS

* For them we have:

$$Ly = 0$$

* Principle of superposition $\equiv L$ operator's linearity

Linear combinations of solutions are solutions as well!!

. y_k (sols. of an eq.):

$$Ly_k = 0 \Rightarrow L\left(\sum_{k=1}^{n} c_k y_k\right) = \sum_{k=1}^{n} c_k \underbrace{(Ly_k)}_{\text{all } 0} = 0$$

↳ So y_k is a sol.

* The dimension of the vector space of solutions is related to the Wronskian.

THEOREM: Let us consider n solutions for an n-dimensional linear homogeneous equation defined in the domain I: $Ly_K = 0$.
Then the following statements are equivalent:

1) The functions are linearly depent in I.
2) The Wronskian for the y_K is identically 0 in I.
3) " " " " " is 0 in one point $x_0 \in I$.

So, what's the dimension of the vector space of sols. of a lin. hom. ode?
(In other words, how many lin. ind. sols. do I need to build the gen. sol.?)

(+ info. in book or notes)

- The uniqueness and existence theorem can be combined with a convenient initial value problem to construct a set of linearly ind. sols.

Such constructions are infinite by choices of constants of the initial value problem. Any such set is called a fundamental system of solutions (its $W \neq 0$).

THEOREM: A linear combination of n linearly ind. sols. of an n-th order linear ode is a solution.

E.g.: The fundamental system for the harmonic oscillator:

$$y'' + \omega^2 y = 0 \longrightarrow \{\cos \omega x, \sin \omega x\}$$

Exercise 3.13: Show that the set $\{1, e^x, e^{-x}\}$ is a fundamental system for the eq.: $y''' - y' = 0$.

For it to be true, we need:

1) The dimension of the set is equal to the order of the eq.
2) The set is lin. independent ($W \neq 0$)
3) The pretended sols. are real sols.

- Apparently 3 eqs. (cond 1 ✓)
- $W = \begin{vmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix} = \begin{vmatrix} e^x & -e^{-x} \\ e^x & e^{-x} \end{vmatrix} = e^0 - (-e^0) = 2 \neq 0$ (cond 2 ✓)
- The gen. sol. should be: $y = A + B \cdot e^x + C \cdot e^{-x}$

$y' = Be^x - Ce^{-x}$

$y'' = Be^x + Ce^{-x}$

$y''' = Be^x - Ce^{-x} = y' \rightarrow$ Eq. is satisfied (cond. 3 ✓)

EQ. ASSOCIATED TO A FUNDAMENTAL SYSTEM

For the y_k in the fundamental system (lin. ind.)

$$W[\{y_1, y_2, \dots, y_n\}] \neq 0$$

If we "extend" our set with a lin. comb. of the others:

$$W[\{y_1, \dots, y_n, \sum_{k=1}^{n} c_k y_k\}] = 0$$

↳ One of the cases that work for this is the gen. sol itself, but typically that's what one wants to discover; let it be an unknown.

So $W[\{y_1, \dots, y_n, y\}] = 0$
↳ gives the diff. eq. for the set.

Example:

Find the eq. associated with $\{x, \frac{1}{x}\}$

$$W = [x, \tfrac{1}{x}, y] = \begin{vmatrix} x & \frac{1}{x} & y \\ 1 & -\frac{1}{x^2} & y' \\ 0 & 2x^{-3} & y'' \end{vmatrix} = -\frac{1}{x}y'' + 2x^{-3}y - 2x^{-2}y' - \frac{1}{x}y'' =$$

$$= 2x^{-3}y - 2x^{-2}y' - 2x^{-1}y'' = 0 \Rightarrow y'' + \frac{y'}{x} - \frac{y}{x^2} = 0$$

Ex. 3.14:

Find the eq. associated with $\{x, e^x\}$

Once simplified (dividing by e^x)

$$W = [\{x, e^x, y\}] = \begin{vmatrix} x & e^x & y \\ 1 & e^x & y' \\ 0 & e^x & y'' \end{vmatrix} = 0 \longrightarrow (x-1)y'' + y - xy' = 0$$

D'ALEMBERT'S METHOD TO SOLVE 2^{nd} ORDER LIN. HOM. ODES

- It requires knowing a particular sol.: y_1.
- One then does the change of vars. $\quad y = y_1 \cdot \underbrace{\int u\,dx}_{int(x)} \qquad (int)' = \left(\int u\,dx\right)' = u$

So, our change of vars. is $\quad y = y_1 \cdot int \quad (y(x) = y_1(x) \cdot int(x))$

- Our initial eq. is:

$$y'' + a_1 y' + a_2 = 0$$

$$y' = (y_1 \cdot int)' = y_1' \cdot int + y_1 \cdot int' = y_1' \cdot int + y_1 \cdot u$$

$$y'' = (y')' = (y_1' \cdot int + y_1 \cdot u)' = y'' \cdot int + y' \cdot int' + y_1' \cdot u + y_1 u' =$$

$$= y_1'' \cdot int + y_1' u + y_1' u + y_1 u' =$$

$$= y_1'' \cdot int + 2y_1' u + y_1 u'$$

$$y'' \quad + \quad a_1 y' \quad + \quad a_2 y \quad = 0$$

$$(y_1''\cdot int + 2\cdot y_1'\cdot u + y_1 u') + a_1(y_1'\cdot int + y_1\cdot u) + a_2\cdot y_1\cdot Int = 0$$

$$\Downarrow$$

$$int\underbrace{\left[y_1'' + a_1 y_1' + a_2 y_1\right]}_{0 \Rightarrow y_1 \text{ is a part. sol.!}} + 2\cdot y_1' u + y_1 u' + a_1 y_1 u = 0$$

↳ We've been left with a separable eq.

$(2y_1' + a_1 y_1)u + y_1 u' = 0 \rightarrow u$ is our new dep. var.

$$\left(2\cdot\frac{y_1'}{y_1} + a_1\right)u = -u' \Longrightarrow u = C_2\cdot\frac{e^{-\int a_1 dx}}{y_1^2}$$

And the gen. sol. of the original eq. is

$$\boxed{y = C_1 y_1 + y_1 \int u\,dx}$$

↳ The 2nd const. is here!

Ex. 3.17:

Solve $(x^2+1)y'' - 2xy' + 2y = 0$

$y_1 = x$ is a sol.

First normalize $(a_0 = 1)$:

$$y'' \overset{a_1}{\boxed{-\frac{2x}{x^2+1}}} y' + \frac{2}{x^2+1} y = 0$$

$$-\int a_1 dx = -\int\left(\frac{-2x}{x^2+1}\right)dx = \ln(x^2+1)$$

$$u = C_2\cdot\frac{e^{-\int a_1 dx}}{y_1^2} = C_2\cdot\frac{e^{\ln(x^2+1)}}{x^2} = C_2\cdot\frac{x^2+1}{x^2}$$

$$y = C_1 x + x\int C_2 \frac{x^2+1}{x^2}\,dx = C_1 x^2 + C_2(x^2-1)$$

USEFUL CHANGES OF VARS. THAT WORK IN SOME CASES

1) If the eq. $y'' + P(x)y' + Q(x)y = 0$ satisfies $2PQ + Q' = 0$,

then $x \to t = \int\sqrt{Q}\,dx$ may easen the search of sols.

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \dot{y}\sqrt{Q}$$

$$y'' = \frac{d}{dt}(\dot{y}\sqrt{Q})\frac{dt}{dx} = \frac{d}{dt}(\dot{y}\sqrt{Q})\sqrt{Q} = \ddot{y}\sqrt{Q} + \frac{\dot{y}}{2\sqrt{Q}}Q$$

$$\ddot{y}Q + \frac{\dot{y}}{2\sqrt{Q}}Q' + P\dot{y}\sqrt{Q} + Qy = 0$$

$$\ddot{y}Q + \cancel{\frac{\dot{y}}{2\sqrt{Q}}\underbrace{(Q' + 2PQ)}_{0}} + Qy = 0 \Rightarrow Q(\ddot{y}+y) = 0$$

↳ harmonic oscillator

Example:

$$xy'' - y' + 4x^3y = 0 \xrightarrow{\text{Normalizing}} y'' - \frac{y'}{x} + 4x^2y = 0$$

$$2PQ + Q' = 2\left(-\frac{1}{x}\right)4x^2 + 8x = 0 \quad \checkmark$$

Change of vars $\Rightarrow t = \int\sqrt{Q}\,dx = \int\sqrt{4x^2}\,dx = \int 2x\,dx = x^2$

$$\ddot{y} + \dot{y} = 0 \to y = A\cdot\cos t + B\cdot \text{sen}\, t = A\cdot\cos(x^2) + B\cdot\sin(x^2)$$

2) If $f = \dfrac{4Q - P^2 - 2P'}{4} = \text{const.}$, then $y = u \cdot e^{-\frac{1}{2}\int P(x)\,dx}$

$u'' + f(x) \cdot u = 0 \rightarrow$ Harmonic oscillator with frequency $\sqrt{f}$

E.g.:

$$xy'' + 2y' + xy = 0$$

Norm.

$$y'' + \frac{2}{x}y' + y = 0$$

$$f = \frac{4Q - P^2 - 2P'}{4} = \frac{4 \cdot 1 - \frac{4}{x^2} + \frac{4}{x^2}}{4} = 1 \quad \checkmark$$

$$y = u \cdot e^{-\frac{1}{2}\int \frac{2}{x}dx} = u \cdot e^{-\ln x} = u \cdot x^{-1} \Rightarrow u'' + u = 0 \rightarrow u = A \cdot \cos x + B \cdot \sin x$$

$$y = \frac{1}{x}(A \cdot \cos x + B \cdot \sin x)$$

3.8. COMPLETE LINEAR EQUATIONS

- They're like this: $Ly = b$

- From linearity: $\left.\begin{array}{l} Ly_1 = b_1 \\ Ly_2 = b_2 \end{array}\right\} L(a_1y_1 + a_2y_2) = a_1b_1 + a_2b_2$

Different particular sols.

Same left hand side

Two different inhomogeneous cases.

$$\left.\begin{array}{l} Ly_1 = 0 \\ Ly_2 = b \end{array}\right\} \Rightarrow L(y_1 + y_2) = Ly_1 + Ly_2 = b$$

$$\left.\begin{array}{l} Ly_1 = b \\ Ly_2 = b \end{array}\right\} \Rightarrow L(y_1 - y_2) = 0$$

Consequently, the gen. sol. of an inhomogeneous linear eq. is the sum of the gen. sol. of the homogeneous eq. plus a particular sol. of the inhomogeneous case:

$$Ly=b \longrightarrow \text{Gen. sol.: } y=\sum_{k=1}^{n} c_k y_k + y_p$$

VARIATION OF PARAMETERS => Method to find part. sols. of the inhom. lin. eq.
(see notes)

2nd order

$$y''+a_1(x)y'+a_2(x)y=b$$

↳ Assume we know the gen. sol. $y=c_1y_1+c_2y_2$ of the case $b=0$

Let c_1, c_2 be functions:

$$\{c_1, c_2\} \longrightarrow \{g, h\}$$

Look for $y_p = \underset{\substack{\uparrow\\ c_1}}{g(x)}y_1 + \underset{\substack{\uparrow\\ c_2}}{h(x)}y_2$

Solve this system:

$$\boxed{\begin{aligned} g'y_1+h'y_2&=0\\ g'y_1'+h'y_2'&=b\end{aligned}}$$

3rd order

$$y'''+a_1(x)y''+a_2(x)y'+a_3(x)y=b$$

↳ Assume we know the gen. sol. $y=c_1y_1+c_2y_2+c_3y_3$ of the case $b=0$

$$\{c_1, c_2, c_3\} \longrightarrow \{g, h, f\}$$

Look for:

$$y_p = g(x)y_1 + h(x)y_2 + f(x)y_3$$

Solve:

$$\boxed{\begin{aligned} g'y_1+h'y_2+f'y_3&=0\\ g'y_1'+h'y_2'+f'y_3'&=0\\ g'y_1''+h'y_2''+f'y_3''&=b\end{aligned}}$$

Don't forget the trial & error method to look for the part. sol. !!

E.g.: Solve $y''' - y' = 1$ $\qquad$ $\boxed{y_{inh} = y_h + y_p}$

We solved $y''' - y' = 0$ in exercise 3.13. $\rightarrow y_h = A + Be^x + Ce^{-x}$

We'll try to look y_p by inspection:

$$\cancel{y = a \rightarrow y' = 0,\ y''' = 0 \rightarrow y''' - y' \neq 1}$$

$$y = ax \rightarrow y' = a,\ y''' = 0 \rightarrow y''' - y' = 0 - a = 1 \rightarrow a = -1 \rightarrow \boxed{y = -x}$$

is a part. sol.

$$y_{inh} = A + Be^x + Ce^{-x} - x$$

Careful! The particular sol of the inhomogeneous case doesn't have free constants.

E.g.:

$$y'' - y = x^2$$

gen. sol. of the hom. eq.

$$y_h = c_1 \underbrace{e^x}_{y_1} + c_2 \underbrace{e^{-x}}_{y_2}$$

$$\left|\begin{array}{l} g' e^x + h' e^{-x} = 0 \\ g' e^x - h' e^{-x} = x^2 \end{array}\right\rangle 2 \cdot g' \cdot e^x = x^2 \rightarrow g' = \frac{e^{-x} x^2}{2} \rightarrow \underline{\underline{g = -\frac{1}{2}(x^2 + 2x + 2)e^{-x}}}$$

$$(1)-(2) \rightarrow 2 \cdot h' \cdot e^{-x} = -x^2 \rightarrow \underline{\underline{h = -\frac{1}{2}(x^2 + 2x + 2)e^x}}$$

$$y_p = g(x) y_1(x) + h(x) y_2(x) = -\frac{1}{2}(x^2 + 2x + 2)e^{-x} \cdot e^x - \frac{1}{2}(x^2 + 2x + 2)e^x \cdot e^{-x} =$$

$$= -x^2 - 2$$

part. sol of the inh. sol

Finally,

$$y_{inh} = \underbrace{c_1 e^x + c_2 \cdot e^{-x}}_{y_{hom}} \underbrace{- x^2 - 2}_{y_p}$$

Ex. 3.2.4: $\sin(2x) = 2\cdot\sin x\cdot\cos x$

Find the gen. sol of $y'' + y = \frac{1}{\cos x}$

$$y_{inh} = y_{hom} + y_p$$

$y_h =$ (gen. sol. of the hom. eq.) $= A\cdot\cos x + B\cdot\sin x$

↳ Harmonic oscillator

$$y_p = g(x)\cdot\cos x + h(x)\sin x$$

$$\begin{cases} g'\cdot\cos x + h'\sin x = 0 & (1) \\ -g'\cdot\sin x + h'\cdot\cos x = \frac{1}{\cos x} & (2) \end{cases}$$

$(1)\cdot\sin x + (2)\cdot\cos x \Rightarrow h'\cdot\sin^2 x + h'\cdot\cos^2 x = 1 \rightarrow h' = 1 \rightarrow h = x$

$g'\cdot\cos x + \sin x = 0 \rightarrow g' = -\tan x \rightarrow g = \ln(\cos x)$

LINEAR HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

- One possible route: Try e^{Kx} (as a part. sol.), solve for K and apply D'Alembert. ($y''' - y = 0$)
- If that doesn't work, try $x^n e^{Kx}$.

E.g.: $y'' - y = 0$

We try e^{Kx} as a part. sol. of this hom. eq.

$y = e^{Kx}$

$y'' = K^2\cdot e^{Kx}$

$\{ \; y'' - y = K^2\cdot e^{Kx} - e^{Kx} = (K^2 - 1)e^{Kx} = 0 \rightarrow K = \pm 1$

$y_1 = e^x$

$y_2 = e^{-x}$

In this case, we've been lucky & we've found two lin. ind. sols.

$$y = A\cdot e^x + B\cdot e^{-x}$$

E.g.:

$$y''' + y'' - y' - y = 0$$

$y = e^{Kx}$

$y' = K \cdot e^{Kx}$ $\Rightarrow K^3 + K^2 - K - 1 = 0 \overset{\text{Ruffini}}{=} (K+1)(K^2-1) \rightarrow K = \pm 1$

$y'' = K^2 \cdot e^{Kx}$

$y''' = K^3 \cdot e^{Kx}$

$$y = A \cdot y_1 + B \cdot y_2 + C \cdot y_3 = A \cdot e^{x} + B \cdot e^{-x} + \_\_\_\_$$

I have not yet found the 3rd one I need. Let's try

$y = x^n e^{Kx} \longleftrightarrow \underline{n = 1, K = -1}$

GENERAL MECHANISM

* The linear operator L is a differentiation polynomial if the linear homogeneous eq. has constant coefficients.

$$L = P(D) = D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n$$

The formal substitution $D \to K$ gives us the characteristic polynomial.

$$P(D) = \prod_{i=1}^{r} (D - K_i)^{m_i} \cdot \prod_{j=1}^{s} (D^2 + K_j^2)^{n_j}$$

r simple root with multiplicity m_i

s "double" roots with multiplicity n_j.

Each factor $(D - K_i)^{m_i}$ gives us sols: $e^{K_i x}, x e^{K_i x}, \dots, x^{m_i - 1} \cdot e^{K_i x} \dots$

Each factor $(D^2 - K_j^2)^{n_j}$ gives us: $\cos(K_j x), \sin(K_j x), x \cdot \cos(K_j x), x \cdot \sin(K_j x) \dots$

E.g.:

$$y''' + y'' - y' - y = 0$$

$$k^3 + k^2 - k - \overset{k^0}{1} = 0 = (k+1)^2 (k-1) \Rightarrow y = A \cdot e^{-x} + B e^{x} + C e^{x}$$

$$(k+1)^2 \to e^{-x}, x \cdot e^{-x} \qquad (k-1) \to e^{x}$$

E.g.:

$$y''' - y' = 0$$

$$k^3 - k = 0 = k(k^2 - 1) = k(k+1)(k-1) \Rightarrow y = A + Be^{-x} + Ce^{x}$$

$$k \to e^{0x} = 1 \qquad (k+1) \to e^{-x} \qquad (k-1) \to e^{x}$$

E.g.: $y'' + y = 0$

$$k^2 + 1 = 0 \Rightarrow y = A \cdot \cos x + B \cdot \sin x$$

$$\{\cos x, \sin x\}$$

4.1. DEFINITION AND PROPERTIES

* In 3D, the intersection of the surfaces $\varphi_1(x,y,z)=0$ and $\varphi_2(x,y,z)=0$ defines a curve

* The intersection of two parameter families of curves defines a congruence, then there will be a single curve going through each point.

* In such case the constants can be isolated: $\left.\begin{matrix}\psi_1(x,y,z)=C_1\\ \psi_2(x,y,z)=C_2\end{matrix}\right\}$

* Differentiation with respect to "x" gives us the odes:

$$\frac{\partial\psi_i}{\partial x}+\frac{\partial\psi_i}{\partial y}y'+\frac{\partial\psi_i}{\partial z}z'=0, \quad i=1,2$$

* There are two ways to present the eqs. of the congruence:

  - normal form:

$$\left\{\begin{matrix}y'=f_1(x,y,z)\\ z'=f_2(x,y,z)\end{matrix}\right.$$

  - Canonical form

$$\frac{dx}{g_1(x,y,z)}=\frac{dy}{g_2(x,y,z)}=\frac{dz}{g_3(x,y,z)}$$

Ex. 4.2.

Find the odes of the circumferences:

$$\begin{cases} x^2+y^2+z^2=A & \leftarrow \text{Sphere} \\ x+y+z=B & \leftarrow \text{Plane} \end{cases}$$

(Both in canonical and normal form)

Differentiating:

$$2x+2yy'+2zz'=0$$

$$1+y'+z'=0$$

Solve for z' and substitute

$$\circledast \; z'=-(1+y')$$

$$x+yy'+z\,[-(1+y')]=0 \rightarrow x+y'(y-z)-z=0$$

$$\boxed{y'=\frac{z-x}{y-z}} \xrightarrow{\circledast} \boxed{z'=\frac{y-x}{z-y}}$$

$$\frac{dy}{dx}=\frac{z-x}{y-z} \quad \xrightarrow[\text{both eqs.}]{\text{Dividing}} \quad \frac{dy}{dz}=\frac{z-x}{x-y}$$

$$\frac{dz}{dx}=\frac{y-x}{z-y}$$

$$\boxed{\frac{dy}{z-x}=\frac{dx}{y-z}=\frac{dz}{x-y}}$$

* For n+1-dimentional systems a more convenient notation is:

$$(\underset{\text{ind. var}}{t}, \underbrace{x_1, \dots, x_n}_{\text{dep. var.}})$$

* Eqs. of the congruences: $\varphi_i(t, x_1, \dots, x_n) = C_i$, $i = 1, \dots, n$

  - Normal form:

  $$\dot{x}_i = f_i(t, x_1, \dots, x_n) \qquad i = 1, \dots, n$$

  - Canonical form

  $$\frac{dt}{g_0} = \frac{dx_1}{g_1} = \dots = \frac{dx_n}{g_n}$$

UNIQUENESS & EXISTENCE

(Normal form) → f_i and $\frac{df_i}{x_j}$ continuous → unique sol. to $\underset{\text{initial value problem}}{X_i(t_0) = X_{i0}}$ $(i = 1, \dots, n)$

4.2 SOLUTION METHODS

* No general method to do it.
* We're going to consider two methods:
  - Reduction to one eq.
  - First integrals.

REDUCTION TO ONE EQUATION

* A system of n first order eqs. can be re-expressed as a diff. eq. of order n (the reverse of what we did in topic 3).

Ex. 4.3

Solve the following system:

$$\dot{x} = 3x - 2y$$

$$\dot{y} = 2x - y$$

We can either choose x or y.

$$(\dot{x} = 3x - 2y)^{\boldsymbol{\cdot}}$$

$$\downarrow$$

$$\ddot{x} = 3\dot{x} - 2\dot{y}$$

We want to have everything expressed in terms of $x, \dot{x}, \ddot{x}$ (as we had 2 eqs, the final ode has to be of 2^{nd} order).

$$\ddot{x} = 3\dot{x} - 2\underbrace{(2x - y)}_{\dot{y}} = 3\dot{x} - 4x + 2y$$

From the first eq.: $2y = 3x - \dot{x}$. So,

$$\ddot{x} = 3\dot{x} - 4x + 3x - \dot{x} \rightarrow \boxed{\ddot{x} - 2\dot{x} + x = 0}$$

constant coefficients!

Characteristic polynomial: $K^2 - 2K + \overset{K^0}{1} = 0 = (K-1)^2$

$\hookrightarrow K = 1$ is a double root.

$$x = Ae^{Kt} + \underbrace{Bte^{Kt}}_{\text{Because there's a double root.}} = Ae^{t} + Bte^{t}$$

Now, for y:

$$2y = 3x - \dot{x} = 3(Ae^t + Bte^t) - (Ae^t + Bte^t)^{\boldsymbol{\cdot}} = \ldots\ldots$$

Ex. 4.4.

Solve $\dot{x} = y$, $\dot{y} = xy$ (by reducing it into a single equation)

$$(\dot{x} = y)^{\cdot} \to \ddot{x} = \dot{y}$$

$$\ddot{x} = xy = x\dot{x} \to \ddot{x} = x\dot{x}$$

$$\int \ddot{x}\,dt = \int x\dot{x}\,dt \to \dot{x} = \frac{x^2}{2} + c_1 \to \frac{\dot{x}}{\frac{x^2}{2}+c_1} \to \begin{cases} c_1 > 0 & 2t + c_3 = \frac{\operatorname{arctg}}{\sqrt{c_2}}\left(\frac{x}{\sqrt{c_1}}\right) \\ c_1 < 0 & 2t + c_3 = \frac{\operatorname{arctg}}{\sqrt{c_2}}\left(\frac{x}{\sqrt{c_1}}\right) \\ c_1 = 0 & 2t + c_1 = -\frac{1}{x} \end{cases}$$

FIRST INTEGRAL METHOD

- If a function $\phi(t, x_1, \dots, x_n)$ is constant throughout the "evolution" of the system, i.e. $\dot{\phi} = 0$, then
  - * $\phi(t, x_1, \dots, x_n)$ is a first integral for the system.
  - * $\phi(t, x_1, \dots, x_n) = C$ is an eq. for different surfaces in $(t, x_1, \dots, x_n)$ for every C.
- Knowing first integrals makes finding solutions easier!
  - * ϕ is a first integral if:

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + \sum_{i=1}^{n} \frac{\partial\phi_i}{\partial x_i} f_i = 0 \quad \text{(identically)}$$

↳ Remember $\dot{x}_i = f_i$

E.g.:

Show that $\phi = e^{-t}(x+y)$ is a first integral for

$$\begin{cases} \dot{x} = y \\ \dot{y} = x \end{cases}$$

$$\dot{\phi} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x}\dot{x} + \frac{\partial\phi}{\partial y}\dot{y} = (-1)\cdot e^{-t}(x+y) + e^{-t}\dot{x} + e^{-t}\dot{y} =$$

$$= \cancel{-e^{-t}(x+y)} + \cancel{e^{-t}y} + \cancel{e^{-t}x} = 0$$

- One can use the constant (in $\phi = \overset{\text{constant}}{c}$) to solve for one of the variables:

$$x_i = \varphi_i(t, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, c)$$

- For each 1^{st} integral, we can solve for one variable.

E.g. (last one):

Solve $\begin{cases} \dot{x} = y \\ \dot{y} = x \end{cases}$ using the 1^{st} integral $\phi = e^{-t}(x+y) = A$.

As ϕ is a first integral, it stays constant.

From ϕ:

$$x = Ae^{t} - y$$

Differentiating (to be able to use the info. in the system)

$$\dot{x} = Ae^{t} - \dot{y} = Ae^{t} - x \Rightarrow \dot{x} + x = Ae^{t}$$

↳ Linear first order inhomogeneous

$$\dot{x} + x = 0 \longrightarrow x_{hom.}?$$

$$k + 1 = 0 \rightarrow k = -1 \rightarrow x_{hom} = Be^{-t}$$

$$x = x_p + x_{hom}$$

↳ part. sol of inhom. eq.

⇓

VARIATION OF CONSTANTS

(Theory)

$$y = c_1 y_1$$

$$y_p = g(x) y_1$$

$$g' y = b$$

$$x_h = Be^{-t} \quad x_1$$

$$x_p = g(t) e^{-t}$$

$$\dot{g} e^{-t} = Ae^{t}$$

$$\dot{g} = Ae^{2t} \rightarrow g = A\frac{e^{2t}}{2}$$

$$x_p = \frac{Ae^{2t}}{2} e^{-t} = \frac{Ae^{t}}{2}$$

• If one can find $\underline{n}$ functionally independent 1^{st} integrals one will be able to write the gen. sol. (all the x_i can be reexpressed as functions of C_i and t).

Functional independence $\Leftrightarrow \left| \dfrac{\partial(\phi_1, \dots, \phi_n)}{\partial(x_1, \dots, x_n)} \right| \neq 0$

Ex. 4.6.:

a) Show that $e^{-t}(x+y)$ and $e^{t}(x-y)$ are functionally independent.

b) Show as well that $x^2-y^2=0$ is not independent of the other two.

a) $\phi_1 = e^{-t}(x+y)$

$\phi_2 = e^{t}(x-y)$

$\phi_3 = x^2-y^2$

Functional dep. of ϕ_1 and ϕ_2: $\begin{vmatrix} \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_1}{\partial y} \\ \frac{\partial \phi_2}{\partial x} & \frac{\partial \phi_2}{\partial y} \end{vmatrix} \overset{?}{=}$

$$= \begin{vmatrix} e^{-t} & e^{-t} \\ e^{t} & -e^{t} \end{vmatrix} = -e^{0} - e^{0} = -2$$

b) $\phi_3 = \phi_1 \cdot \phi_2 \rightarrow$ Clear dependence

• How can 1^{st} integrals be found?

* Look for symmetries (e.g. physical conservation forces)

* Practice, inspection

E.g.:

Find two 1^{st} integrals for $\begin{cases} \dot{x} = y - z \\ \dot{y} = z - x \\ \dot{z} = x - y \end{cases}$

$$\dot{x} + \dot{y} + \dot{z} = 0 \rightarrow \boxed{x + y + z = A}$$

Let's do now:

$$x\dot{x} = x(y-z)$$
$$y\dot{y} = y(z-x)$$
$$z\dot{z} = z(x-y)$$

$$x\dot{x} + y\dot{y} + z\dot{z} = 0 \rightarrow \boxed{x^2 + y^2 + z^2 = B}$$

- Usually, the canonical form of systems can be used to find symmetries that allow finding first integrals.

E.g.:

$$\begin{cases} \dot{x} = \dfrac{2tx}{t^2 - x^2 - y^2} \rightarrow \dfrac{dx}{x} = \dfrac{2t}{t^2 - x^2 - y^2} dt \\ \dot{y} = \dfrac{2ty}{t^2 - x^2 - y^2} \rightarrow \dfrac{dy}{y} = \dfrac{2t}{t^2 - x^2 - y^2} dt \end{cases}$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{2t\,dt}{t^2 - x^2 - y^2}$$

$$\frac{dx}{x} = \frac{dy}{y} \rightarrow \ln y = \ln x + \ln A \rightarrow y = Ax$$

As we have a 2D system, we need a second 1^{st} integral.

Let us use the following property: $\dfrac{a}{b} = \dfrac{c}{d} \Leftrightarrow \dfrac{a+c}{b+d} = \dfrac{a}{b}$

Using it and multiplying the fractions by $2x$, $2y$ and 1 respectively and then summing we get:

$$\frac{2x\,dx+2y\,dy+2t\,dt}{2x^2+2y^2+t^2-x^2-y^2}=\frac{2x\,dx+2y\,dy+2t\,dt}{x^2+y^2+t^2}=\frac{2x\,dx}{x^2}$$

$$d\ln(t^2+x^2+y^2)=d\ln x$$

$$\boxed{x^2+y^2+t^2=Bx}$$

Ex. 4.9

Solve:

$$\begin{cases}\dot{x}=\dfrac{y}{x+y}\to\dfrac{dx}{y}=\dfrac{dt}{x+y}\\[2ex]\dot{y}=\dfrac{x}{x+y}\to\dfrac{dy}{x}=\dfrac{dt}{x+y}\end{cases}\qquad\frac{dx}{y}=\frac{dy}{x}=\frac{dt}{x+y}$$

First 1^{st} integral:

$$\frac{dx}{y}=\frac{dy}{x}\to\frac{y^2}{2}=\frac{x^2}{2}+C\to y^2-x^2=C$$

Second 1^{st} int.:

$$dx+dy=\frac{y}{x+y}dt+\frac{x}{x+y}dt=\frac{(x+y)}{x+y}dt=dt$$

$$dx+dy-dt=0\to x+y-t=D$$

Ex. 4.⊛

$$\begin{cases}\dot{x}=\dfrac{ty}{y^2-x^2}\to\dfrac{dx}{y}=\dfrac{t\cdot dt}{y^2-x^2}\\[2ex]\dot{y}=\dfrac{-tx}{y^2-x^2}\to\dfrac{dy}{-x}=\dfrac{t\cdot dt}{y^2-x^2}\end{cases}$$

$$\frac{dx}{y} = \frac{dy}{-x} = \frac{t \cdot dt}{y^2 - x^2}$$

First 1^{st} int.:

$$\frac{dx}{y} = \frac{dy}{-x} \rightarrow y^2 + x^2 = C$$

Second 1^{st} int.:

$$dx + dy = \frac{yt}{y^2 - x^2} dt - \frac{xt}{y^2 - x^2} dt = \frac{y-x}{y^2-x^2} t\, dt = \frac{t\,dt}{y+x}$$

$$(x+y)(dx+dy) - t\,dt = 0 \qquad \begin{matrix} x+y = z \\ dx+dy = dz \end{matrix}$$

$$\boxed{(x+y)^2 - t^2 = B}$$

4.3. SYSTEMS OF 1^{st} ORDER EQUATIONS

* Now, we'll focus on systems of the form:

$$\dot{x}_i = \sum_{i=1}^{n} a_{ij}(t) x_j + b_i(t)$$

We'll demand these are continuous in some I (for existence & uniqueness)

NOTATION

$$\dot{\vec{X}}_i = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} \qquad \vec{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

vector $\qquad$ vector

$$\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \qquad A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & & a_{nn} \end{pmatrix}$$

$$\Rightarrow \dot{\vec{X}} = A\vec{X} + \vec{b}$$

(Likewise, $L\vec{x} = \dot{\vec{x}} - A\vec{x} = \vec{b}$)

And, because of linearity, $L(c_1\vec{x}_1 + c_2\vec{x}_2) = c_1 L\vec{x}_1 + c_2 L\vec{x}_2$

Ex. 4.10

Write the system $\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$ in matrix form and write it as one eq:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\ddot{x} = \dot{y} \to \ddot{x} = -x \to \ddot{x} + x = 0 \begin{cases} x = A\cos t + B \sin t \\ y = \dot{x} = -A\sin t + B \cdot \cos t \end{cases}$$

4.4. LINEAR HOMOGENEOUS SYSTEMS

- Let us begin with $L\vec{x} = 0$

  Linearity ⇒ Superposition principle

$$L(\underbrace{\vec{x}_i}_{\text{Solution}}) = 0 \Rightarrow L\left(\sum_{i=1}^{n} c_i \vec{x}_i\right) = \sum_{i=1}^{n} c_i L \vec{x}_i = 0$$

The group of solutions of a linear system is a vector space.

- The vectors are linearly dependent if the system

$$\sum_{j=1} c_j \vec{x}_j = 0 \Leftrightarrow \sum_{j=1}^{\infty} x_{ij} c_j = 0 \quad \forall t \in I$$

  ↳ the same, but row by row

  has non-zero solutions.

- If the system (of vectors or solutions) is dependent, the Wronskian

$$W[\vec{x}_1, \ldots, \vec{x}_n] = \begin{vmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ x_{12} & x_{22} & \cdots & x_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{vmatrix} = 0 \text{ in all points of } I$$

↳ First solution vector

- In general, for any set of vectors

$$W=0 \not\Rightarrow \text{linear dependence}$$

  - It only applies when they are solutions of a linear system of eqs.
  - In this case, if $W(t_0)=0$ for some $t_0 \in I$, then $W(t)=0 \ \forall t \in I$

- Existence & uniqueness → The dimension of the space of solutions of an n-th dimensional system cannot be less than n.

- Fundamental system of equations ≡ Group of n linearly independent solutions of the system.

- Any sol. of $L\vec{x}=0$ can be written as

  Mirar PPT.

Ex. 4.11

Prove that the vectors $\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$ and $\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ form a fundamental system of the system of eqs: $\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$

$\dot{\vec{X}} = \mathbb{A} X$

$\mathbb{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\vec{X}_1 = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$

$\vec{X}_2 = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

Are they sols.?

$\dot{\vec{X}}_1 = \mathbb{A}\vec{X}_1$

$\dot{\vec{X}}_2 = \mathbb{A}\vec{X}_2$

Are they indep.?

$W = \begin{vmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{vmatrix} \neq 0 ?$

$\vec{X} = C_1 \vec{X}_1 + C_2 \vec{X}_2$

FUNDAMENTAL MATRICES

- The n vectors of a fundamental system will be the columns of a fundamental matrix

$$\mathbb{F}(t) = (\vec{x}_1 \ \vec{x}_2 \ \vec{x}_3 \cdots \vec{x}_n) = \begin{pmatrix} x_{11} & \cdots & x_{n1} \\ x_{12} & \cdots & x_{n2} \\ \vdots & & \vdots \\ x_{1n} & \cdots & x_{nn} \end{pmatrix}$$

- Fundamental matrices are non-singular by construction.

$$\det \mathbb{F}(t) = W[\vec{x}_1, \ldots, \vec{x}_n] \neq 0$$

Besides, the fundamental matrix is a solution of a linear system

$$L\mathbb{F} = 0 \iff \boxed{\dot{\mathbb{F}} = A\mathbb{F}}$$

$$\mathbb{F} = \begin{pmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & & \vdots \\ x_{1n} & \cdots & x_{nn} \end{pmatrix} \longrightarrow \dot{\mathbb{F}} = \begin{pmatrix} \dot{x}_{11} & \cdots & \dot{x}_{n1} \\ \vdots & & \vdots \\ \dot{x}_{1n} & \cdots & \dot{x}_{nn} \end{pmatrix}$$

1^{st} sol. $\quad n^{th}$ sol.

Ex. 4.12
Find the fundamental matrix for $\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$

In a previous exercise we saw the two (vector) solutions are

$$\vec{x}_1 = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \quad \vec{x}_2 = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \quad \text{so} \quad \mathbb{F} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

(Just to be sure)

$$\dot{\mathbb{F}} = A \cdot \mathbb{F} \ ?$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\dot{\mathbb{F}} = \begin{pmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{pmatrix} \stackrel{?}{=} \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}}_{\mathbb{F}} = \begin{pmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{pmatrix} = \dot{\mathbb{F}}$$

Exam question (sep. 08)

Consider the matrix $\mathbb{F} = \begin{pmatrix} t & 1 \\ -1 & t \end{pmatrix}$.

What system is it a fundamental system of?

$\dot{\mathbb{F}} = \mathbb{A} \cdot \mathbb{F} \longrightarrow$ As fundamental matrices are non-singular $\mathbb{F}^{-1}$ is well defined

$$\dot{\mathbb{F}} \cdot \mathbb{F}^{-1} = \mathbb{A} \cdot \mathbb{F} \cdot \mathbb{F}^{-1} \Rightarrow \mathbb{A} = \dot{\mathbb{F}} \cdot \mathbb{F}^{-1}$$

$$\mathbb{F}^{-1} = \frac{\operatorname{adj}(\mathbb{F})^t}{\det(\mathbb{F})}$$

$$\mathbb{F} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \operatorname{adj}(\mathbb{F}) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \to (\operatorname{adj}(\mathbb{F}))^t = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\mathbb{F}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \xrightarrow[\text{problem}]{\text{for our}} \mathbb{F}^{-1} = \frac{1}{t^2+1} \begin{pmatrix} t & -1 \\ 1 & t \end{pmatrix}$$

$$\mathbb{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{t^2+1} \begin{pmatrix} t & -1 \\ 1 & t \end{pmatrix} = \frac{1}{t^2+1} \begin{pmatrix} t & -1 \\ 1 & t \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbb{A} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} \dot{x} = \dfrac{1}{t^2+1}(tx+y) \\ \dot{y} = \dfrac{1}{t^2+1}(-x+ty) \end{cases}$$

- The general solution of an homogeneous (1^{st} order) linear system of eqs. is a linear combination of the elements of the fundamental system.

$$\vec{X} = \sum_{j=1}^{n} c_j \vec{X_j} \Rightarrow X_i = \underset{\substack{\downarrow \\ \text{row by row}}}{\sum_{j=1}^{n}} X_{ij} c_j$$

For instance, for our example, $\dot{x} = y$, $\dot{y} = -x$ we have

$$\vec{X} = \underbrace{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}}_{\mathbb{F}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

as its general solution

4.5 COMPLETE LINEAR SYSTEMS

$\hookrightarrow$ inhomogeneous

- The complete solution is obtained by summing the solution to the homogeneous system and a particular sol. of the hom. system.

- $$\left. \begin{array}{l} L\vec{x}_1 = 0 \\ L\vec{x}_2 = b \end{array} \right\} L(\vec{x}_1 + \vec{x}_2) = b$$

- $$\left. \begin{array}{l} L\vec{x}_1 = b \\ L\vec{x}_2 = b \end{array} \right\} L(\vec{x}_1 - \vec{x}_2) = 0$$

VARIATION OF PARAMETERS

- The procedure is similar to the one for single eqs:
  - Let us suppose $\dot{\vec{x}} = \mathbb{A}\vec{x} + \vec{b}$ is the system we're studying, and let us assume a solution to its homogeneous part has been found:

$$\vec{X} = \mathbb{F} \cdot \vec{c}$$

  - We propose a trial vector

$$\vec{c} \to \vec{g}(t)$$

so that

$$\vec{X}_p(t) = \mathbb{F}(t)\,\vec{g}(t)$$

We need this to hold

$$\dot{\vec{X}}_p = \mathbb{A}\vec{X}_p + \vec{b}$$

$$\dot{\vec{x}}_p = (\mathbb{F}\cdot\vec{g})^{\cdot} = \dot{\mathbb{F}}\cdot\vec{g} + \mathbb{F}\cdot\dot{\vec{g}} \underset{\downarrow}{=} (\mathbb{A}\mathbb{F})\vec{g} + \mathbb{F}\dot{\vec{g}} = \mathbb{A}\cdot\vec{x}_p + \mathbb{F}\cdot\dot{\vec{g}}$$

As $\mathbb{F}$ is a fund. matrix $\dot{\mathbb{F}} = \mathbb{A}\cdot\mathbb{F}$

$\vec{x}_p$

$\dot{\vec{x}}_p = \mathbb{A}\vec{x}_p + \vec{b}$

So we partially conclude : $\vec{b} = \mathbb{F}\cdot\dot{\vec{g}}$

Now,

$$\mathbb{F}^{-1}\cdot\vec{b} = \mathbb{F}^{-1}\cdot\mathbb{F}\cdot\dot{\vec{g}} \Rightarrow \mathbb{F}^{-1}\cdot\vec{b} = \dot{\vec{g}} \Rightarrow \vec{g} = \int \mathbb{F}^{-1}\cdot\vec{b}$$

Finally, the gen. sol is :

$$\vec{X} = \mathbb{F}\cdot\vec{c} + \mathbb{F}\cdot\vec{g} = \underbrace{\mathbb{F}\cdot\vec{c}}_{X_{hom}} + \underbrace{\mathbb{F}\int\mathbb{F}^{-1}\cdot\vec{b}}_{X_{part.}}$$

P. 135 → 33.15 (

EXPONENTIAL OF A MATRIX

$$e^{\mathbb{A}} = \lim_{n\to\infty}\left(\mathbb{1} + \frac{\mathbb{A}}{n}\right)^n = \sum_{k=0}^{\infty}\frac{1}{k!}\mathbb{A}^k$$

SOLVING HOMOGENEOUS SYSTEMS WITH CONSTANT COEFFICIENTS

Remember (topic 3):

- If we have $\dot{x} = ax$, we try $x = Ce^{kt} \rightarrow \dot{x} = kCe^{kt} = ax = aCe^{kt} \rightarrow$
$\rightarrow \boxed{k = a}$

- Or for $\ddot{x} - ax = 0$, try again $x = Ce^{kt}$:
$ck^2 e^{kt} - ace^{kt} = 0 \rightarrow (k^2 - a) = 0 \Rightarrow \boxed{k = \pm\sqrt{a}}$

For systems we have:

$$\dot{\vec{X}} = \mathbb{A}\vec{X}$$

So now we'll go for $\vec{x} = e^{kt}\vec{C} = \begin{pmatrix} C_1 e^{kt} \\ \vdots \\ C_n e^{kt} \end{pmatrix}$

$$\dot{\vec{X}} = \mathbb{A}\vec{X} \Rightarrow ke^{kt}\vec{C} = \mathbb{A}(e^{kt}\vec{C})$$

Then, finally, what remains to be solved is

$$(\mathbb{A} - k\mathbb{1})\cdot\vec{C} = 0$$

the typical eigenvalue problem.

In general, each eigenvalue combined with its eigenvectors gives us a linearly independent solution.

Example:

$$\left.\begin{aligned} \dot{x} &= x + 2y \\ \dot{y} &= 4x + 3y \end{aligned}\right\} \rightarrow \mathbb{A} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

$\det(\mathbb{A} - k\cdot\mathbb{1}) = 0$ gives us the eigen values.

$$\begin{vmatrix} 1-k & 2 \\ 4 & 3-k \end{vmatrix} = (1-k)(3-k) - 8 = -5 - 4k + k^2 = (k-5)(k+1) = 0$$

eigenvalues $\begin{cases} K_1 = -1 \\ K_2 = 5 \end{cases}$

eigenvectors

$A\vec{V_1} = K_1 \vec{V_1}$

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1 \cdot \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{cases} x + 2y = -x \rightarrow 2x = -2y \\ 4x + 3y = -y \rightarrow 4x = -4y \end{cases}$$

I take x as I like, e.g. $x = 1$.

$$\vec{V_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightarrow \vec{X_1} = \vec{V_1}\, e^{K_1 t} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

Is it true?

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} = \begin{pmatrix} 1-2 \\ 4-3 \end{pmatrix} e^{-t} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} \checkmark$$

$A\vec{V_2} = K_2 \vec{V_2}$

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{cases} x + 2y = 5x \rightarrow y = -2x \\ 4x + 3y = 5y \rightarrow 4x = -2y \end{cases} \rightarrow \vec{V_2} = (1\ \ -2)$$

$$\vec{X_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{5t}$$

H.W. → Check that $\dot{\vec{X}}_2 = A \cdot \vec{X_2}$ is satisfied

If you have fewer eigenvectors than the dimension of the system, that is, your eigenvalues are multiple, propose a solution of the form

$$\vec{X} = \left(\vec{C_1} + \vec{C_2}\, t + \dots + \vec{C_m}\, t^{m-1} \right) e^{Kt}$$

and deduce the constants.

Example:

$$\begin{cases} \dot{x} = x - y \\ \dot{y} = y \end{cases} \rightarrow \mathbb{A} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1-K & -1 \\ 0 & 1-K \end{vmatrix} = (1-K)^2 = 0 \Rightarrow K = 1 \text{ is a double eigenvalue}$$

So we propose:

$K=1$ in this case

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} = (\vec{c}_1 + \vec{c}_2 t) e^{Kt} = \begin{pmatrix} A \\ C \end{pmatrix} e^t + \begin{pmatrix} B \\ D \end{pmatrix} t e^t = \begin{pmatrix} A + Bt \\ C + Dt \end{pmatrix} e^t$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \dot{\vec{x}} = \mathbb{A}\vec{x}$$

$$\left[\begin{pmatrix} A + Bt \\ C + Dt \end{pmatrix} e^t \right]^{\cdot} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A + Bt \\ C + Dt \end{pmatrix} e^t$$

$$\begin{pmatrix} B \\ D \end{pmatrix} e^t + \begin{pmatrix} A + Bt \\ C + Dt \end{pmatrix} e^t = \begin{pmatrix} A + Bt - C - Dt \\ C + Dt \end{pmatrix} e^t$$

$$\begin{pmatrix} \cancel{A} + B + \cancel{Bt} \\ \cancel{C} + D + \cancel{Dt} \end{pmatrix} e^t = \begin{pmatrix} \cancel{A} + \cancel{Bt} - C - Dt \\ \cancel{C} + \cancel{Dt} \end{pmatrix} e^t$$

lin. indep.

$$\begin{cases} D = 0 \\ B = -C \\ A \text{ free} \end{cases} \Rightarrow \vec{x} = \begin{pmatrix} A + Bt \\ -B \end{pmatrix} e^t = \underbrace{\begin{pmatrix} A \\ -B \end{pmatrix} e^t}_{\vec{x}_1} + \underbrace{\begin{pmatrix} B \\ 0 \end{pmatrix} t e^t}_{\vec{x}_2}$$

If the roots are conjugated pairs $(\alpha \pm i\omega)$ try:

$$\vec{x} = (\vec{a}_1 + \vec{a}_2 t + \dots + \vec{a}_m t^{m-1}) e^{\alpha t} \cos \omega t +$$
$$+ i (\vec{b}_1 + \vec{b}_2 t + \dots + \vec{b}_m t^{m-1}) e^{\alpha t} \sin \omega t$$

Exam question (Jan. 07)

If $\mathbb{F}$ is a fundamental matrix, is $\mathbb{F}^{-1}$ a fundamental matrix for another system? which one?

We have $\dot{\mathbb{F}} = \mathbb{A}\mathbb{F}$ from the statement.

Let us suppose for another system $\tilde{\mathbb{A}}$

$$(\mathbb{F}^{-1})^\cdot = \tilde{\mathbb{A}}\,\mathbb{F}^{-1}$$

Trick:

$$(\mathbb{F}\cdot\mathbb{F}^{-1}) = \mathbb{1}$$

$$(\mathbb{F}\cdot\mathbb{F}^{-1})^\cdot = \dot{\mathbb{F}}\mathbb{F}^{-1} + \mathbb{F}\dot{\mathbb{F}}^{-1} = 0$$

$$\mathbb{F}(\mathbb{F}^{-1})^\cdot = -\dot{\mathbb{F}}\mathbb{F}^{-1}$$

multiplying both sides by $\mathbb{F}^{-1}$:

$$\mathbb{F}^{-1}\left[\mathbb{F}(\mathbb{F}^{-1})^\cdot\right] = -\mathbb{F}^{-1}\dot{\mathbb{F}}\mathbb{F}^{-1}$$

$$\dot{\mathbb{F}}^{-1} = \underbrace{\boxed{-\mathbb{F}^{-1}\dot{\mathbb{F}}}}_{\mathbb{A}}\mathbb{F}^{-1}$$

Exam question (Feb. 04)

Is $F(t) = \begin{pmatrix} t & 3t \\ 0 & t \end{pmatrix}$ a fundamental matrix of a system in the intervals $t \in (1,2)$ or $t \in (-1,1)$?

$W \equiv \det F = t^2 \rightarrow \det\mathbb{F} = 0$ if $t = 0 \rightarrow \mathbb{F}$ is not invertible in $t \in (-1,1)$, not a fund. matrix.

It's invertible in $(1,2)$, so it's a fund. matrix.

LAPLACE TRANSFORMATION $\Rightarrow F(s)=\int_0^\infty e^{-st} f(t)\,dt$

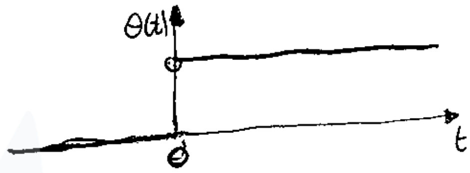
E.g.

$f(t)=1$

$$F(s)=\int_0^\infty e^{-st}\,dt = -\frac{1}{s}\,e^{st}\Big|_0^\infty = \frac{1}{s}$$

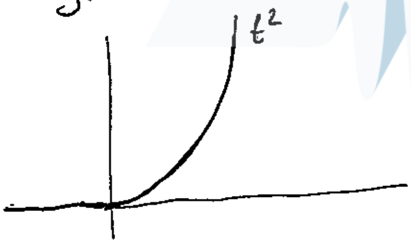
Step function (Heviside func.)

$$\theta(t)=\begin{cases}1 & t>0\\ 0 & t<0\end{cases}$$



E.g.:

$f(t)=t^2\,\theta(t)$



$f(t)=\sin t\cdot\theta(t)$

F(α) space

$f(t)$ defined by pieces & with an exponential "order α".

$f(t)$ is finite in $[0,a] \to \int_0^a e^{-st} f(t)\,dt < \infty$

$\forall a\to\infty$, $\int_0^\infty e^{-st} f(t)\,dt$ converges if $f(t)$ is an exponential of order α, which means:

$$e^{-\alpha t}|f(t)| \leq M \quad \forall t>t_0$$

$$\hookrightarrow |f(t)| \leq M e^{\alpha t}$$

E.g.:

$f(t)=1$ $\qquad \in F(\alpha=2)$

$|f(t)|\leq 1 \quad \forall t>0 \Rightarrow \boxed{\alpha=0}$

$f(t)=e^{\sin t}$

$|\sin t|\leq 1$

$\frac{1}{e}\leq e^{\sin t}\leq e \Rightarrow e^{\sin t}\leq M=e \Rightarrow \boxed{\alpha=0}$ $\qquad \in F(\alpha=0)$

$f(t)=e^{(1+\cos t)t}$ $\qquad \in F(\alpha=2)$

$(1+\cos t)t\leq(1+|\cos t|)t\leq 2t$

$e^{(1+\cos t)t}\leq e^{2t}\Rightarrow e^{-2t}e^{(1+\cos t)}\leq M=1 \Rightarrow \boxed{\alpha=2}$

$(\forall t>0.739)$

$f(t)$ continuous (finite ammount of discontinuities); $\forall t\in[0,t_0]: |f(t)|\leq L$

$$\int_0^a e^{-st}f(t)\,dt\leq\int_0^{t_0}e^{-st}|f(t)|\,dt+\int_{t_0}^{a}e^{-st}|f(t)|\,dt\leq$$

$$\leq L\int_0^{t_0}e^{-st}\,dt+M\int_{t_0}^{a}e^{-st}e^{\alpha t}\,dt=L\left[-\frac{e^{-st}}{s}\right]_0^{t_0}-M\left[\frac{e^{(-s+\alpha)t}}{\alpha-s}\right]_{t_0}^{a}$$

$$=L\left(-\frac{e^{-st_0}}{s}+\frac{1}{s}\right)+M\left(-\frac{e^{(-s+\alpha)a}}{\alpha-s}+\frac{e^{(-s+\alpha)t_0}}{\alpha-s}\right)$$

If we do the limit $a\to\infty$:

Laplace transf. exists $\forall s>\alpha$

E.g.:

$$f(t)=e^{at}\in F(a)$$

$$F(s)=\int_0^{\infty} e^{-st}e^{at}\,dt=\lim_{R\to\infty}\int_0^{R} e^{-(s-a)t}\,dt=\lim_{R\to\infty}\left(\frac{1}{s-a}-\frac{e^{-(s-a)R}}{s-a}\right)\ \underline{\underline{\text{if } s>a}}$$

$$=\frac{1}{s-a}$$

$$\mathcal{L}[e^{at}]=\frac{1}{s-a}\quad(s>a)$$

* Check exponential order of $f(t)=t^n$
* $\mathcal{L}[t^n]$

LINEARITY

$$\left.\begin{array}{l}\mathcal{L}[f(t)]=F(s)\\ \mathcal{L}[g(t)]=G(s)\end{array}\right\}\ \mathcal{L}[af(t)+bg(t)]=aF(s)+b\,G(s)$$

E.g.:

$$\mathcal{L}[\cosh(at)]=\mathcal{L}\left[\frac{e^{at}+e^{-at}}{2}\right]=\frac{1}{2}\mathcal{L}[e^{at}]+\frac{1}{2}\mathcal{L}[e^{-at}]=$$

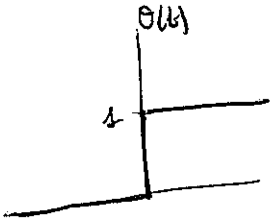
$$=\frac{1}{2}\frac{1}{s-a}+\frac{1}{2}\frac{1}{s+a}=\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}s}{s^2-a^2}=\frac{s}{s^2-a^2}$$

$$\mathcal{L}[\sinh(at)]=\frac{a}{s^2-a^2}$$

THEOREM OF DISPLACEMENT

$$\underbrace{\mathcal{L}[f(t)]=F(s)}_{s>\alpha}\rightarrow\mathcal{L}[e^{at}f(t)]=F(s-a)\quad\forall s>\alpha+a$$

E.g.: $\mathcal{L}[e^{at}\cosh(bt)] = \dfrac{s-a}{(s-a)^2-b^2}$



$$\frac{d\theta(t)}{dt} = \delta(t)$$

$$\delta(t-t_0) = \begin{cases} 1 & \forall t = t_0 \\ 0 & \forall t \neq t_0 \end{cases}$$

↓
DIRAC'S DELTA

$$\boxed{\mathcal{L}[\theta(t-a)f(t-a)]} = \int_0^\infty dt\, \theta(t-a) f(t-a)\, e^{-st} = \int_0^\infty dt\, f(t-a)\, e^{-st} \overset{\;}{\underset{J=t-a}{=\!=}}$$

$$= e^{as}\int_0^\infty dJ\, f(J)\, e^{-sJ} = \boxed{e^{as}F(s)}$$

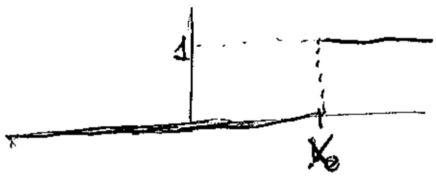
Try: $\mathcal{L}[\theta(t-a)\,e^{b(t-a)}]$

$$\boxed{\mathcal{L}[f(at)]} = \int_0^\infty dt\, e^{-st} f(at) \underset{J=at}{=\!=} \frac{1}{a}\int_0^\infty dJ\, e^{-\overset{\tilde{s}}{\frac{s}{a}}J} f(J) =$$

$$= \frac{1}{a}F(\tilde{s}) = \boxed{\frac{1}{a}F\left(\frac{s}{a}\right)}$$

HEAVISIDE FUNCTION & DIRAC'S DELTA

$$\theta(t-t_0) = \begin{cases} 1 & x > x_0 \\ 0 & x < x_0 \end{cases}$$



Generalized derivative

$$\int_{-\infty}^{\infty} f g'\,dx = fg\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'g\,dx \overset{\lim_{x\to\pm\infty}|fg| = 0}{=\!=} -\int_{-\infty}^{\infty} f'g\,dx$$

Applying that to the Heaviside func.:

$$\int_{-\infty}^{\infty} \theta'(x) f(x)\,dx = -\int_{-\infty}^{\infty} \theta(x) f'(x)\,dx = -\int_{0}^{\infty} f'(x)\,dx = -f(x)\Big|_{0}^{\infty} \underset{\lim_{x\to\infty} f = 0}{=} f(0)$$

$$\int_{-\infty}^{\infty} \theta(x-x_0) f(x)\,dx = f(x_0) = \int_{-\infty}^{\infty} \theta'(x-x_0) f(x)\,dx$$

$$\int_{-\infty}^{\infty} \delta(x-x_0)\,dx = 1 \Rightarrow \delta(x-x_0) = \begin{cases} \infty & x = x_0 \\ 0 & x \neq x_0 \end{cases}$$

HEAVISIDE FORMULA $\Rightarrow$ Mathematica $\Rightarrow$ Apart$\left[\frac{P(x)}{Q(x)}\right]$

$$\frac{P(s)}{Q(s)} = \frac{A}{(s-a)^n} + \frac{B}{(s-a)^{n-1}} + \frac{C}{(s-a)^{n-2}} + \cdots + \frac{N}{(s-a)} + \frac{M}{(s-b)}$$

$$A = \lim_{s\to a} (s-a)^n \frac{P(s)}{Q(s)}$$

$Q(s) = (s-a)\tilde{Q}(a)$

$$B = \lim_{s\to a} \frac{d}{ds}\left((s-a)^n \frac{P(s)}{Q(s)}\right)$$

$\vdots$

$$N = \lim_{s\to a} \frac{d^{n-1}}{ds^{n-1}}\left((s-a)^n \frac{P(s)}{Q(s)}\right)$$

$$M = \lim_{s\to b} (s-b) \frac{P(s)}{Q(s)}$$

SUMMARY

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt \quad \forall s > \alpha \;:\; f(t) \in F(\alpha)\ \forall t > t_0$$

$*\ \mathcal{L}[e^{at} f(t)] = F(s-a)$

$*\ \mathcal{L}[\theta(t-a) f(t-a)] = e^{-as} F(s)$

$*\ \mathcal{L}[t^n f(t)] = (-1)^n \dfrac{d^n F(s)}{ds^n}$

$*\ \mathcal{L}[f(at)] = \dfrac{1}{a} F\left(\dfrac{s}{a}\right)$

$*\ \mathcal{L}\left[\dfrac{d^n f}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$

E.g.: $\mathcal{L}[\theta(t)] = \dfrac{1}{s}$

$$\frac{1}{s-b} = \mathcal{L}[e^{bt}\theta(t)]$$

$$\frac{1}{(s-b)^2} = -\frac{d}{ds}\frac{1}{(s-b)} = \mathcal{L}[e^{bt}\, t\, \theta(t)]$$

$*\ \dfrac{e^{-bs}}{s-a} = \mathcal{L}\left[\theta(t-b) e^{a(t-b)}\right]$

E.g.: $\mathcal{L}[\sin(at)] = \displaystyle\int_0^\infty e^{-st}\sin(at)\,dt = \frac{1}{2i}\int_0^\infty e^{-st}\left(e^{iat} - e^{-iat}\right)dt = \frac{1}{2i}\int_0^\infty \left(e^{-st+iat} - e^{-st-iat}\right)dt =$

$$= \frac{1}{2i}\left[\frac{e^{-(s-ia)t}}{-(s-ia)} - \frac{e^{-(s+ia)t}}{-(s+ia)}\right]_0^\infty = \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right] = \frac{1}{2i}\left[\frac{s+ia-s+ia}{(s^2+a^2)}\right] = \frac{a}{s^2+a^2}$$

$$\mathcal{L}[\cos(at)] = \frac{s}{s^2+a^2}$$

Ex. 6.3

$$P(x) = \frac{2}{x(x-1)^2(x+1)}$$

$$Q(x) = \frac{1}{x^2(x+1)^2(x-1)^2}$$

$\underline{\underline{x=0}}$

$$\left.\begin{array}{l} \lim_{x\to 0} xP(x) = 2 \\ \lim_{x\to 0} x^2 Q(x) = 1 \end{array}\right\} \text{Regular singular}$$

$\underline{\underline{x=-1}}$

$$\left.\begin{array}{l} \lim_{x\to -1} (x+1)P(x) = -1/2 \\ \lim_{x\to -1} (x+1)^2 Q(x) = 1/4 \end{array}\right\} \text{Regular singular}$$

$\underline{\underline{x=1}}$

$$\left.\begin{array}{l} \lim_{x\to 1} (x-1)P(x) = +\infty \\ \lim_{x\to -1} (x-1)^2 Q(x) = 1/4 \end{array}\right\} \text{Irregular singular}$$

E.g.:

Schrödinger eq.

$\Psi(x) \equiv$ wave function

$$-\frac{\hbar}{2m}\frac{\partial^2\Psi(x)}{\partial x^2} + V(x)\,\Psi(x) = +i\hbar\frac{\partial\Psi}{\partial t} \xrightarrow{\Psi=\Psi_x(x)\cdot\Psi_t(t)}$$

$$\rightarrow -\underbrace{\frac{\hbar}{2m}\Psi_t\frac{d^2\Psi_x}{dx^2} + V(x)\Psi_x\Psi_t}_{E\Psi_x\Psi_t} = \underbrace{i\hbar\Psi_x\frac{d\Psi_t}{dt}}_{E\Psi_x\Psi_y} \rightarrow$$

$$\rightarrow -\frac{\hbar}{2m}\frac{d^2\Psi_x}{dx^2} + V(x)\Psi_x = E\Psi_x$$

$V(x) = \frac{1}{2}m\omega^2x^2$

$$-\frac{\hbar}{2m}\frac{d^2\Psi}{dx^2} + \frac{1}{2}m\omega^2x^2\Psi = E\Psi$$

$\tilde{x} = \frac{x}{x_0}, \quad x_0 = \sqrt{\frac{\hbar}{m\omega^2}}, \quad \varepsilon = \frac{E}{E_0}, \quad E_0 = \frac{1}{2}\hbar\omega$

$$\frac{d^2\Psi}{dx^2} + (\varepsilon - x^2)\Psi = 0 \Longrightarrow x\to\infty \Rightarrow \Psi(x) \sim Ae^{x^2/2} + Be^{-x^2/2}$$

$\Psi = e^{-x^2/2}y(x)$

$$\boxed{y'' - 2xy' + \lambda y = 0} \quad \left(\lambda = \frac{E}{E_0} - 1\right)$$

Hermite eq.

Sol:

$n+2 = 2k \Rightarrow a_{2k} = \frac{4(k-1)-\lambda}{2k(2k-1)}a_{2(k-1)}$

$n+2 = 2k+1 \Rightarrow a_{2k+1} = \frac{4k-2-\lambda}{2k(2k+1)}a_{2k-1}$

$$a_{n+2} = \frac{2n-\lambda}{(n+2)(n+1)}a_n \longrightarrow y(x) = a_0\left[1 - \frac{\lambda}{2}x^2 + \dots\right] + a_1\left[x + \frac{2-\lambda}{6}x^3 + \dots\right]$$

$$|x|\to\infty \Rightarrow a_{2n} \approx \frac{a_0}{n!} \Rightarrow y(x) = e^{x^2}$$

$\therefore \lambda=0, \quad a_1=0 \to y=a_0$

$\lambda=2, \quad a_0=0 \to y=a_1 x$

$\lambda=4, \quad a_1=0 \to y=a_0\left(1-\frac{\lambda}{2}x^2\right)$

$\dots$

$$\lambda=2n=\frac{E}{E_0}-1 \longrightarrow \boxed{E=(2n+1)E_0}$$

$n=0,1,2,\dots$

E.g.: $(1-x^2)y''-2xy'+\nu(\nu+1)y=0$

Let's consider the solution around $x=0$. → Ordinary point.

$$\boxed{y=\sum_{n=0}^{\infty} a_n x^n} \to y'=\sum_{n=1}^{\infty} n a_n x^{n-1} \to y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$\underbrace{\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}}_{\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n}-\sum_{n=0}^{\infty} n(n-1)a_n x^n-2\sum_{n=0}^{\infty} n a_n x^n+\nu(\nu+1)\sum_{n=0}^{\infty} a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[a_{n+2}(n+2)(n+1)-a_n n(n-1)-2n a_n+\nu(\nu-1)a_n\right]x^n=0$$

$$a_{n+2}=\frac{n(n-1)+2n-\nu(\nu-1)}{(n+2)(n+1)}a_n=\frac{n(n+1)-\nu(\nu+1)}{(n+2)(n+1)}a_n$$

$$\begin{cases} n=2(k-1) \to a_{2k}=\dfrac{2(k-1)[2(k-1)+1]-\nu(\nu+1)}{2k(2k-1)}a_{2(k-1)} \to \nu(\nu+1)=2(k-1)(2(k-1)+1) \quad (a_1=0) \\[2ex] n=2k-1 \to a_{2k+1}=\dfrac{(2k-1)2k-\nu(\nu+1)}{2k(2k+1)}a_{2k-1} \to \nu(\nu+1)=2k(2k-1) \quad (a_0=0) \end{cases}$$

ν	$(1-x^2)y''-2xy'+\nu(\nu+1)y=0$
0	$y=a_0=a_0P_0$
1	$y=a_1x=a_1P_1$
2	$y=(1-3x^2)a_0=a_0(-2P_2)$
3	$y=(x-\frac{5}{3}x^3)a_1=-\frac{2}{3}a_1P_3$
4	$y=(1-10x^2+\frac{35}{3}x^4)a_0=a_0P_4$

Legendre polynomials

$$P_\ell(x)=\frac{1}{\ell!2^\ell}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell\ ,\ \ell\in\mathbb{Z}^+$$

E.g.: FROBENIUS METHOD

$$x(x-1)y''+3y'-2y=0$$

$$\boxed{y=\sum_{n=0}^{\infty}a_n x^{n+\lambda}}$$

$$\Downarrow$$

$$(x^3-x^2)y''+3xy'-2xy=0$$

$$x^3y''=\sum_{n=0}^{\infty}(n+\lambda)(n+\lambda-1)a_n x^{n+\lambda+1}=\sum_{n=1}^{\infty}(n+\lambda-1)(n+\lambda-2)a_{n-1}x^{n+\lambda}$$

$$x^2y''=\sum_{n=0}^{\infty}(n+\lambda)(n+\lambda-1)a_n x^{n+\lambda}$$

$$3xy'=3\sum_{n=0}^{\infty}a_n(n+\lambda)x^{n+\lambda}$$

$$2xy=2\sum_{n=0}^{\infty}a_n x^{n+\lambda+1}=2\sum_{n=1}^{\infty}a_{n-1}x^{n+\lambda}$$

$$\sum_{n=1}^{\infty}\left[(n+\lambda-1)(n+\lambda-2)-2\right]a_{n-1}x^{n+\lambda}+\sum_{n=0}^{\infty}\left[-(n+\lambda)(n+\lambda-1)+3(n+\lambda)\right]a_n x^{n+\lambda}=0$$

$n=0 \to -\lambda(\lambda-1)+3\lambda=0\to -\lambda^2+4\lambda=0\to\lambda(-\lambda+4)=0\to\lambda=\begin{cases}0\\4\end{cases}$

↳ Every term must be zero, so we analyze the lowest one.

$\lambda = 4$

$$a_n = \frac{2-(n+3)(n+2)}{3(n+4)-(n+4)(n+3)}\,a_{n-1} = \frac{2-(n+3)(n+2)}{n(n+4)}\,a_{n-1} = \frac{n+1}{n}\,a_{n-1}$$

$$\left.\begin{aligned} a_1 &= 2a_0 \\ a_2 &= 3a_0 \\ &\dots \end{aligned}\right\} \longrightarrow y_1 = \sum_{n=0}^{\infty} a_0(n+1)x^{n+4} = x^4 a_0 \sum_{n=0}^{\infty}(n+1)x^n = \underbrace{a_0}_{c_1} x^4 \frac{1}{(1-x)^2}$$

By D'Alembert's method we'll get the second solution.

* Solving diff. eqs. exactly is hard
* Sometimes only qualitative behaviour is needed (typically $t \to \pm\infty$)
* Qualitative dynamics: stability is the key concept.

* Notation for a system of equations:

$$\dot{\vec{X}} = \vec{f}(t, \vec{x}) \rightarrow$$ Sols. denoted by $\vec{x}^*(t)$

* Invariant set: $\left\{ \begin{array}{l} \text{- initial conditions given in the set.} \\ \text{- the system remains in the set throughout evolution} \end{array} \right.$

$\hookrightarrow$ TYPES $\left\{ \begin{array}{l} \text{- fixed point} \\ \text{- limit cycles} \end{array} \right.$

For a fixed point, equilibrium point or critical point,

$$\vec{X} = \vec{X}^* \cancel{(t)}$$

$$\vec{f}(t, \vec{X}^*) = 0$$

Ex. 7.1

Find the fixed points of the systems:

(1) $\dot{X} = ax \longrightarrow f(x,t) = ax$

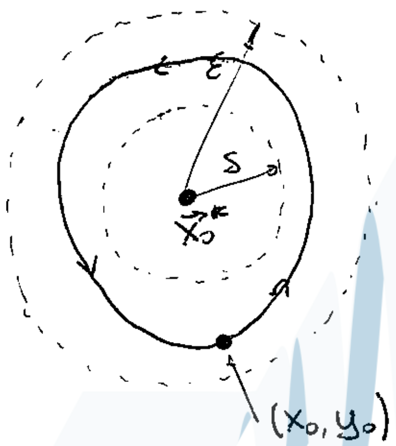
(2) $\dot{X} = ax - x^3 \longrightarrow f(x,t) = ax - x^3$

1) $ax=0 \rightarrow x=0$

2) $ax-x^3 = x(a-x^2)=0 \rightarrow \begin{cases} x=0 \\ x=\sqrt{a} \\ x=-\sqrt{a} \end{cases}$

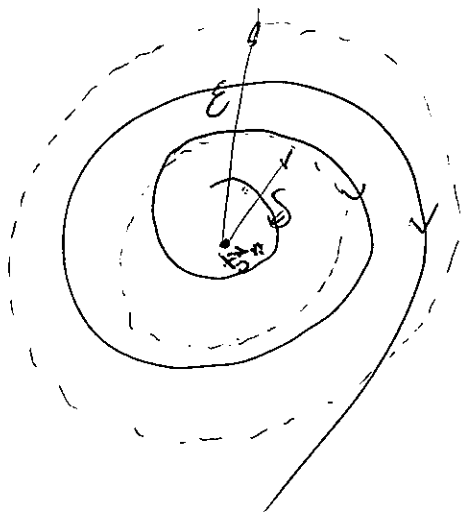
Typically, in physical situations we have errors in the initial values. So, what happens to our system if the in. cond. are changed a bit?

Stable point



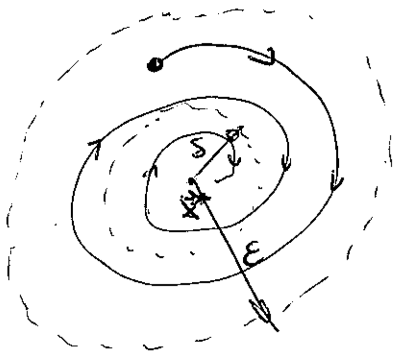
Giving initial cond. between δ and ε will guarantee I will remain there forever

Unstable point



Even if I give initial conditions inside δ (no matter how small) eventually $\varepsilon > \delta$ will be crossed.

Asymptotically stable point



Giving in. cond. between S an ε guarantees in the asymptotic future I will reach $\vec{x}^*$

* The properties of an equilibrium point can be changed when a parameter changes; that effect is called bifurcation

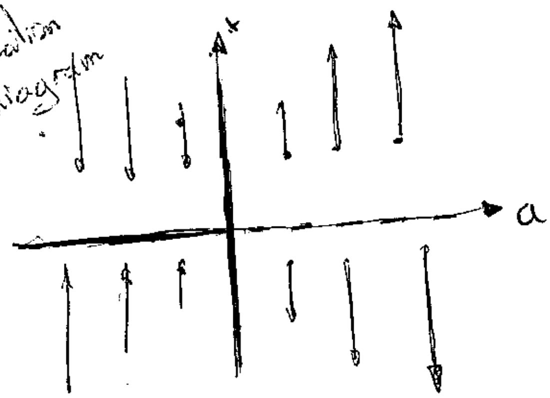
↳ Bifurcation diagrams allow to study the effect.

E.g.: $\dot{x} = ax \rightarrow x = Ce^{at} \Rightarrow t(0) = C,\ t(\infty) = \begin{cases} a>0 \rightarrow \infty \\ a<0 \rightarrow 0 \end{cases}$

↳ Should not really solve it
just for pedagogical purposes.

But, if we don't know how to solve it...

Bifurcation diagram



— stability

- - - instability

Fixed point location:

$ax = 0 \rightarrow \begin{cases} a = 0 \quad (\forall x) \\ x = 0 \end{cases}$

<u>a>0</u>

$x>0 \rightarrow \dot{x}>0$

$x<0 \rightarrow \dot{x}<0$

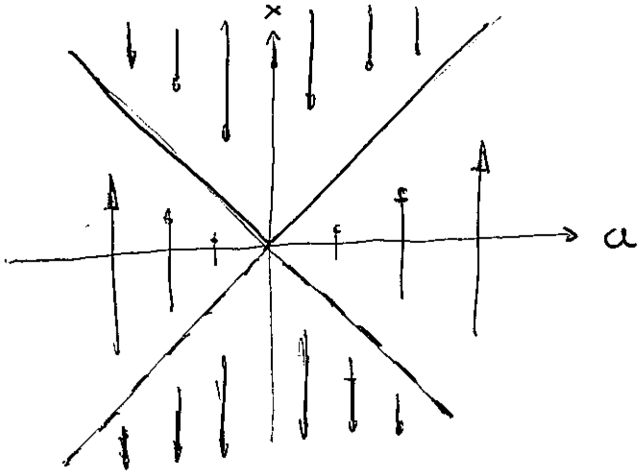
<u>a<0</u>

$x>0 \rightarrow \dot{x}<0$

$x<0 \rightarrow \dot{x}>0$

E.g.: $\dot{x} = a^2 - x^2$ fixed points $\to$ $a - x^2 = 0$

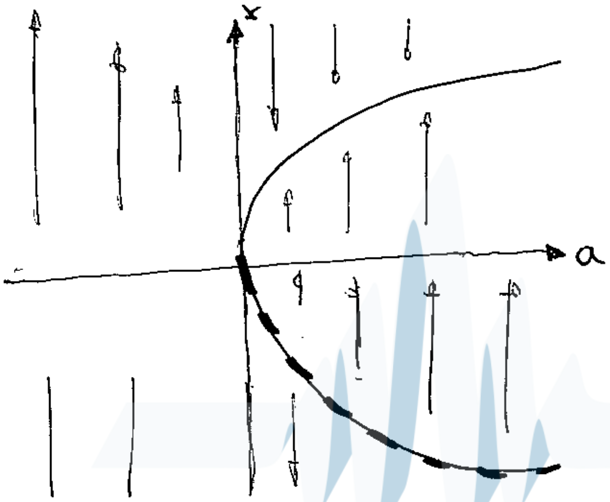
$\hookrightarrow x = \pm a$



x very big (negative or positive)

$\hookrightarrow$ $\underline{\dot{x} < 0}$

E.g.: $\dot{x} = ax - x^3 \longrightarrow$ fixed points $\to x(a - x^2) = 0 \begin{cases} x = 0 \\ x = \pm\sqrt{a} \to a^2 = x \end{cases}$



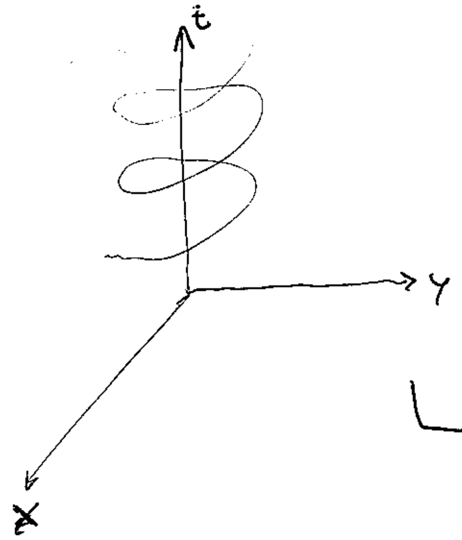
7.2. TWO-DIMENSIONAL AUTONOMOUS SYSTEMS

They look like this!

$$\begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y) \end{cases}$$ (indep. var. is absent)

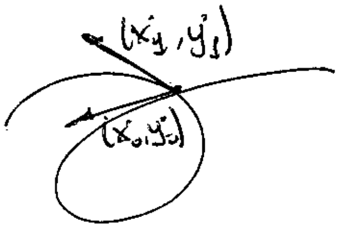
$\hookrightarrow$ $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ will be the sol.

$\hookrightarrow$ This well be a congruence in the space (t, x, y).



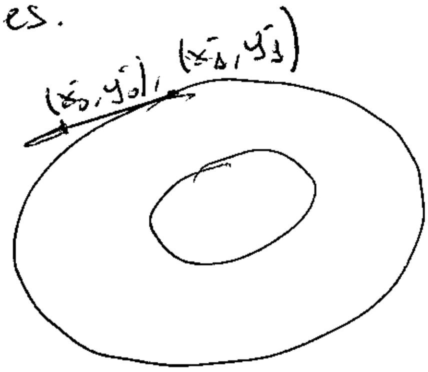
Autonomy enforces that the projection onto the plane xy is a congruence too.

↳ As we move up (in time) we can go back to the same (x, y) values.



NOT POSSIBLE

↳ Not a congruency



CONGRUENCY

The space (x, y) is called the phase space.

Such systems would be solved in principle integrating

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} \rightarrow \ldots$$

but we will rather do a qualitative analysis.

(Prior to full theoretical considerations ...)

UNIDIMENSIONAL MECHANICAL SYSTEMS

Newton's law $\rightarrow$ $\ddot{x} = f(x, \dot{x})$ → 2nd order ODE

↳ mass has been absorbed

↓

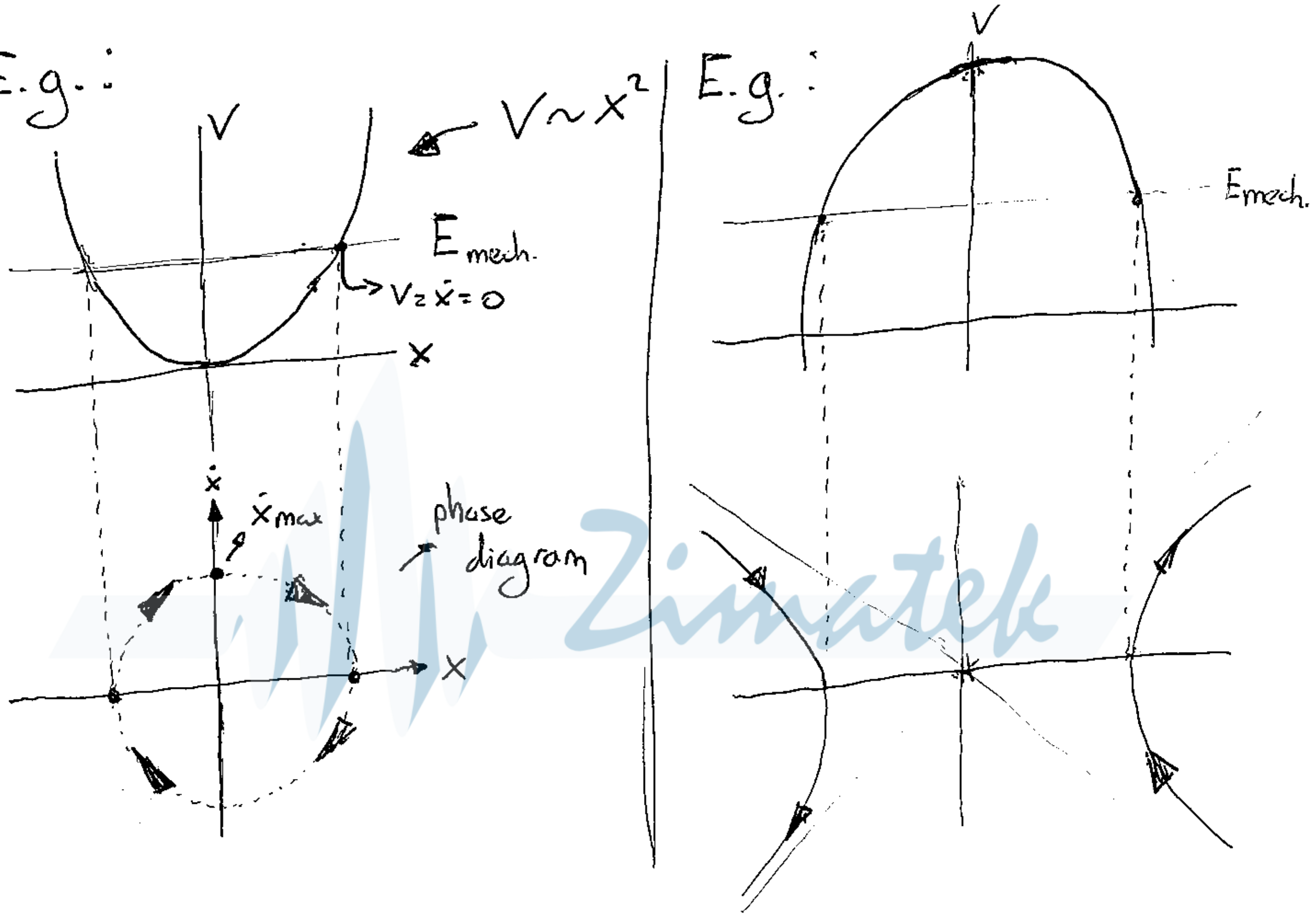
Transform to a 1st order system

$$\rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = \ddot{x} = f(x,\dot{x}) = f(x,y) \end{cases} \longrightarrow \begin{cases} \dot{x} = y \\ \dot{y} = f(x,y) \end{cases}$$

If "f" is the force, then for a conservative system:

$$\vec{f} \equiv -\vec{\nabla} V \rightarrow f = \frac{-\partial V}{\partial x} = -V' \rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = -V' \end{cases}$$

E.g.: E.g.:



7.3 QUASILINEAR SYSTEMS

- Hypothesis: the equilibrium point is at the origin

$$\begin{aligned} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \end{aligned} \longrightarrow \begin{aligned} P(0,0) &= 0 \\ Q(0,0) &= 0 \end{aligned}$$

- DEF.: Quasilinear means a Taylor expansion around the fixed point is accurate.

$$\dot{x} \approx a_{11}x + a_{12}y$$
$$\dot{y} \approx a_{21}x + a_{22}y$$

Another check: $\lim\limits_{\sqrt{x^2+y^2}\to 0} \dfrac{P-(a_{11}x+a_{12}y)}{\sqrt{x^2+y^2}} = 0$

$\lim\limits_{\sqrt{x^2+y^2}\to 0} \dfrac{Q-(a_{21}x+a_{22}y)}{\sqrt{x^2+y^2}} = 0$

} Non-linear terms are weaker

$$\mathbb{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix}_{(x,y)=(0,0)}$$

- We will only study cases with isolated fixed points

↳ $\boxed{\det \mathbb{A} \neq 0}$

- The characteristic roots of $\mathbb{A}$ will be

$$\boxed{K_{1,2} = \frac{1}{2}\left(\mathrm{tr}\,\mathbb{A} \pm \sqrt{\Delta}\right)}$$

↳ trace of $\mathbb{A}$ → Sum of the diagonal

$$\boxed{\Delta = \mathrm{tr}^2\,\mathbb{A} - 4\det \mathbb{A}}$$

LIAPUNOV'S METHOD → The behaviour of the quasilinear system and the full system is the same near the point.

CLASSIFICATION OF FIXED POINTS

- Different, real roots
  - Both negative
  - " positive
  - One neg, the other pos.

- Complex roots
  - Negative real part
  - Positive real part
  - Purely imaginary (real part null)

- Same, real roots
  - $a_{12} = a_{21} = 0$
  - $|a_{12}| + |a_{21}| \neq 0$

Different real roots : $\Delta > 0 \Rightarrow k_1 > k_2$

$\dot{\vec{X}} = \mathbb{A}\vec{X} \rightarrow$ eigenvectors : $\vec{X}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\vec{X}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

k_1 k_2 eigenvalues

gen. sol. $\Rightarrow \vec{X} = C_1 \vec{X}_1 + C_2 \vec{X}_2$

$$\boxed{\begin{aligned} x &= C_1 x_1 e^{k_1 t} + C_2 x_2 e^{k_2 t} \\ y &= C_1 y_1 e^{k_1 t} + C_2 y_2 e^{k_2 t} \end{aligned}}$$

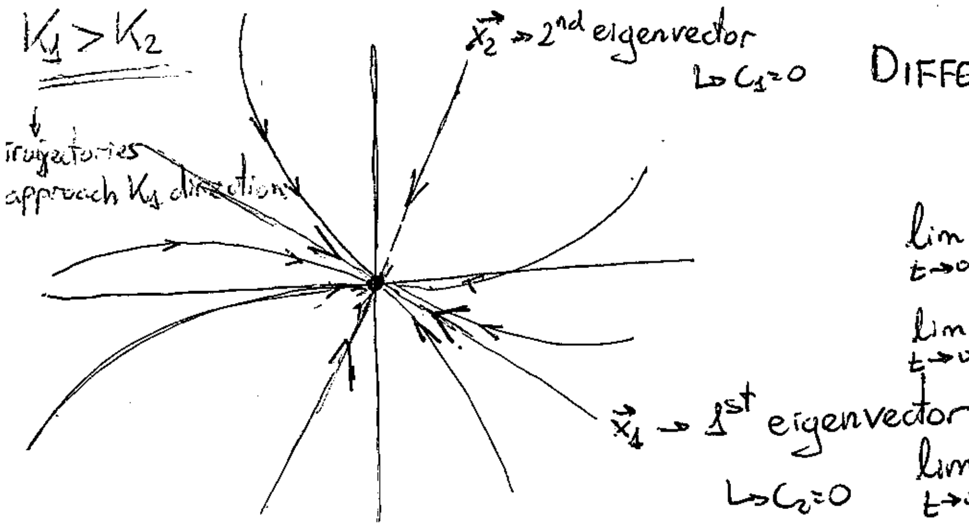
$\rightarrow$ Elements to do a sketch? a cualitative plot?

Slope of the sols : $\dfrac{y}{x} = \dfrac{C_1 y_1 e^{k_1 t} + C_2 y_2 e^{k_2 t}}{C_1 x_1 e^{k_1 t} + C_2 x_2 e^{k_2 t}}$ $\rightarrow$ General for all real, different roots.

$\hookrightarrow$ Two part. sols. $\rightarrow$ $\begin{cases} C_2 = 0 \rightarrow \dfrac{y}{x} = \dfrac{y_1}{x_1} \\ C_1 = 0 \rightarrow \dfrac{y}{x} = \dfrac{y_2}{x_2} \end{cases}$ $\rightarrow$ There will always be 2 part. sols. parallel to the eigenvectors

$K_1 > K_2$

↓ trajectories approach K_1 direction



ASYMPTOTICALLY STABLE NODE

DIFFERENT, REAL, NEGATIVE

$K_1, K_2 < 0$

$$\lim_{t\to\infty} x = 0$$

$$\lim_{t\to\infty} y = 0$$

$$\lim_{t\to\infty} \frac{y}{x} \underset{K_2<K_1<0}{=\!=\!=} y_1/x_1$$ → except for $C_1 = 0$

↓

$$\lim_{t\to\infty} \frac{y}{x} = y_2/x_2$$

DIFFERENT, REAL, POSITIVE : $K_1, K_2 > 0$

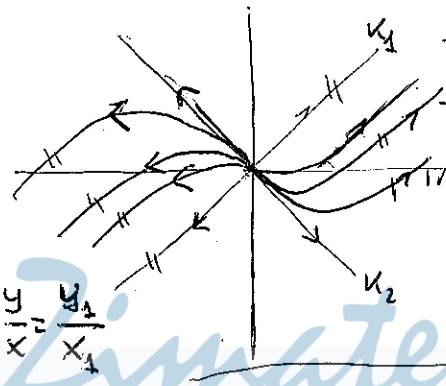
$$\lim_{t\to\infty} |x| = \infty$$

$$\lim_{t\to\infty} |y| = \infty$$

Trajectories escape from the origin

$K_1 > K_2$

↓ Trajectories escape in the K_1 dir.

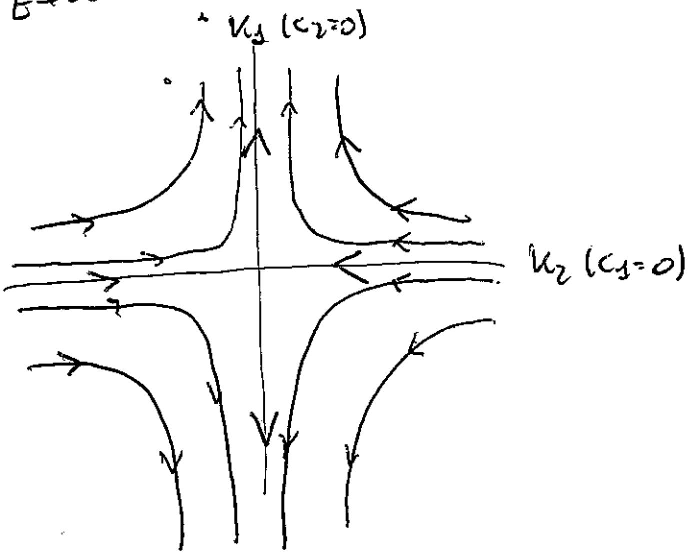


$$\lim_{t\to\infty} \frac{y}{x} \underset{K_1>K_2>0}{=\!=\!=} \frac{y_2}{x_2}$$ → except for $C_2 = 0$ → $\lim_{t\to\infty} \frac{y}{x} = \frac{y_1}{x_1}$

UNSTABLE NODE

DIFFERENT, REAL, $-/+$

$$\lim_{t\to\infty} (x,y) = (0,0) \text{ if } C_1 = 0, \quad \lim_{t\to\infty} (|x|,|y|) = (\infty,\infty) \text{ if } C_2 = 0.$$



$\overset{(+)}{K_1} > \overset{(-)}{K_2}$

positive means repulsion
negative means attraction

SADDLE POINT

COMPLEX CHARACTERISTIC ROOTS

$\Delta < 0 \Rightarrow K_+ = \alpha + i\omega, \; K_- = \alpha - i\omega$ $\begin{cases} + \text{ real part} \\ - \text{ " } \\ 0 \text{ " } \end{cases}$

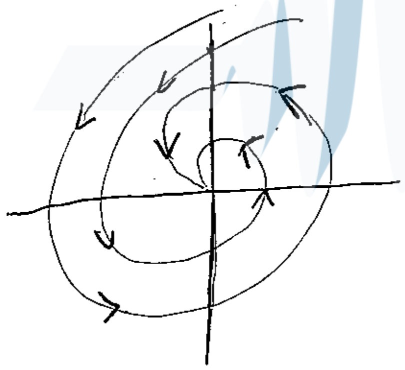
$$x = e^{\alpha t}(c_1 x_1 \cos \omega t + c_2 x_2 \sin \omega t)$$
$$y = e^{\alpha t}(c_1 y_1 \cos \omega t + c_2 y_2 \sin \omega t)$$

Either for $\alpha \neq 0$ or $\alpha = 0$, the slope rotates!

$$\frac{y}{x} = \frac{c_1 y_1 \cos \omega t + c_2 y_2 \sin \omega t}{c_1 x_1 \cos \omega t + c_2 x_2 \sin \omega t}$$

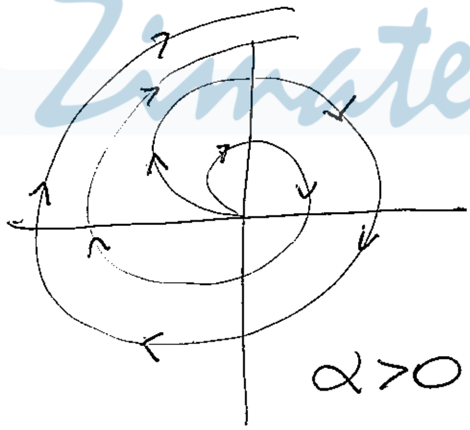
$\mathrm{tr}\,\mathbb{A} < 0 \Rightarrow \alpha < 0 \Rightarrow \lim_{t\to\infty} (x,y) = (0,0) \Rightarrow$ trajectories approach the origin

$\mathrm{tr}\,\mathbb{A} > 0 \Rightarrow \alpha > 0 \Rightarrow \lim_{t\to-\infty} (x,y) = (0,0) \Rightarrow$ trajectories escape from the origin



$\alpha < 0$

STABLE FOCUS
" SPIRAL POINT



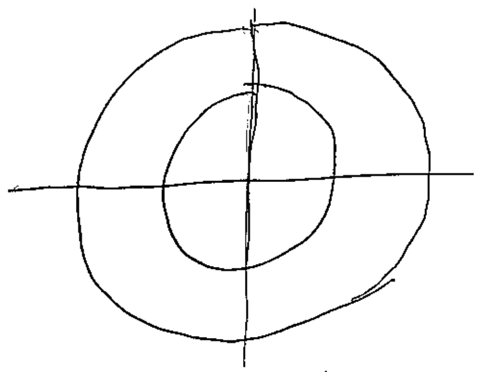
$\alpha > 0$

UNSTABLE FOCUS
" SPIRAL POINT

$\mathrm{tr}\,\mathbb{A} = 0 \to \alpha = 0$

x & y are periodic $\to$

$\to$ trajectories are circles/ellipses



CENTER

Non-linear terms can make a circle become a focus.

E.g.:

$$\dot{x} = -y$$
$$\dot{y} = x - ay^n \quad (n=2,3)$$

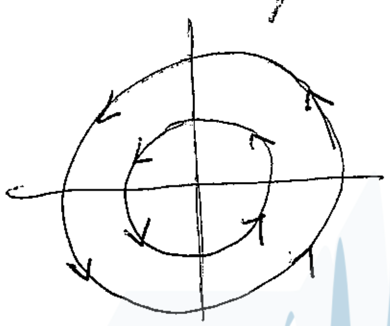
$\longrightarrow$ LINEAR APPROX. : $\begin{cases} \dot{x} \sim -y \\ \dot{y} \sim x \end{cases} \rightarrow A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

FIXED POINT: (0, 0)

y^2 or y^3 are non-linear so we drop them

Look for eigenvalues $\rightarrow \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \rightarrow \lambda = \pm i$

$\hookrightarrow$ $\boxed{\text{CIRCLE}}$



Let us do a symmetries analysis with the original system:

$\boxed{n=2}$

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x - ay^2 \end{cases} \Rightarrow \begin{cases} -\dot{x} = -y \\ -\dot{y} = x - ay^2 \end{cases} \xrightarrow{y \to -y} \begin{cases} \dot{x} = -y \\ \dot{y} = x - ay^2 \end{cases}$$

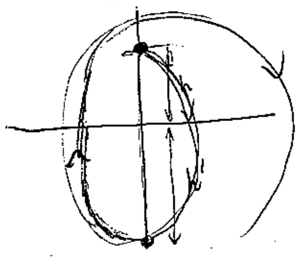
$t \to t' \begin{cases} \dot{x} \to -\dot{x} \\ \dot{y} \to -\dot{y} \end{cases}$

This is not our original system!

So, circle in the linear approx. is still a circle if I have a y^2 term

$\boxed{n=3}$

$\begin{cases} \dot{x} = -y \\ \dot{y} = x - ay^3 \end{cases} \rightarrow$ spoils the $y \to -y$ symetry

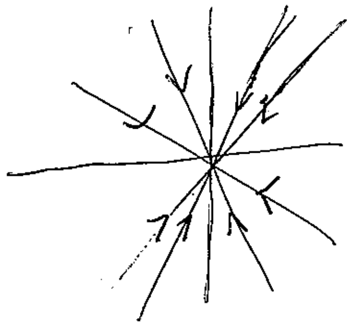


SAME CHARACTERISTIC REAL ROOTS

$\Delta = 0 \rightarrow K_1 = K_2$

1st case:

$a_{21} = a_{12} = 0$
$\hookrightarrow a_{11} = a_{22} = a$ $\rightarrow \begin{cases} x = C_1 e^{at} \\ y = C_2 e^{at} \end{cases}$ $\rightarrow$ All slopes are different



$K_1 = K_2 < 0$

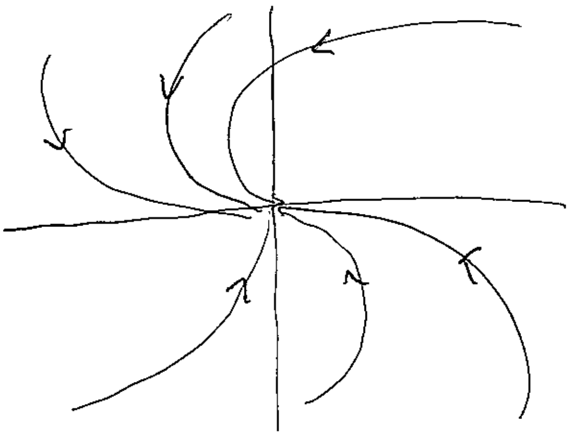
PROPER NODE OR STAR NODE

2nd case:

$|a_{12}| + |a_{21}| \neq 0$

$x = (C_1 x_1 + C_2(x_1 t + x_2))e^{kt}$
$y = (C_1 y_1 + C_2(y_1 t + y_2))e^{kt}$ $\rightarrow$ All the slopes have the same asyntote

$$\lim_{t \to \infty} \frac{y}{x} = \frac{y_1}{x_1}$$



DEGENERATE NODE

E.g.: JANUARY 2005

$$\dot{x} = x^2y - y + 1 - x^2$$

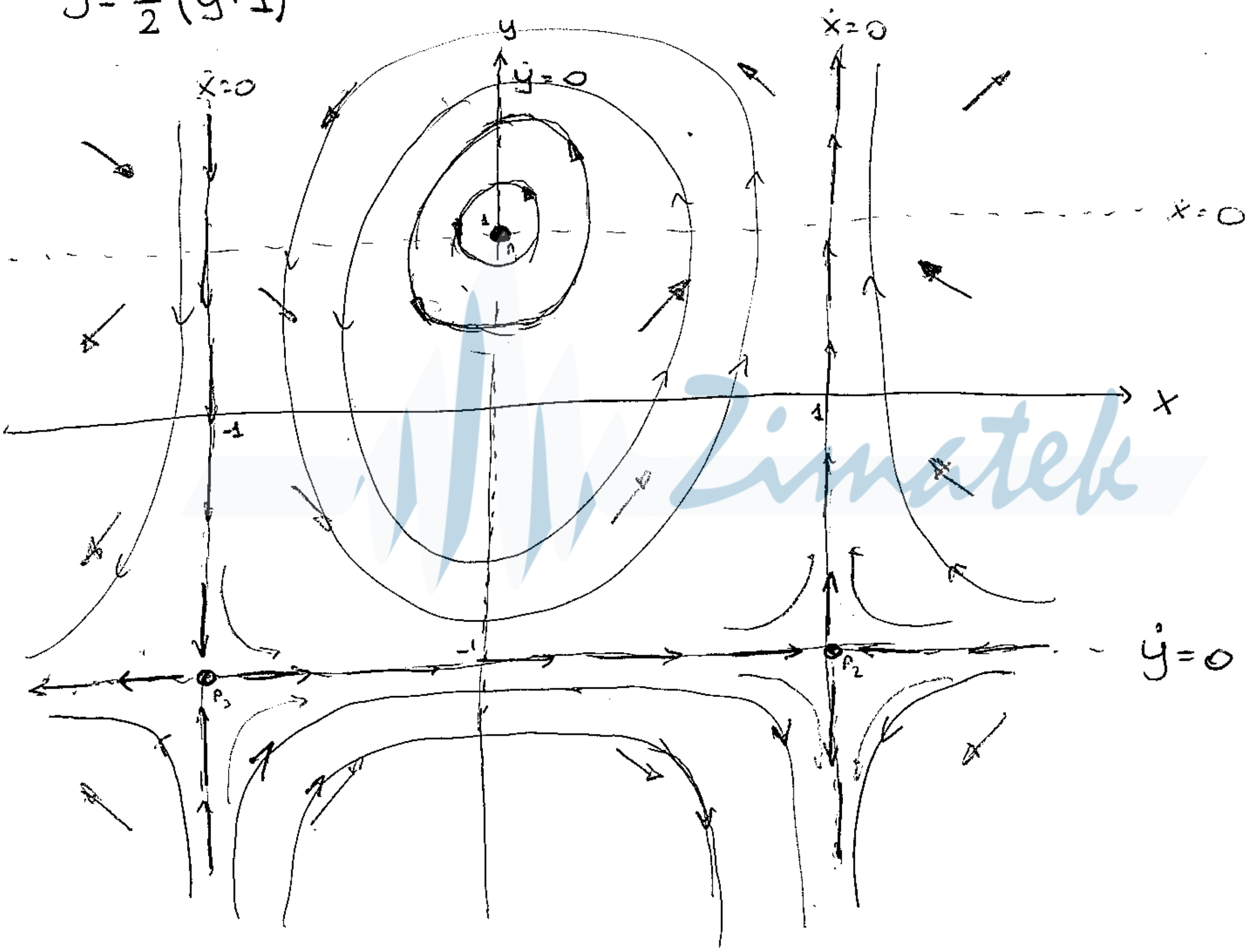
$$\dot{y} = \frac{1}{2}(xy + x)$$

$$\dot{x} = y(x^2-1) + (1-x^2) = (y-1)(x^2-1)$$

$$\dot{y} = \frac{x}{2}(y+1)$$

- FACTORIZE IF YOU CAN
- IDENTIFY THE $\dot{x}=0$ AND $\dot{y}=0$ CURVES $\Rightarrow$ fixed points are at their crossings
- FIND PARTICULAR SOLUTIONS
- LOOK FOR SYMMETRIES

● → FIXED POINT



$\dot{x}=0$	$\dot{y}=0$
$y=1$	$\rightarrow \frac{x}{2}(1+1) \rightarrow \boxed{x=0}$
$x=1$	$\rightarrow \frac{1}{2}(y+1) \rightarrow \boxed{y=-1}$
$x=-1$	$\rightarrow -\frac{1}{2}(y+1) \rightarrow \boxed{y=-1}$

FIXED POINTS $\longrightarrow$
$$\begin{cases} P_1 = \{x=0, y=1\} \\ P_2 = \{x=1, y=-1\} \\ P_3 = \{x=-1, y=-1\} \end{cases}$$

$$\mathbb{A} = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix} = \begin{pmatrix} 2xy-2x & x^2-1 \\ \frac{1}{2}(y+1) & \frac{1}{2}x \end{pmatrix}$$

$\dot{x} = P$
$\dot{y} = Q$

$$\mathbb{A}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (x,y)=(0,1)$$

$$\mathbb{A}_2 = \begin{pmatrix} -4 & 0 \\ 0 & 1/2 \end{pmatrix} \quad (x,y)=(1,-1)$$

$\lambda_1 = -4 \rightarrow V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ stable

$\lambda_2 = 1/2 \rightarrow V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ unstable

SADDLE POINT

$$\mathbb{A}_3 = \begin{pmatrix} 4 & 0 \\ 0 & -1/2 \end{pmatrix} \quad (x,y)=(-1,-1)$$

$\lambda_1 = 4 \rightarrow V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ unstable

$\lambda_2 = -1/2 \rightarrow V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ stable

SADDLE POINT

$$\mathbb{A}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0 \rightarrow \lambda^2+1=0 \rightarrow \lambda = \pm i$$

CENTER

In order to know if there's a circle around P_1, we have to look for symmetries.

Let's have a look at the eqs:

This suggests to check $x \to -x$

$$\begin{cases} \dot{x} = (x^2-1)(y-1) \\ \dot{y} = \frac{1}{2}x(y+1) \end{cases}$$

$x \rightarrow -x$ | $t \rightarrow -t$

$\dot{x} \rightarrow -\dot{x}$ | $\dot{x} \rightarrow -\dot{x}$

$\dot{x} \rightarrow \dot{x}$

$$\hookrightarrow -\dot{y} = \frac{1}{2}(-x)(y+1) \rightarrow \dot{y} = \frac{1}{2}x(y+1)$$

It's a circle, because of the symmetries.

But the symmetric across $y=1$, and so instead of circles...we have ellipses

Particular solutions

- 1^{st} eq. is automatically satisfied at the point that fall in that sol./curve

If $y \rightarrow y_0^{p.ct.}$ $\Rightarrow$ $0 = \frac{1}{2}(xy_0 + x) \rightarrow \boxed{y_0 = -1}$ Part. sol.

$$\boxed{\dot{x} = x^2(-1) - (-1) + 1 - x^2}$$

I have managed to get one eq. that depends on x

We'll do the same for x:

$x \longrightarrow x_0 \longrightarrow \dot{x}_0 = 0$

$0 = x_0^2 y - y + 1 - \dot{x}_0^2 \longrightarrow \boxed{x_0 = \pm 1}$ are part. sols.

$\dot{y} = \pm \frac{1}{2} (y \pm 1)$

STURM-LIOUVILLE THEORY

$[a,b] \subseteq \mathbb{R}$

$f : [a,b] \longrightarrow \mathbb{C}$

f is a C^n class function

↳ n times differentiable

$\left\{ \forall f_1, f_2 \in C^n([a,b]) : \lambda f_1 + \mu f_2 \in C^n([a,b]) \right.$ $\forall \lambda, \mu \in \mathbb{C}$

<u>DEF.</u>: Given $f, g \in C^n([a,b])$, the scalar product relative to the weight $\rho(x)$ is defined as follows:

↳ $\rho(x) : [a,b] \to \mathbb{R}$, real, cont. and $\rho > 0$ on $[a,b]$

$$\langle f | g \rangle \equiv \int_a^b \bar{f} g \rho \, dx$$

↳ complex conjugate

Properties:

1) $\langle f | g \rangle = \overline{\langle g | f \rangle}$ The scalar prod. of function is Hermitian

2) $\langle f_1 + f_2 | g \rangle = \langle f_1 | g \rangle + \langle f_2 | g \rangle$

3) $\langle f | \lambda g \rangle = \lambda \langle f | g \rangle$

$\langle \lambda f | g \rangle = \bar{\lambda} \langle f | g \rangle$

$\langle f | g_1 + g_2 \rangle = \langle f | g_1 \rangle + \langle f | g_2 \rangle$

4) $\langle f | f \rangle \equiv \| f^2 \| \geqslant 0$ ($=0$ only if $f = 0$ almost everywhere)

$f \perp g \Longleftrightarrow \langle f | g \rangle = 0$

↑ orthogonal

$\langle f | g \rangle = 0 \ \forall g \in C^n([a,b]) \Rightarrow f = 0$

$f_n \to f$ if $\| f_n - f \| \to 0$ $n \to \infty$

Convergence in mean square

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = f$$

$a_0(x),\ a_1(x),\ a_2(x)$

$$L = a_0 \frac{d^2}{dx^2} + a_1 \frac{d}{dx} + a_2$$

operator

$L^{\dagger} \equiv$ formal adjoint $\Rightarrow \langle L^{\dagger} f | g \rangle = \langle f | L g \rangle$

self-adjoint operator $\Rightarrow L^{\dagger} = L$
(Hermitian)

Whether or not L is self-adjoint depends on the BOUNDARY CONDS.

Now, we ask ourselves...

$$Ly \overset{?}{=} \frac{d}{dx}\left(P \frac{dy}{dx}\right) + Qy \quad \left(= Py'' + P'y' + Qy\right)$$

$$\alpha \frac{d}{dx}\left(P \frac{dy}{dx}\right) + \alpha Q y = Ly$$

$$\underset{a_0}{\alpha P} y'' + \underset{a_1}{\alpha P'} y' + \alpha Q y = Ly \longrightarrow P = \exp \int \frac{a_1}{a_0} dx$$

$$\alpha = \frac{a_0}{P} = 1/\rho$$

Summarising...

$$a_0 y'' + a_1 y' + a_2 y = Ly \longrightarrow \rho = \frac{1}{a_0} \exp\left(\int \frac{a_1}{a_0} dx\right)$$

$a_0 = P/\rho$

$a_1 = P'/\rho$

$$Ly = \frac{1}{\rho}\left[(Py')' + Qy\right]$$

Is "L" self-adjoint, then? $\Rightarrow \langle Lf | g \rangle - \langle f | Lg \rangle \overset{?}{=} 0$ *

$$\int_a^b \overline{Lf}\, g \rho\, dx - \int_a^b \bar{f} (Lg) \rho\, dx = \int_a^b \left(\frac{1}{\rho} \frac{d}{dx}\left(P \frac{d\bar{f}}{dx}\right) + \frac{Q}{\cancel{\rho}} \bar{f}\right) g \rho\, dx -$$

$$- \int_a^b \bar{f} \left(\frac{1}{\rho} \frac{d}{dx}\left(P \frac{dg}{dx}\right) + \frac{Q}{\cancel{\rho}} g\right) \rho\, dx = \int_a^b \left(\frac{d}{dx}\left(P \frac{d\bar{f}}{dx}\right) g - \bar{f} \frac{d}{dx}\left(P \frac{dg}{dx}\right)\right) dx =$$

$$= \int_a^b \left[\frac{d}{dx}\left(p \frac{d\bar{f}}{dx} g\right) - \cancel{p \frac{d\bar{f}}{dx}\frac{dg}{dx}} - \frac{d}{dx}\left(p\frac{dg}{dx}\bar{f}\right) + \cancel{p\frac{dg}{dx}\frac{d\bar{f}}{dx}} \right] dx =$$

$$= \int_a^b \frac{d}{dx}\left(p\frac{d\bar{f}}{dx} g - p\frac{dg}{dx}\bar{f}\right) dx = \boxed{\left[p \underbrace{\left(g\frac{d\bar{f}}{dx} - \bar{f}\frac{dg}{dx}\right)}_{W[g,\bar{f}]} \right]_a^b \overset{*}{=} 0}$$

L is self-adjoint if the boundary conditions are such that this holds

BOUNDARY CONDITIONS

$a, b \rightarrow y(a), y(b), y'(a), y'(b)$ (for 2^{nd} order ODEs)

Boundary conds. must be linear so sols. are linear as well:

$$\begin{cases} A_1 y(a) + B_1 y(b) + C_1 y'(a) + D_1 y'(b) = E_1 \\ A_2 y(a) + B_2 y(b) + C_2 y'(a) + D_2 y'(b) = E_2 \end{cases}$$

> Non-homogeneous boundary conditions (B.C.) if $E_1^2 + E_2^2 \neq 0$

Not good!! → Must be avoided

$A_1, A_2, B_1, \dots, E_2$ are FIXED numbers.

Non-hom. B.C. are ill-defined, as we'll lose linearity.

Example:

$y(0) \overset{*}{=} 1$

$y'(1) = 0$ $\rightarrow y_1, y_2$

$$(y_1 + y_2)(0) = y_1(0) + y_2(0) = 1 + 1 = 2 \overset{*}{\neq} 1 \,!!$$

Thus, one must always try to redefine the problem to achieve homogeneous B.C.. This might destroy "nice" properties of the ODE, but that's FINE!!

Example:

$y'' + \lambda y = 0$ $\quad$ $z = y - 1$ $\quad \longrightarrow \quad z'' + \lambda z = -\lambda$

$y(0) = 1$ $\quad$ $z(0) = y(0) - 1 = 0$ ✓

$y'(1) = 0$ $\quad$ $z'(1) = y'(1) = 0$

Types of B.C.:

1) REGULAR (B.C. of separated type)

$$W(a)=0 \iff A_1 y(a) + C_1 y'(a) = 0$$

$$W(b)=0 \iff A_2 y(b) + C_2 y'(b) = 0$$

2) PSEUDO-PERIODIC

$$\begin{pmatrix} y(a) \\ y'(a) \end{pmatrix} = \underset{\substack{\uparrow \\ \text{constant matrix}}}{\mathbb{A}} \begin{pmatrix} y(b) \\ y'(b) \end{pmatrix} \Rightarrow P(b)\, W[g,\bar{g}](b) - P(a)\, W[g,\bar{g}](a) = (\dots) =$$

$$= \left(P(a) \det \mathbb{A} - P(b)\right) W[g,\bar{g}](b) = 0$$

$$\underline{P(a) \det \mathbb{A} = P(b)}$$

PERIODIC if $\mathbb{A} = \mathbb{I}$ and $P(a) = P(b)$

$$\hookrightarrow \begin{cases} y(a) = y(b) \\ y'(a) = y'(b) \end{cases}$$

3) SINGULAR

i) $P(a) = 0$, $W[g,\bar{g}](a) = 0$ AND $y(a), y'(a)$ are bounded

ii) $P(b) = 0$, $W[g,\bar{g}](b) = 0$, $y(b), y'(b)$ are bounded.

iii) $P(a) = P(b) = 0$, y, y' are bounded on a & b.

If $a = -\infty$ or $b = \infty$ or both, the problem is singular.

In a STURM-LIOUVILLE problem, $\underset{\oplus \text{B.C.}}{Ly + \lambda y = 0}$, L self-adjoint, the values of λ which allow for solutions are called EIGENVALUES. The corresponding solutions are the EIGENFUNCTIONS.

Example:

$$y'' + \lambda y = 0 \ , \ y(0)=0, \ y(\pi)=0$$

$\hookrightarrow L = \frac{d^2}{dx^2} \rightarrow \rho = 1$

$a_0 y'' + a_1 y' + a_2$

$a_0 = 1, a_1 = 0, a_2 = 0$

$\lambda = 0 \rightarrow$ Gen. sol.: $y = Ax + B$ ⟨ $y(0) = B = 0$; $y(\pi) = A\pi = 0 \rightarrow A = 0$ $\rightarrow y = 0$

$\hookrightarrow$ Trivial sol.

$\downarrow$

$\lambda = 0$ not an eigenvalue

$\lambda < 0 \rightarrow$ No solutions

$\lambda > 0 \rightarrow$ Gen. sol.: $y = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$

$\hookrightarrow y(0) = 0 = A$

$\hookrightarrow y(\pi) = 0 = B\sin(\sqrt{\lambda}\pi)$

Secular equation

$$\sin(\sqrt{\lambda}\,\pi) = 0 \longrightarrow \sqrt{\lambda_n}\,\pi = n\pi \rightarrow \boxed{\lambda_n = n^2} \ , \ n = 1, 2, 3, \dots$$

eigenvalues

For each λ_n, the solution is $y_n = \sin(nx)$. $\rightarrow$ We chose $B=1$

$$\langle \sin(nx) | \sin(mx) \rangle = \int_0^{\pi} \sin(nx)\sin(mx)\, dx = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases}$$

$$\| \quad \delta_{nm} \frac{\pi}{2}$$

$\hookrightarrow \delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$

Eigenfunctions for different eigenvectors are orthogonal.

$\|\sin(nx)\|^2 = \frac{\pi}{2} \rightarrow \|\sin(nx)\| = \sqrt{\frac{\pi}{2}}$

Orthonormal set of eigenfunctions: $\boxed{u_n = \sqrt{\frac{2}{\pi}}\sin(nx)}$

For a self-adjoint L,

i) All eigenvalues are real.

ii) Eigenfunctions corresponding to different eigenvalues are orthogonal.

<u>Proof</u>: Let λ, μ be two different eigenvalues : $\begin{cases} Ly_\lambda + \lambda y_\lambda = 0 \\ Ly_\mu + \mu y_\mu = 0 \end{cases}$ ⊕ B. C.

$$\begin{cases} \frac{d}{dx}\left(P\frac{dy_\lambda}{dx}\right) + Qy_\lambda + \lambda\rho y_\lambda = 0 & (\cdot\, \bar{y}_\mu) \\ \frac{d}{dx}\left(P\frac{dy_\mu}{dx}\right) + Qy_\mu + \mu\rho y_\mu = 0 & (\cdot\, y_\lambda) \end{cases}$$

$$\rightarrow \bar{y}_\mu \frac{d}{dx}\left(P\frac{dy_\lambda}{dx}\right) + \cancel{Qy_\lambda\bar{y}_\mu} + \lambda\rho y_\lambda\bar{y}_\mu -$$

$$-\left(y_\lambda \frac{d}{dx}\left(P\frac{d\bar{y}_\mu}{dx}\right) + \cancel{Qy_\lambda\bar{y}_\mu} + \bar{\mu}\rho y_\lambda\bar{y}_\mu\right) = 0$$

$$\int_a^b \left\{ \bar{y}_\mu \frac{d}{dx}\left(P\frac{dy_\lambda}{dx}\right) + \lambda\rho y_\lambda\bar{y}_\mu - y_\lambda\frac{d}{dx}\left(P\frac{d\bar{y}_\mu}{dx}\right) - \bar{\mu}\rho y_\lambda\bar{y}_\mu \right\} dx =$$

$$= \int_a^b \left\{ \frac{d}{dx}\left(P\bar{y}_\mu\frac{dy_\lambda}{dx} - Py_\lambda\frac{d\bar{y}_\mu}{dx}\right) + (\lambda-\bar{\mu})\rho y_\lambda\bar{y}_\mu \right\} dx =$$

$$= \underset{\substack{\| \\ 0}}{\cancel{\left[P\,W[\bar{y}_\mu, y_\lambda]\right]_a^b}} + (\lambda-\bar{\mu})\int_a^b \bar{y}_\mu y_\lambda \rho\, dx = 0$$

because L is self-adjoint

$$\boxed{(\lambda-\bar{\mu})\langle y_\mu | y_\lambda\rangle = 0}$$

- If $\lambda=\mu \rightarrow (\lambda-\bar{\lambda})\underset{\neq 0}{\|y_\lambda\|^2}=0 \rightarrow \lambda=\bar{\lambda} \rightarrow \lambda$ is REAL

- If $\lambda\neq\mu \rightarrow \underset{\neq 0}{(\lambda-\bar{\mu})}\langle y_\lambda|y_\mu\rangle=0 \rightarrow \langle y_\lambda|y_\mu\rangle=0$

$\Downarrow$ ORTHOGONALITY

Example:

$$y''+\lambda y=0, \quad y(0)=y(\ell), \quad y'(0)=y'(\ell)$$

↳ Periodic B.C.

A & B might actually take complex values.
↳ "i"s can be absorbed into them.

$\lambda<0 \rightarrow$ No solution

$\lambda=0 \rightarrow y=Ax+B \quad \begin{cases} y(0)=B=y(\ell)=A\ell+B \rightarrow A=0 \\ y'=A=0 \rightarrow y'(0)=0=y'(\ell) \text{ OK} \end{cases}$

$\lambda=0$ is an eigenvalue whose eigenfunction is $y_0=1$.

$\lambda>0 \rightarrow y=A\cos(\sqrt{\lambda}x)+B\sin(\sqrt{\lambda}x)$

$\rightarrow y(0)=A=y(\ell)=A\cos(\sqrt{\lambda}\ell)+B\sin(\sqrt{\lambda}\ell)$

$\rightarrow y'(0)=\cancel{\sqrt{\lambda}}B=y'(\ell)=\cancel{\sqrt{\lambda}}\left(-A\sin(\sqrt{\lambda}\ell)+B\cos(\sqrt{\lambda}\ell)\right)$

$\lambda\neq 0$

We get two eqs:

$$(*)\begin{cases} A\left(1-\cos(\sqrt{\lambda}\ell)\right)-B\sin(\sqrt{\lambda}\ell)=0 \\ A\sin(\sqrt{\lambda}\ell)+B\left(1-\cos\sqrt{\lambda}\ell\right)=0 \end{cases}$$

In order to have solutions with non-vanishing A, B:

$$\begin{vmatrix} 1-\cos(\sqrt{\lambda}\ell) & -\sin(\sqrt{\lambda}\ell) \\ \sin(\sqrt{\lambda}\ell) & 1-\cos(\sqrt{\lambda}\ell) \end{vmatrix}=0$$

With the determinant, we get our secular equation:

$$\left(1-\cos(\sqrt{\lambda}\ell)\right)^2+\sin^2(\sqrt{\lambda}\ell)=0$$

$$2\left[1-\cos(\sqrt{\lambda}\ell)\right]^2=0$$

$$\cos(\sqrt{\lambda}\,\ell) = 1$$

$$\sqrt{\lambda_n}\,\ell = 2\pi n \quad , \quad n = 1, 2, 3, \ldots$$

So then, our eigenvalues will be:

$$\lambda_0 = 0 \quad , \quad \lambda_n = \frac{4\pi^2 n^2}{\ell^2} \quad , \quad n = 1, 2, \ldots$$

(*) For each λ_n:

$$\begin{cases} A\cdot(0) + B\cdot(0) = 0 \\ A\cdot(0) + B\cdot(0) = 0 \end{cases} \longrightarrow A, B \text{ are free}$$

Finally,

$$\lambda_0 = 0 \longrightarrow y_0 = 1$$

$$\lambda_n = \frac{4\pi^2 n^2}{\ell^2} \xrightarrow{\text{degenerate}} \left\{ y_n^{(1)} = \cos\left(\frac{2\pi n}{\ell} x\right) , \ y_n^{(2)} = \sin\left(\frac{2\pi n}{\ell} x\right) \right\}$$

each eigenvalue gives more than one eigenfunction.

Given a Sturm-Liouville problem and a self-adjoint operator L, we know that the eigenvalues will be real and different eigenfunctions corresponding to diff. eigenvalues will be orthogonal.

Moreover, if the S-L problem is also REGULAR, $\{\lambda_n\}$ are non-degenerate and the eigenfunctions $\{y_n\}$ can be chosen to be real and have exactly $n-1$ zeroes on (a, b)

Example: $y'' + \frac{1}{x} y' + \lambda y = 0$, $y(1) = 0$ $\quad y(0)$?

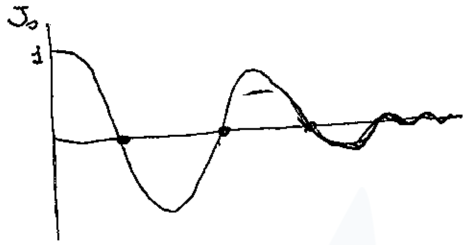
$a_0 = 1$, $a_1 = \frac{1}{x}$ $\rightarrow$ $P = \exp \int \frac{a_1}{a_0} dx = x$ $\rightarrow$ P vanishes at $x = 0$

$$\rho(x) = x$$

$t = \sqrt{\lambda}\, x \rightarrow$ Bessel eq of order $\nu = 0 \rightarrow y = A\, J_0(\sqrt{\lambda}\, x) + B\, Y_0(\sqrt{\lambda}\, x)$

But, as Y_0 has a divergence at $x = 0 \rightarrow B = 0$

Applying $y(1) = 0$, $J_0(\sqrt{\lambda}) = 0$.



α_n is the n^{th} zero of J_0

$\sqrt{\lambda_n} = \alpha_n \rightarrow \underline{\underline{\lambda_n = \alpha_n^{\,2}}}$

EIGENVALUES

Eigenfunctions $\rightarrow$ $y_n = J_0(\alpha_n x)$

$$\int_0^1 J_0(\alpha_n x)\, J_0(\alpha_m x)\, x\, dx = 0$$

$\{\lambda_n\}_{n=1,2,3,\dots} \rightarrow \{y_n\} \rightarrow$ normalize $\rightarrow \{u_n\}$ orthonormal set

$\quad\hookrightarrow \langle u_n | u_m \rangle = \delta_{nm}$

$\forall f \in C^n$ / $\in L^2([a,b])$ (f does not need to satisfy B.C.)

$\Downarrow$

$S = \sum_{n=1}^{\infty} \langle u_n | f \rangle u_n$ $\underline{\text{converges}}$ to f $\Rightarrow$ $\boxed{f = \sum_{n=1}^{\infty} \langle u_n | f \rangle u_n}$

FOURIER SERIES

$f : \mathbb{R} \longrightarrow \mathbb{C}$, T is a period for f if $f(x+T) = f(x)$, $\forall x \in \mathbb{R}$.
↳ real number

There are 3 possibilities
- i) only 0 is a period.
- ii) every real no. is a period. → f = ctant.
- iii) all periods are of the form kT, $k \in \mathbb{Z}$

↓ fundamental period

$\left.\begin{matrix}\sin x \\ \cos x\end{matrix}\right\}$ are periodic with $T = 2\pi$

↳ $e^{ix} = \cos x + i \sin x \rightarrow e^{i\lambda x}$ periodic with T ?

$$e^{i\lambda T} = \underset{f(0)}{\overset{\shortparallel}{1}} \rightarrow \boxed{\lambda = \frac{2\pi}{T} k}, \ k \in \mathbb{Z}$$

↓ ω

$\left\{ e^{i \frac{2\pi}{T} k x} \right\}_{k \in \mathbb{Z}}$ → orthonormal set (O.N.)

$L^2(T) \equiv \{$ set of $f : \mathbb{R} \to \mathbb{C}$ periodic with fundamental T and $\int_0^T |f|^2 < \infty$

$y'' + \lambda y = 0$, $y(a) = y(b)$
$y'(a) = y'(b)$

$\Downarrow$

$\lambda_n = \left(\frac{2\pi}{T} n\right)^2$, $T = b - a$

eigenfunctions $\left\{ \begin{matrix} \sin\left(n \frac{2\pi}{T} x\right) \\ \cos\left(n \frac{2\pi}{T} x\right) \end{matrix} \right.$ → Degeneracy

$$\boxed{e^{\pm i \frac{2\pi}{n} T x}}$$

$$\langle f | g \rangle = \int_0^T \bar{f} g \frac{dx}{T} \left(= \int_a^{a+T} \bar{f} g \frac{dx}{T} \right)$$

↳ weight

We'll prove that $\{e^{i\frac{2\pi}{T}nx}\}$ is an orthonormal set:

$$\langle e^{i\frac{2\pi}{T}kx} | e^{i\frac{2\pi}{T}nx} \rangle = \int_0^T e^{-i\frac{2\pi}{T}kx} \cdot e^{i\frac{2\pi}{T}nx} \frac{dx}{T} =$$

$$= \int_0^T e^{i\frac{2\pi}{T}(n-k)x} \frac{dx}{T} \begin{cases} \xrightarrow{n\neq k} \left. \dfrac{e^{i\frac{2\pi}{T}(n-k)x}}{i\frac{2\pi}{T}(n-k)} \right|_0^T = 0 \\ \xrightarrow{n=k} 1 \end{cases} = \delta_{kn} \quad \checkmark$$

$$f \in L^2(T) \rightarrow \text{FOURIER SERIES of } f: \sum_{k=-\infty}^{\infty} c_k(f)\, e^{ik\omega x}$$

$$c_k(f) = \langle e^{ik\omega x} | f \rangle = \int_0^T e^{-ik\omega x} f \frac{dx}{T}$$

This definition can be rewritten is terms of the sine and cosine:

$$\sum_{k=-\infty}^{\infty} c_k(f)\, e^{ik\omega x} = a_0 + \sum_{k=1}^{\infty} a_k(f) \cos(k\omega x) + \sum_{k=1}^{\infty} b_k(f) \sin(k\omega x)$$

$$\begin{aligned} a_k &= c_k + c_{-k} \\ b_k &= i(c_k - c_{-k}) \end{aligned} \iff \begin{aligned} c_k &= \frac{a_k - ib_k}{2} \\ c_{-k} &= \frac{a_k + ib_k}{2} \end{aligned}$$

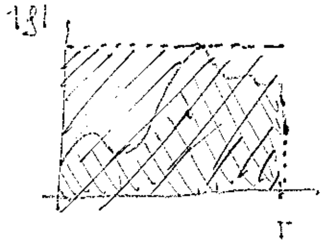
$$k = 1, 2, 3, \dots$$

$$a_0 = c_0$$

$$\left. \begin{aligned} a_k &= 2 \cdot \int_0^T f(x) \cos(k\omega x) \frac{dx}{T} \\ b_k &= 2 \int_0^T f(x) \sin(k\omega x) \frac{dx}{T} \end{aligned} \right\} k = 1, 2, 3, \dots$$

$$a_0 = \int_0^T f(x) \frac{dx}{T}$$

$$|c_k(f)| \leq \int_0^T |f(x)| \frac{dx}{T} \leq \underset{x\in[0,T]}{\text{ext}} (|f(x)|)$$



If f is even $\rightarrow f(x)\sin(k\omega x)$ is odd

$$\underbrace{\int_{-T/2}^{T/2} f(x)\sin(k\omega x)\frac{dx}{T}}_{b_k} = 0 \rightarrow b_k = 0$$

If f is odd $\rightarrow f(x)\cos(k\omega x)$ is odd $\rightarrow a_k = 0 \quad \forall k = 1,2,3,\ldots$

$a_0 = 0$

- - - - - - -

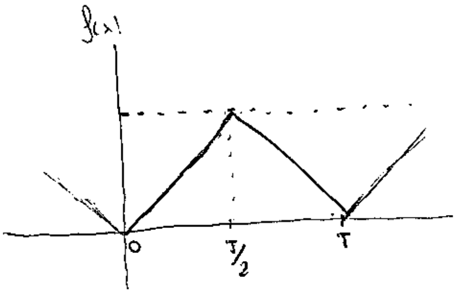
$f \in L^2(T) \rightarrow f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega x}$ at any x where $f(x)$ is continuous.

If f has a jump at $\tilde{x}$, the FOURIER SERIES CONVERGES to $\dfrac{f(\tilde{x}^+) + f(\tilde{x}^-)}{2}$.

$$\boxed{\langle f|f\rangle = \|f\|^2 = \int_0^{\infty} |f|^2 dx = \sum_{k=-\infty}^{\infty} |c_k|^2} \rightarrow \lim_{k\to\infty} |c_k| = 0$$

Example:

$$f(x) = \begin{cases} x, & 0 \leq x \leq T/2 \\ T-x, & T/2 \leq x \leq T \end{cases}$$



$$a_0 = \int_0^T f(x)\frac{dx}{T} = \frac{1}{T}\cdot\frac{T^2}{4} = \frac{T}{4}$$

$$a_k = 2\int_0^T f(x)\cos(k\overset{2\pi}{\overset{\uparrow}{\omega}} x)\frac{dx}{T} =$$

$b_k = 0$

$f(x)$ is periodic

$$= \frac{2}{T}\int_0^{T/2} x\cos(k\omega x)\,dx + \frac{2}{T}\int_{T/2}^{T} (T-x)\cos(k\omega x)\,dx$$

$$= (\ldots) = \frac{T}{k^2\pi^2}\left[(-1)^k - 1\right]$$

If $k = 2n \rightarrow a_{2n} = 0$

If $k = 2n+1 \rightarrow a_{2n+1} = \dfrac{-2T}{(2n+1)^2\pi^2}$

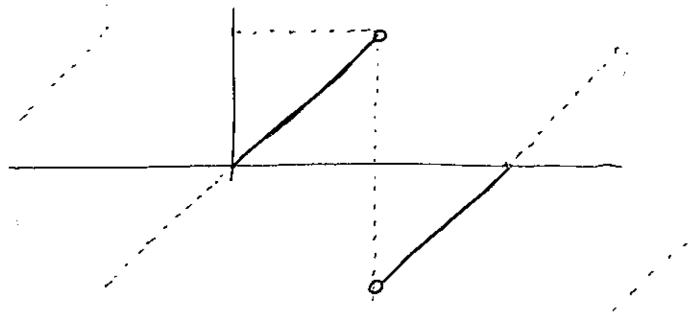
$n = 0, 1, \ldots$

Fourier series : $\dfrac{T}{4} - \dfrac{2T}{\pi^2} \sum_{n=0}^{\infty} \dfrac{1}{(2n+1)^2} \cos\big((2n+1)\omega x\big) = f(x) \quad , \; x \in [0,T]$.

or $\forall x \in \mathbb{R}$ for $f(x)$ extended periodically

Example :

$$f(x) = \begin{cases} x \,, & 0 \leq x < T/2 \\ x - T \,, & T/2 < x \leq T \end{cases}$$



→ $f(x)$ is odd

$a_0 = a_k = 0 \quad \forall k \geq 1$

$$b_k = 2 \int_0^T f(x) \sin(k\omega x) \frac{dx}{T} = (\ldots) = \frac{T}{k\pi} (-1)^{k+1}$$

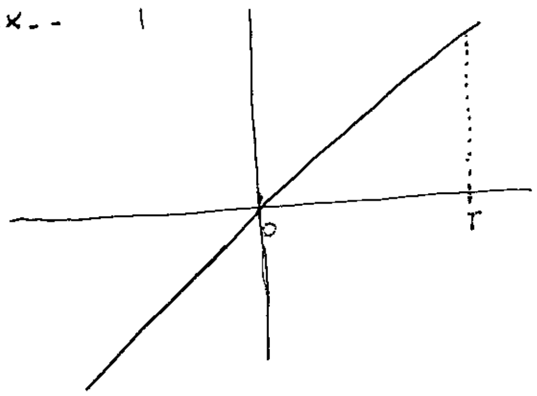
$$f(x) = \frac{T}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(k\omega x) \quad \forall x \neq (2n+1)\frac{T}{2} \quad , n \in \mathbb{Z} \setminus \{0\}$$

At $x = T/2$ the sum converges to 0.

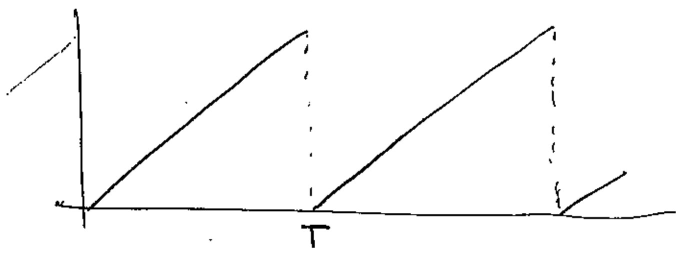
Fourier series are unique AS LONG AS THE period is defined.
(Both examples give the same result between 0 and $T/2$)

Any function can be expanded in FOURIER series.

Ex.:



$\Longrightarrow$



$b_k = -\dfrac{T}{k\pi}$

$a_0 = T/2$

$a_k = 0 \,, \; k \geq 1$

$$x = \frac{T}{2} - \sum_{k=1}^{\infty} \frac{T}{k\pi} \sin(k\omega x) \,, \quad x \in (0,T)$$

$$Ly + \underset{\substack{\uparrow \\ \text{constant}}}{\mu} y = f(x) \quad ① \quad \text{B.C. (on } a \text{ and } b\text{)}$$

(homogeneous)

Assume that we know the sol. for $Ly_n + \lambda_n y_n = 0$; $n = 0, 1, 2, \dots$ with the same B.C.

→ eigenvalues

↳ eigenfunctions

→ eigenfunctions are normalised

$$\langle y_n | y_m \rangle = \delta_{nm}$$

$$y(x) = \sum_{n=0}^{\infty} c_n y_n$$

unknown, unknown

$$Ly + \mu y = L\left(\sum_{n=0}^{\infty} c_n y_n\right) + \mu\left(\sum_{n=0}^{\infty} c_n y_n\right) = \sum_{n=0}^{\infty} c_n L y_n + \sum_{n=0}^{\infty} \mu c_n y_n =$$

$$= \sum_{n=0}^{\infty} (-c_n \lambda_n y_n) + \sum_{n=0}^{\infty} \mu c_n y_n = \sum_{n=0}^{\infty} c_n (\mu - \lambda_n) y_n = f(x)$$

$$\boxed{c_n (\mu - \lambda_n) = \langle y_n | f(x) \rangle \quad \forall n}$$

↳ y_n must be normalized!!

FREDHOLM'S THEOREM

① μ is NOT an eigenvalue ($\mu \neq \lambda_n$ for all n) → $c_n = \dfrac{\langle y_n | f(x) \rangle}{\mu - \lambda_n}$

$$\boxed{y(x) = \sum_{n=0}^{\infty} \frac{\langle y_n | f \rangle}{\mu - \lambda_n} y_n(x)} \quad (*)$$

② μ is one of the eigenvalues, say $\mu = \lambda_p$:

i) $\langle y_p | f \rangle \neq 0$ → No solution.

ii) $\langle y_p | f \rangle = 0$ → c_p is free → $y(x) = \sum_{\substack{n=0 \\ n \neq p}}^{\infty} \frac{\langle y_n | f \rangle}{\mu - \lambda_n} y_n + C_p y_n(x)$

arbitrary constant

FREDHOLM'S ALTERNATIVE

Inhom. prob. μ	Homog. prob. $\cancel{\mu}$
Unique solution	No SOLUTION (only trivial sol.)
No solution or ∞ solutions	Solution

Example: i) $y'' + \sqrt{2}\, y = \sin x$
ii) $y'' + 4y = \sin x$
iii) $y'' + y = \sin x$
$\Big\}$ ⊕ $y(0) = y(\pi) = 0$

$$Ly + \lambda y = 0, \quad L = \frac{d^2}{dx^2} \longrightarrow \lambda_n = n^2 \quad n = 1, 2, 3, \ldots$$
$y(0) = y(\pi) = 0$
$$y_n = \sqrt{\frac{2}{\pi}} \sin(nx)$$

i) $\sqrt{2} = \mu$, $\sqrt{2} \neq \lambda_n \ \forall n$

orthogonal

$$y = \sum_{n=1}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\langle \sin(nx) | \sin x \rangle}{\sqrt{2} - n^2} \sqrt{\frac{2}{\pi}} \sin(nx) = \frac{\cancel{\pi}}{\cancel{2}} \cdot \frac{1}{\sqrt{2}-1} \cdot \frac{\cancel{2}}{\cancel{\pi}} \sin x =$$

$$= \frac{1}{\sqrt{2}-1} \sin x$$

ii) $\mu = 4 \rightarrow \mu = \lambda_2$

$$\langle y_2 | f \rangle = \sqrt{\frac{2}{\pi}} \langle \sin(2x) | \sin x \rangle = 0 \,!!$$

$$y = \sum_{\substack{n=1 \\ n \neq 2}}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\langle \sin(nx) | \sin x \rangle}{4 - n^2} \sqrt{\frac{2}{\pi}} \sin(nx) + C_2 y_2 =$$

$$= \frac{1}{3} \sin(x) + \underbrace{C_2 \sqrt{\frac{2}{\pi}}}_{A} \sin(2x)$$

iii) $\mu = 1 = \lambda_1$

$\langle \underset{y_1}{\sin x} | \underset{f}{\sin x} \rangle \neq 0 \rightarrow$ No solution

Let's assume that $\mu \neq \lambda_n, \forall n$

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{\mu - \lambda_n} \cdot y_n(x) \left(\int_a^b \rho(\xi) \overline{y_n}(\xi) f(\xi) \, d\xi \right) =$$

$$= \int_a^b \underbrace{\left[\sum_n \rho(\xi) \frac{1}{\mu - \lambda_n} y_n(x) \overline{y_n}(\xi) \right]}_{G(x,\xi)} f(\xi) \, d\xi$$

$$\boxed{G(x,\xi) = \rho(\xi) \sum_n \frac{y_n(x) \overline{y_n}(\xi)}{\mu - \lambda_n}}$$ GREEN FUNCTION

$$y(x) = \int_a^b G(x,\xi) f(\xi) \, d\xi$$

PROPERTIES

(i) $\dfrac{\overline{G}(x,\xi)}{\rho(x)} = \dfrac{G(\xi,x)}{\rho(\xi)}$

(ii) $G(x,\xi)$ satisfies the B.C. for all ξ

(iii) $G(x,\xi)$ is continuous but not differentiable at $x = \xi$
everywhere

compute $Ly + \mu y = \int_a^b \underbrace{(LG + \mu G)}_{\delta(x-\xi)} f(\xi) \, d\xi = f(x)$

$$\boxed{LG + \mu G = \delta(x - \xi)}$$ $\rightarrow$ "$(L+\mu)^{-1}$" $\sim G$

ELEMENTARY SOLUTION

$LG+\mu G=\delta(x-\xi)$ ⊕ B.C. ⊕ continuity - fully determines the Green function.

Example:

$$y''=f(x)\quad,\quad y(0)=y(\pi)=0\qquad\left(L=\frac{d^2}{dx^2},\ \mu=0,\ \text{B.C. ok}\right)$$

1st method

$$y''+\lambda y=0 \oplus y(0)=y(\pi)=0 \longrightarrow \lambda_n=n^2,\ y_n=\sqrt{\frac{2}{\pi}}\sin(nx),\ n\in\mathbb{N}$$

$$\boxed{G(x,\xi)=-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx)\sin(n\xi)}{n^2}}$$

2nd method

$$G''+\cancel{0\cdot G}=\delta(x-\xi)\ ,\quad G(0,\xi)=G(\pi,\xi)=0 \ \oplus\ G \text{ is continuous}$$

We fix ξ:

$0<x<\xi$

$$G''=0\longrightarrow G=Ax+B\longrightarrow G(0,\xi)=\underline{B=0}\longrightarrow G=Ax$$

$\xi<x<\pi$

$$G''=0\longrightarrow G=Cx+D\longrightarrow G(\pi,\xi)=C\pi+D=0\longrightarrow D=-C\pi$$

$$G=C(x-\pi)$$

We apply the continuity condition at $x=\xi$:

$$G(\xi^-,\xi)=G(\xi^+,\xi)$$

$$A\xi=C(\xi-\pi)\longrightarrow A=C\,\frac{\xi-\pi}{\xi}$$

$$G(x,\xi)=C\left[\frac{\xi-\pi}{\xi}x\,\theta(\xi-x)+(x-\pi)\,\theta(x-\xi)\right]$$ → We've written in one line what we could've written in two.

$$G' = C\left[\frac{\xi-\pi}{\xi}\,\theta(\xi-x) - \frac{\xi-\pi}{\xi}\,x\,\delta(x-\xi) + \theta(x-\xi) + (x-\pi)\,\delta(x-\xi)\right]$$

At this stage, deltas must go away!!! $\quad f(x)\,\delta(x-a) =$ [illegible]

$$G' = C\left[\frac{\xi-\pi}{\xi}\,\theta(\xi-x) - \cancel{\frac{\xi-\pi}{\xi}\,\xi\,\delta(x-\xi)} + \theta(x-\xi) + \cancel{(\xi-\pi)\,\delta(x-\xi)}\right]$$

$$G' = C\left[\frac{\xi-\pi}{\xi}\,\theta(\xi-x) + \theta(x-\xi)\right]$$

$$G'' = \underbrace{C\left(-\frac{\xi-\pi}{\xi} + 1\right)}_{=1}\delta(x-\xi) \rightarrow \text{Remember initial equation: } G'' = \delta(x-\xi)$$

$$\rightarrow \boxed{C = \frac{\xi}{\pi}}$$

$$\boxed{G(x,\xi) = \frac{\xi-\pi}{\pi}\,x\,\theta(\xi-x) + \frac{\xi}{\pi}(x-\pi)\,\theta(x-\xi)} \quad x \in (0,\pi)$$

$$\hookrightarrow G(x,\xi) = \begin{cases} \dfrac{\xi-\pi}{\pi}\,x, & 0 < x < \xi \\ \dfrac{x-\pi}{\pi}\,\xi, & \xi < x < \pi \end{cases}$$

$$y(x) = \int_0^\pi G(x,\xi) f(\xi)\,d\xi = \int_0^x G(x,\xi) f(\xi)\,d\xi + \int_x^\pi G(x,\xi) f(\xi)\,d\xi =$$

$\xi < x$!! $\quad \rightarrow \xi > x$

$$= \int_0^x \frac{x-\pi}{\pi}\,\xi\, f(\xi)\,d\xi + \int_x^\pi \frac{\xi-\pi}{\pi}\,x\, f(\xi)\,d\xi$$

If $L = a_0 \dfrac{d^2}{dx^2} + a_1 \dfrac{d}{dx} + a_2$

$G = f_1\,\theta(x-\xi) + f_2(\xi - x)$, f_1 & f_2 sols. to the homogeneous problem

$$LG = \delta(x-\xi) \longrightarrow f_1'(\xi) - f_2'(\xi) = \frac{1}{a_0(\xi)}$$

LAPLACIAN OPERATOR $\rightarrow$ $\nabla^2 f = \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

In spherical coordinates:

$$\nabla^2 f = \frac{1}{r^2 \sin\theta} \left\{ \sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{\sin\theta} \frac{\partial^2 f}{\partial \varphi^2} \right\}$$

In cilindrical coordinates:

$$\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\boxed{\nabla^2 \phi = -C \rho(r)} \quad \left(C = \frac{1}{\varepsilon_0}, \; C = -4\pi G \right)$$

POISSON EQUATION

$$\boxed{\nabla^2 \phi = 0}$$

LAPLACE EQUATION

$$\boxed{\nabla^2 \phi + a\phi = 0, \quad a = \text{cst.}}$$

HELMHOLTZ EQ. $(a > 0)$

$$\boxed{\nabla^2 \phi = \frac{1}{a^2} \frac{\partial \phi}{\partial t}}$$

HEAT EQUATION

$$\boxed{\nabla^2 \phi = \frac{1}{v^2} \frac{\partial^2 \phi}{\partial t^2}}$$

WAVE EQ.

$$\boxed{\frac{\hbar}{2m} \nabla^2 \phi + V(\vec{r})\phi = i\hbar \frac{\partial \phi}{\partial t}}$$

SCHRÖDINGER EQUATION

They're all, linear, second order partial differential eqs. (PDE).

Ansatz: $\phi = X(x)\,Y(y)\,Z_1(z)$ → We want to find sols. of that type.

Helmoltz eq.

$$\frac{d^2X}{dx^2}YZ_1 + X\frac{d^2Y}{dy^2}Z_1 + XY\frac{d^2Z_1}{dz^2} + aXYZ_1 \stackrel{?}{=} 0$$

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{only depends on } x} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\text{" " } y} + \underbrace{\frac{1}{Z_1}\frac{d^2Z_1}{dz^2}}_{\text{" " } z} + a = 0$$

If the sum of all of them is a cstant, then each one is a constant.

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2} = l = \text{cnt.}}\,, \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = m = \text{cstant}, \quad \frac{1}{Z_1}\frac{d^2Z_1}{dz^2} = n = \text{cnt.}$$

&

$$\boxed{l + m + n = -a}$$

↳ $l, m, n \equiv$ separation constants

$$X'' - lX = 0$$

arbitrary constants.

$$\boxed{\phi = \sum_{l,m,n} X_l(x)\,Y_m(y)\,Z_{1_n}(z)\,A_{lmn}}$$

~~Example~~:

$$L_t y = L_x y \quad , \quad L_t = \frac{1}{\rho(x)}\left[\frac{\partial}{\partial x}\left(P(x)\frac{\partial}{\partial x}\right) - Q(x)\right]$$

$$L_x = \alpha(t)\frac{\partial^2}{\partial t^2} + \beta(t)\frac{\partial}{\partial t} + \gamma(t)$$

$$\begin{cases} A_1 y(t,a) + A_2 y_x(t,a) + B_1 y(t,b) + B_2 y_x(t,b) = 0 \\ C_1 \;\;\text{"} \;\; + C_2 \;\;\text{"} \;\; + D_1 \;\;\text{"} \;\; + D_2 \;\;\text{"} \;\; = 0 \end{cases} \longrightarrow \text{B.C.}$$

Initial conditions: $y(t_0, x) = y_0(x)$ → given function

$(\alpha \neq 0)$: $y_t(t_0, x) = y_1(x)$

Separation of vars. → $y = T(t)X(x)$

Go to PDE:

$$X(x) L_t T = T(t) L_x X$$

$\cdot \frac{1}{y}$

$$\frac{L_t T}{T} = \frac{L_x X}{X} = -\lambda$$ ← separation constant

$$L_x X + \lambda X = 0 \quad \oplus \quad \begin{cases} A_1 X(a) + A_2 X'(a) + B_1 X(b) + B_2 X'(b) = 0 \\ C_1 \quad " \quad + C_2 \quad " \quad + D_1 \quad " \quad + D_2 \quad " \quad = 0 \end{cases}$$

STURM-LIOUVILLE PROBLEM $\Rightarrow \langle \mid \rangle$, $\boxed{\{\lambda_n\} \text{ eigenvalues}, \{X_n\} \text{ eigenfunctions}}$

$$y = \sum_n T_n(t) X_n(x) \longrightarrow L_t y = L_x X \Rightarrow \sum_n (L_t T_n + \lambda_n T_n) X_n = 0$$

$L_t T_n + \lambda_n T_n = 0 \quad \oplus$ Initial conditions

↳ $\begin{cases} 2^{nd} \text{ order ODE} \\ 1^{st} \quad " \quad " \quad \text{for } \alpha(t) = 0 \end{cases}$

$$T_n(t) = \underbrace{C_n \tau_n(t) + D_n \theta_n(t)}$$

Linearly indep. sols.

$$y = \sum_n \left(C_n \tau_n(t) + D_n \theta_n(t) \right) X_n(x)$$

$$\left.\begin{aligned} \sum_n \left(C_n \tau_n(t_0) + D_n \theta_n(t_0) \right) X_n(x) &= y_0(x) \\ \sum_n \left(C_n \dot{\tau}_n(t_0) + D_n \dot{\theta}_n(t_0) \right) X_n(x) &= y_1(x) \end{aligned}\right\}$$ → Using initial conds.

$$\frac{\langle X_n | y_0 \rangle}{\underbrace{\langle X_n | X_n \rangle}_{\text{Norm}}} = C_n T_n(t_0) + D_n \theta_n(t_0)$$

$$\frac{\langle X_n | y_1 \rangle}{\langle X_n | X_n \rangle} = C_n \dot{T}_n(t_0) + D_n \dot{\theta}_n(t_0)$$

For each "n", we get two eqs. to fix C_n and D_n (if $\alpha = 0 \rightarrow$ one eq. to fix C_n)

Example:

$$\frac{\partial^2 y}{\partial x^2} - \alpha \frac{\partial^2 y}{\partial t^2} = 0 \quad , \quad \alpha = \text{ctant.} \neq 0 .$$

$$y(t,0) = y(t,\pi) = 0$$

$$y(0,x) = 0 , \quad y_t(0,x) = 1$$

Separation constant

$$y = T(t) X(x) \rightarrow T X'' = \alpha X \ddot{T} \rightarrow \frac{X''}{X} = \alpha \frac{\ddot{T}}{T} = -\lambda$$

⊛ $X'' + \lambda X = 0 \ , \ X(0) = X(\pi) = 0$

S-L prob.

$\lambda_n = n^2 ; \ n = 1, 2, 3, \ldots$

$X_n = \sin(nx)$

⊛ $\alpha \ddot{T}_n + \lambda_n T_n = 0 \rightarrow \ddot{T}_n + \frac{n^2}{\alpha} T_n = 0$

$\alpha > 0$: $T_n = C_n \cos\left(\frac{n}{\sqrt{\alpha}} t\right) + D_n \sin\left(\frac{n}{\sqrt{\alpha}} t\right)$

$$y = \sum_{n=1}^{\infty} \left(\cancel{C_n \cos\left(\frac{n}{\sqrt{\alpha}} t\right)} + D_n \sin\left(\frac{n}{\sqrt{\alpha}} t\right) \right) \sin(nx)$$

$$y(0,x) = \sum_{n=1}^{\infty} C_n \sin(nx) = 0 \rightarrow C_n = 0 \ \forall n$$

$$y_t = \sum_{n=1}^{\infty} D_n \frac{n}{\sqrt{\alpha}} \cos\left(\frac{n}{\sqrt{\alpha}} t\right) \sin(nx)$$

$$y_t(0,x) = \sum_{n=1}^{\infty} \boxed{D_n \frac{n}{\sqrt{\alpha}}} \sin(nx) = 1 \xrightarrow{\text{Fourier}} D_n = \begin{cases} 0, & n = 2K \\ \frac{4\sqrt{\alpha}}{\pi(2K+1)^2}, & n = 2K+1 \end{cases}$$

$$\boxed{y = \sum_{K=0}^{\infty} \frac{4\sqrt{\alpha}}{\pi} \cdot \frac{1}{(2K+1)^2} \sin\left(\frac{2K+1}{\sqrt{\alpha}} t\right) \sin[(2K+1)x]} \quad \begin{matrix} t > 0 \\ 0 < x < \pi \end{matrix}$$

$\alpha < 0$:

$$\boxed{y = \sum_{K=0}^{\infty} \frac{4\sqrt{-\alpha}}{\pi} \cdot \frac{1}{(2K+1)^2} \sinh\left(\frac{2K+1}{\sqrt{-\alpha}} t\right) \sin[(2K+1)x]}$$

Example:

$$u_t = \alpha^2 u_{xx} \rightarrow \frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$u(0,t) = T_1$, $u(L,t) = T_2$ → NON-HOMOGENEOUS BOUNDARY CONDS. !

$u(x,0) = f(x)$ → known function

We define the following:

$$\boxed{V(x,t) = u(x,t) + g(x)}$$

→ In the exam, we'll have to calculate $g(x)$.

$$\left.\begin{aligned} V(0,t) &= T_1 + g(0) \\ V(L,t) &= T_2 + g(L) \end{aligned}\right\} \quad g(x) \text{ with } g(0) = -T_1, \; g(L) = -T_2$$

(ONE CHOICE)

$$g(x) = -T_1 - (T_2 - T_1)\frac{x}{L}$$

New PDE:

$$V_t = \alpha^2 (V_{xx} - g'')$$

with new B.C.: $V(0,t) = V(L,t) = 0$

and new I.C.: $V(x,0) = f(x) + g(x)$

Next step: SOLVE THE HOMOGENEOUS PDE:

$$z_t = \alpha^2 z_{xx} \quad ④ \quad \begin{cases} z(t,0) = 0 \\ z(t,L) = 0 \end{cases}$$

Separation of variables : $z = X(x)\,\mathcal{T}(t)$

Separation constant

$$\dot{\mathcal{T}} X = \alpha^2 \mathcal{T} X'' \rightarrow \frac{\dot{\mathcal{T}}}{\mathcal{T}} = \alpha^2 \frac{X''}{X} = -\alpha^2 \lambda$$

$$X'' + \lambda X = 0 \oplus X(0) = X(L) = 0$$

$$\hookrightarrow \begin{cases} \lambda_n = \dfrac{n^2\pi^2}{L^2} & n = 1, 2, 3, \dots \\ X_n = \sin\left(n \dfrac{\pi}{L} x\right) \end{cases}$$

$$V(t,x) = \sum_{n=1}^{\infty} \mathcal{T}_n(t) \sin\left(n\frac{\pi}{L}x\right) \left\langle \begin{array}{l} V_t = \displaystyle\sum_{n=1}^{\infty} \dot{\mathcal{T}}_n(t) \sin\left(n\frac{\pi}{L}x\right) \\ V_{xx} = -\displaystyle\sum_{n=1}^{\infty} \mathcal{T}_n(t) \frac{n^2\pi^2}{L^2} \sin\left(n\frac{\pi}{L}x\right) \end{array} \right.$$

Go to PDE:

$$\sum_{n=1}^{\infty} \dot{\mathcal{T}}_n(t) \sin\left(n\frac{\pi}{L}x\right) = -\alpha^2 \sum_{n=1}^{\infty} \mathcal{T}_n \frac{n^2\pi^2}{L^2} \sin\left(n\frac{\pi}{L}x\right) - \alpha^2 g''(x)$$

Expand $g(x)$: $\sum_{n=1}^{\infty} g_n \sin\left(n\frac{\pi}{L}x\right)$ constants

$$'' \quad = - \quad '' \quad + \alpha^2 \sum_{n=1}^{\infty} \frac{n^2\pi^2}{L^2} g_n \sin\left(n\frac{\pi}{L}x\right)$$

$$\downarrow$$

$$\boxed{\dot{\mathcal{T}}_n(t) = -\alpha^2 \frac{n^2\pi^2}{L^2} \left(\mathcal{T}_n(t) + g_n\right)} \quad \forall n \quad \text{(ODEs)}$$

Back to $V(t,x) = u(t,x) + g(x)$: $\oplus$ I.C. :

$$\sum_{n=1}^{\infty} \mathcal{T}_n(0) \sin\left(n\frac{\pi}{L}x\right) = f(x) + \sum_{n=1}^{\infty} g_n \sin\left(n\frac{\pi}{L}x\right)$$

$$\hookrightarrow f(x) = \sum_{n=1}^{\infty} \overset{\text{constants}}{f_n} \sin\left(n\frac{\pi}{L}x\right)$$

$$\boxed{\mathcal{T}_n(0) = f_n + g_n}$$

LAPLACE EQ. IN SPHERICAL COORDINATES

$$\nabla^2\psi = 0 = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial^2\psi}{\partial r^2}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right]$$

S.V. $\rightarrow \psi = R(r)Z(\theta)\Phi(\varphi)$

$$\frac{Z\Phi}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R\Phi}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dZ}{d\theta}\right) + \frac{RZ}{r^2\sin\theta}\frac{d^2\Phi}{d\varphi^2} = 0$$

$$\frac{1}{RZ\Phi} : \frac{1}{R}(r^2R')' + \frac{1}{Z\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dZ}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = 0$$

$' \equiv \frac{d}{dr}$

$$\frac{1}{R}(r^2R')' = Q$$

Separation constant

S.L. problem

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = \text{constant} = -m^2$$

because of periodic B.C. (implicit in the physical prob.)

$\hookrightarrow m \in \mathbb{N}$

$\Phi_m = C_m\cos(m\varphi) + D_m\sin(m\varphi) = \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$

$$\boxed{\frac{1}{\sin\theta}\frac{1}{Z}\frac{d}{d\theta}\left(\sin\theta\frac{dZ}{d\theta}\right) + Q - \frac{m^2}{\sin^2\theta} = 0}$$

ASSOCIATED LEGENDRE EQ.

$$\begin{cases} x = \cos\theta, \ |x| \leq 1 \\ y(x) = Z(\theta) \end{cases} \rightarrow (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(Q - \frac{m^2}{1-x^2}\right)y = 0$$

(*)

Case $m=0$

$$\boxed{(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + Qy = 0}$$

LEGENDRE EQUATION

The series will converge up to the next singular point:

At $x=0$, the radius of conv. will be $|x| < 1$.

We try the following solution:

$$y=\sum_{n=0}^{\infty} a_n x^n \Rightarrow a_{n+2}(n+2)(n+1) = a_n\left[(n+1)n - Q\right], \quad n\geq 0$$

We choose a solution with the following characterisics:

$\begin{cases} a_0 = 1 \\ a_1 = 0 \end{cases} \rightarrow$ Only even powers

$\begin{cases} a_0 = 0 \\ a_1 = 1 \end{cases} \rightarrow$ Only odd powers

We'll want polynomial solutions, so they don't diverge at $x=\pm 1$.

If $Q = \ell(\ell+1)$, $\ell \in \mathbb{N}$, there will be polynomial sols.

$\hookrightarrow a_{\ell+2} = 0 \Rightarrow a_{\ell+2k} = 0 \quad \forall k \geq 1.$

* If ℓ is even $\rightarrow$ Choose $a_1 = 0 \rightarrow$ Polynomial of ℓ^{th} degree (and even powers)

* If ℓ is odd $\rightarrow$ " $a_0 = 0 \rightarrow$ " " " " (" odd ")

Solutions:

$$\boxed{P_\ell(x) = \frac{1}{2^\ell}\sum_{j=0}^{2j\leq \ell} \frac{(-1)^j}{j!}\frac{[2(\ell-j)]!}{(\ell-2j)!\,(\ell-j)!}\, x^{\ell-2j}}$$

LEGENDRE POLYNOMIALS

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{2}(5x^3-3x)$

$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

Properties:

① Parity: $P_\ell(-x) = (-1)^\ell P_\ell(x)$

② Formula de Rodrigues: $P_\ell(x) = \dfrac{1}{2^\ell \ell!} \dfrac{d^\ell}{dx^\ell}\left[(x^2-1)^\ell\right]$

③ $P_\ell(\pm 1) = (\pm 1)^\ell$

④ Orthogonality: $\displaystyle\int_{-1}^{1} P_\ell(x) P_n(x)\, dx = 0, \quad \ell \neq n$

⑤ $\displaystyle\int_{-1}^{1} P_\ell^2(x)\, dx = \frac{2}{2\ell+1}$

⑥ Recurrence:
$$\begin{cases} \dfrac{dP_{\ell+1}}{dx} - x\dfrac{dP_\ell}{dx} - (x+1)P_\ell = 0 \\ (\ell+1)P_{\ell+1} - (2\ell+1)xP_\ell + \ell P_{\ell+1} = 0 \end{cases}$$

⑦ Generating function:
$$G(x,z) = \underbrace{\sum_{n=0}^{\infty} P_n(x) z^n}_{\text{Taylor expansion}} = \frac{1}{\sqrt{1+z^2-2xz}}$$

⑧ $\{P_\ell(x)\}_{\ell\in\mathbb{N}}$ is a complete set on $L^2_{[-1,1]}$.

$$L^2_{[-1,1]} = \{ f : [-1,1] \longrightarrow \mathbb{R} \mid \text{square integrable} \}$$

$$\forall f \in L^2_{[-1,1]}, \quad f = \sum_{n=0}^{\infty} c_\ell P_\ell(x), \quad c_\ell = \frac{2\ell+1}{2}\int_{-1}^{1} f(x) P_\ell(x)\, dx$$

Back to (*):

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \left(Q - \frac{m^2}{1-x^2}\right)y = 0$$

$$\downarrow \quad y(x) = (1-x^2)^{m/2}\, u(x)$$

$$(1-x^2)\frac{d^2u}{dx^2} - 2x(1+m)\frac{du}{dx} + \left[Q - m(m+1)\right]u = 0$$

$$u=\sum_{n=0}^{\infty} a_n x^n \longrightarrow a_{n+2}(n+2)(n+1)=a_n\left[(n+m)(n+m+1)-Q\right]$$

Polynomial sols? Yes, if $Q=\ell(\ell+1)$, $\ell\in\mathbb{N}$. $\longrightarrow a_{\ell-m+2}=0 \rightarrow a_{\ell-m+2k}=0$, $\forall k=1,2,\ldots$

$\boxed{|m|\leq \ell} \rightarrow m\in\mathbb{Z}$

If $\ell-m$ is even $\rightarrow a_1=0 \rightarrow$ Polynomial with even powers.
" " " odd $\rightarrow a_0=0 \rightarrow$ " " odd " .

$$u(x)=\frac{1}{2^\ell}\sum_{j=0}^{2j\leq \ell-m}\frac{(-1)^j}{j!}\frac{[2(\ell-j)]!}{(\ell-j)!(\ell-m-2j)!}x^{\ell-m-2j}$$

$$u(x)=\frac{d^m}{dx^m}P_\ell(x) \longrightarrow y(x)=\boxed{P_\ell^m(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x)=\frac{1}{2^\ell \ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}\left[(x^2-1)^\ell\right]}$$

ASSOCIATED LEGENDRE POLYNOMIALS

$P_1^1(x)=(1-x^2)^{1/2}$ $(=\sin\theta)$

$P_2^1(x)=3x(1-x^2)^{1/2}$ $(=3\sin\theta\cos\theta)$

$P_2^2(x)=3(1-x^2)$ $(=3\sin^2\theta)$

$P_3^1(x)=\frac{3}{2}(5x^2-1)(1-x^2)^{1/2}$ $\left(=\frac{3}{2}(5\cos^2\theta-1)\sin\theta\right)$

$P_3^2(x)=15x(1-x^2)$ $(=15\cos\theta\sin^2\theta)$

$P_3^3(x)=15(1-x^2)^{3/2}$ $(=15\sin^3\theta)$

PROPERTIES

① $P_\ell^{-m}(x)=(-1)^m\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(x)$

② $P_\ell^0(x)=P_\ell(x)$

③ $P_\ell^m(-x)=(-1)^{\ell+m}P_\ell^m(x)$

④ $P_\ell^m(\pm 1)=0$, $\forall m\neq 0$

⑤ The initial eq. can be rewritten as:

$$\frac{d}{dx}\Big[\underbrace{(1-x^2)}_{\text{"}p\text{"}}\frac{dy}{dx}\Big]-\frac{m^2}{1-x^2}y+\underbrace{\ell(\ell+1)}_{\lambda}y=0$$

$\rho=1$

$$\int_{-1}^{1}P_\ell^m(x)\,P_{\ell'}^m(x)\,dx=0 \quad \forall \ell\neq\ell' \rightarrow \text{ORTHOGONALITY}$$

$$\int_{-1}^{1}\left(P_\ell^m\right)^2dx=\frac{(\ell+m)!}{(\ell-m)!}\cdot\frac{2}{2\ell+1}$$

There are two different scalar products.

Moreover, it may be rewritten as:

$$\underbrace{(1-x^2)}_{1/\rho}\frac{d}{dx}\Big[\underbrace{(1-x^2)}_{\text{"}p\text{"}}\frac{dy}{dx}\Big]+\ell(\ell+1)(1-x^2)y-\underbrace{m^2}_{\lambda}y=0$$

$$\int_{-1}^{1}P_\ell^m\,P_\ell^{m'}\,\frac{dx}{1-x^2}=0\,,\quad m\neq m'$$

$$\int_{-1}^{1}(P_\ell^m)^2dx=\frac{(\ell+m)!}{(\ell-m)!\,m}$$

We've proved that $Z\rightarrow P_\ell^m(\cos\theta)$.

Now, we have to solve $(r^2R')'-QR=0$, with $Q=\ell(\ell+1)$.

$$r^2R''+2rR'-\ell(\ell+1)R=0\rightarrow R=C_1r^\ell+C_2\frac{1}{r^{\ell+1}}$$

equidimensional (CAUCHY-EULER eq.)

If we want to make $r\rightarrow\infty$, we'll have to remove this sol.

If we want to make $r\rightarrow 0$, we'll have to remove this sol.

$$\boxed{Y_{\ell,m}(\theta,\varphi)=\sqrt{\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\;P_\ell^m(\cos\theta)\,e^{im\varphi}}$$

SPHERICAL HARMONICS

↳ If it doesn't depend on $\varphi\rightarrow m=0\rightarrow$ Solutions are the Legendre polynomials

$$\boxed{\langle Y_{\ell m} | Y_{\ell' m'} \rangle = \int \overline{Y_{\ell m}(\theta,\varphi)}\, Y_{\ell' m'}(\theta,\varphi) \sin\theta \, d\theta \, d\varphi = \delta_{\ell\ell'} \delta_{mm'}}$$

ORTHO NORMALITY

$$\boxed{\Psi = \sum_{\ell=0}^{\infty} \left(a_\ell r^\ell + \frac{b_\ell}{r^{\ell+1}} \right) \sum_{m=-\ell}^{\ell} c_{\ell m} Y_{\ell m}(\theta,\varphi)}$$

SOLUTION TO $\nabla^2 \Psi = 0$

Regular at axis,

2π periodicity in φ.

THE GAMMA FUNCTION : $\Gamma(z)$, $z \in \mathbb{C}$

$$\Gamma(z) = \lim_{n \to \infty} \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}{z \cdot (z+1) \cdot (z+2) \cdot \ldots \cdot (z+n)} n^z \quad , \; z \neq 0, -1, -2, -3, \ldots$$

Easy to check:

$$\boxed{\Gamma(z+1) = z\,\Gamma(z)}$$

$$\Gamma(1) = \lim_{n \to \infty} \frac{n}{n+1} = 1$$

$$\Gamma(2) = \Gamma(1+1) = 1\,\Gamma(1) = 1$$

$$\Gamma(3) = \Gamma(2+1) = 2\,\Gamma(2) = 2$$

$$\ldots$$

$$\boxed{\Gamma(n) = (n-1)! \quad , \; n \in \mathbb{N}}$$

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt \quad , \; \mathrm{Re}(z) > 0$$

$$\Gamma(1/2) \underset{t=u^2}{=\!=} 2 \int_0^\infty e^{-u^2} du = \sqrt{\pi}$$

LAPLACE EQ. IN CYLINDRICAL COORDINATES

$$\nabla^2\psi = 0 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2}$$

S.V.: $\psi = R(\rho)\,\Phi(\varphi)\,Z(z)$

$$\left(\cdot\frac{1}{\psi}\right) \rightarrow \frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2}\underbrace{\boxed{\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}}}_{=-m^2 \Rightarrow \text{ sep. const.}} + \overbrace{\boxed{\frac{1}{Z}\frac{d^2 Z}{dz^2}}}^{=\chi \Rightarrow \text{ sep. const.}} = 0$$

$$\begin{cases} \dfrac{d^2 Z}{dz^2} - \chi Z = 0 \,, \quad \chi > 0 \quad \begin{cases} e^{\sqrt{\chi} z} \\ e^{-\sqrt{\chi} z} \end{cases} \\[2ex] \dfrac{d^2\Phi}{d\varphi^2} + m^2\Phi = 0 \begin{cases} e^{im\varphi} \\ e^{-im\varphi} \end{cases} , \quad m \in \mathbb{Z} \\[2ex] \dfrac{1}{\rho}\dfrac{d}{d\rho}\left(\rho\dfrac{dR}{d\rho}\right) + \left(\chi - \dfrac{m^2}{\rho^2}\right)R = 0 \quad (*) \end{cases}$$

CHANGE OF VARS.: $\begin{cases} x = \sqrt{\chi}\,\rho \\ y(x) = R(\rho) \end{cases}$

$$\rightarrow \frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{m^2}{x^2}\right)y = 0 \quad (*)$$

$$\boxed{x^2 y'' + x y' + (x^2 - m^2)y = 0} \Rightarrow \text{BESSEL EQ.}$$

$y = x^\lambda \sum_{n=0}^{\infty} a_n x^n$ $\overset{\text{To}}{\longrightarrow}$ ODE

⊛ $\lambda^2 - m^2 = 0$ (indicial eq.) $\begin{cases} \lambda_1 = m \\ \lambda_2 = -m \end{cases}$

⊛ $a_1 = 0$

⊛ $a_{n+2} = \dfrac{a_n}{m^2 - (n+\lambda+2)^2}$, $n \geq 0$

$\Bigg\} \Rightarrow a_{2k+1} = 0 \quad \forall k \in \mathbb{N}$

$$\lambda_1 = m \rightarrow a_{2l} = \frac{(-1)^l}{2^{2l}} \cdot \frac{a_0}{l!\,(m+l)(m+l-1)\cdot\ldots\cdot(m+2)(m+1)}$$

$$\frac{\Gamma(l+m+1)}{\Gamma(m+1)} = (m+1)(m+2)\ldots(m+l)$$

choose $a_0 = \dfrac{1}{2^m\,\Gamma(m+1)}$

$$a_{2l} = \frac{(-1)^l}{2^{2l+m}} \cdot \frac{1}{l!\,\Gamma(l+m+1)}$$

SOLUTION:

$$\boxed{J_m(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\,\Gamma(l+m+1)} \left(\frac{x}{2}\right)^{2l+m}}$$

BESSEL FUNCTION OF ORDER m.

$\lambda_1 - \lambda_2 \notin \mathbb{N} \rightarrow J_{-m}$ 2^{nd} sol.

$\lambda_1 - \lambda_2 \in \mathbb{N}$ $\begin{cases} 2m \text{ is odd} \rightarrow \boxed{m = k + 1/2},\ k \in \mathbb{N} \rightarrow \begin{cases} J_{k+1/2} \\ J_{-(k+1/2)} \end{cases} \\ 2m \text{ is even} \end{cases}$ SPHERICAL BESSEL EQ.

If $2m$ is even, $m>0$:

$$J_{-m}(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\,\Gamma(l-m+1)} \left(\frac{x}{2}\right)^{2l+m} = \sum_{l=m}^{\infty} (\text{same}) \underset{l = l' + m}{=} \ldots$$

$$= J_m(x)\,(-1)^m \rightarrow \text{lin. dep.}$$

2^{nd} solution: $N_m(x) \underset{x \to 0}{\sim} J_m(x) \ln(x)$

(BESSEL EQ. OF 2^{nd} KIND)

If we're working near the axis $(r=0)$, the 2^{nd} sol. is not acceptable, as it diverges in the axis.

Final solution:

$$\boxed{\Psi = \sum_{m,\varkappa} \left(A_m e^{\sqrt{\varkappa} z} + B_m e^{-\sqrt{\varkappa} z}\right)\left(C_m e^{im\varphi} + D_m e^{-im\varphi}\right) J_m(\sqrt{\varkappa}\rho)}$$

Properties of J_m

① Generating function:

$$G(x,s) = e^{\frac{x}{2}\left(s - \frac{1}{s}\right)} \overset{\text{prove!}}{=} \sum_{n=-\infty}^{n=\infty} s^n J_n(x)$$

② Recurrence relations:

$$\frac{2m}{x} J_m = J_{m-1} + J_{m+1}$$

$$J_m' = \frac{1}{2}\left(J_{m-1} - J_{m+1}\right)$$

$$\frac{x^2}{4}\left(J_{m-2} + J_{m+2} + 2J_m\right) + \frac{x}{2}\left(J_{m-1} - J_{m+1}\right) - m^2 J_m = 0$$

$$x J_m' + m J_m = x J_{m-1} \qquad \left(x^m J_m\right)' = x^m J_{m-1}$$

$$x J_m' - m J_m = -x J_{m+1} \qquad \left(x^{-m} J_m\right)' = -x^{-m} J_{m+1}$$

$$J_m(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin\beta - n\beta)\, d\beta$$

$$1 = J_0^2 + 2\sum_{m=1}^{\infty} J_m^2 \Rightarrow \begin{cases} |J_0(x)| \leq 1 & \forall x \\ |J_m(x)| \leq \frac{1}{\sqrt{2}} & , m = 1,2,3,\dots \end{cases}$$

Given (*), applying $y(x)=\frac{u(x)}{\sqrt{x}}$:

$$\frac{d^2u}{dx^2}+\left[1+\frac{1-4m}{x^2}\right]u=0 \Longrightarrow J_{\pm 1/2} \sim \begin{cases} \frac{\sin x}{\sqrt{x}} \\ \frac{\cos x}{\sqrt{x}} \end{cases}$$

$x >> 1$:

$$\frac{d^2u}{dx^2}+u\simeq 0 \rightarrow J_m(x) \underset{x\to\infty}{\sim} \alpha \frac{\cos x}{\sqrt{x}}+\beta\frac{\sin x}{\sqrt{x}} \quad \forall m$$

Occasionally,

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\left(1+\frac{m^2}{x^2}\right)y=0$$

MODIFIED BESSEL EQUATIONS

↳ SOLUTION → $\boxed{I_m(x)=i^{-m}J_m(ix)}$

$$= \sum_{l=0}^{\infty}\frac{1}{l!\,\Gamma(l+m+1)}\left(\frac{x}{2}\right)^{2l+m}$$

Back to (*),

Weight: $\rho(P)=P$ } Just like in a S-L problem!!

$x \rightarrow$ "λ" (eigenvalue)

Given, for example, the following B.C.: regularity at axis $\rho=0$ ⊕ $R(a)=0$

Remember, in a S-L problem: $P\,W[\bar{f},g]\Big|_a^b=0$.

$P=\rho$

$$\cancel{P(0)}\cdot W(f,g)(0)-\underbrace{P(a)\left(\bar{f}(a)\,g'(a)-\bar{f}'(a)\,g(a)\right)}_{=0\,!!!}=0$$

$P(0)=0$

$$R=A\,J_m(\sqrt{x}\rho)+\cancel{B\,N_m(\sqrt{x}\rho)} \overset{B.C.}{\Longrightarrow} J_m(\sqrt{x}a)=0 \Rightarrow \boxed{\sqrt{x}\,a=\alpha_n^{(m)}}$$

n^{th} zero of J_m.

↳ Non-regular at $\rho=0$.

Eigenfunctions: $J_m\left(\frac{\alpha_n^{(m)}}{a}\rho\right)$

ORTHOGONALITY:

$$\int_0^a J_m\left(\frac{\alpha_n^{(m)}}{a}\rho\right) J_m\left(\frac{\alpha_K^{(m)}}{a}\rho\right)\rho\, d\rho = 0 \quad , n \neq K$$

NORM:

$$\int_0^a J_m^2\left(\frac{\alpha_n^{(m)}}{a}\rho\right)\rho\, d\rho$$

$$\left[x^2\left(J_m^2 - J_{m+1}J_{m-1}\right)\right]' = 2x J_m^2$$

$$\left(x^2 J_m^2\right)' = 2x J_m^2 + x^2 \cdot 2 J_m J_m' = 2x J_m^2 + x^2 J_m\left(J_{m-1} - J_{m+1}\right)$$

$$\left(x^2 J_{m+1}J_{m-1}\right)' = 2x J_{m+1}J_{m-1} + x^2 J_{m+1}' J_{m-1} + x^2 J_{m+1} J_{m-1}' =$$

$$= 2x J_{m+1} J_{m-1} + x J_{m-1}\underbrace{\left(x J_m - (m+1) J_{m+1}\right)}_{x J_{m+1}'} +$$

$$+ x^2 J_{m+1}\underbrace{\left(-x J_m + (m-1) J_{m-1}\right)}_{x J_{m-1}'} =$$

$$= x J_{m+1} J_{m-1}\underbrace{\left(2 - (m+1) + m - 1\right)}_{0} + x^2 J_m\left(J_{m-1} - J_{m+1}\right)$$

$$x = \frac{\alpha_n^{(m)}}{a}\rho \rightarrow \quad \frac{a^2}{\alpha_n^{2(m)}}\int_0^{\alpha_n^{(m)}} x J_m^2(x)\, dx = \frac{a^2}{\alpha_n^{2(m)}}\frac{1}{2}\int_0^{\alpha_n^{(m)}}\left[x^2\left(J_m^2 - J_{m+1}J_{m-1}\right)\right]' dx =$$

$$= \frac{a^2}{2\alpha_n^{2(m)}}\left[x^2\left(J_m^2 - J_{m+1}J_{m-1}\right)\right]\Big|_0^{\alpha_n^{(m)}} = \frac{a^2}{2\alpha_n^{2(m)}}\Big[\alpha_n^{2(m)} \underbrace{J_m^2\left(\alpha_n^{(m)}\right)}_{=0!} -$$

$$- \alpha_n^{(m)2} J_{m+1}\left(\alpha_n^{(m)}\right) J_{m-1}\left(\alpha_n^{(m)}\right)\Big] =$$

$\frac{2m}{x} J_m = J_{m-1} + J_{m+1}$; take at $x = \alpha_n^{(m)} \rightarrow$

$\rightarrow J_{m-1}\left(\alpha_n^{(m)}\right) + J_{m+1}\left(\alpha_n^{(m)}\right) = 0$

$$= -\frac{a^2}{2} J_{m+1}\left(\alpha_n^{(m)}\right) J_{m-1}\left(\alpha_n^{(m)}\right) \Longrightarrow \frac{a^2}{2} J_{m+1}^2\left(\alpha_n^{(m)}\right)$$

Bessel-Fourier series:

$$\boxed{f = \sum_{n=1}^{\infty} b_n J_m\left(\frac{\alpha_n^{(m)}}{a}\rho\right) \quad , \quad b_n = \frac{2}{a^2 J_{m+1}^2(\alpha_n^{(m)})} \int_0^a J_m\left(\frac{\alpha_n^{(m)}}{a}\rho\right) f(\rho)\rho\, d\rho}$$

Final solution for this particular problem:

$$\psi = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} J_m\left(\frac{\alpha_n^{(m)}}{a}\rho\right)\left[C_m \sin(m\varphi) + D_m \cos(m\varphi)\right]\left(A_{mn} e^{\frac{\alpha_n^{(m)}}{a}z} + B_{mn} e^{-\frac{\alpha_n^{(m)}}{a}z}\right)$$

To motivate: $u(x,y)$; $\partial_x u = 0 \rightarrow u = f(y)$

Cauchy problem (initial value prob.) : i) $u|_{y=x} = x^2$

$$u(y=x) = f(x) = x^2 \longrightarrow \boxed{u = y^2}$$

ii) $u|_{x=0} = 3 \rightarrow f(y) = 3 \rightarrow \boxed{u=3}$

iii) $u|_{y=0} = 3 \rightarrow f(0) = 3 \rightarrow$?

General case:

$$A(x,y)\, u_x + B(x,y)\, u_y + C(x,y)\, u = E(x,y) \quad \rightarrow \text{linear}$$

Initial data: $u|_{\Gamma} = f(s)$ ← Known

Γ → curve

s → parameter describing Γ → $\begin{cases} x = X(s) \\ y = Y(s) \end{cases}$

$$u(x,y) \longrightarrow u|_{\Gamma} = u(X(s), Y(s)) = f(s)$$

$$\Downarrow$$

$$\begin{cases} u_x \dot{X}(s) + u_y \dot{Y}(s) = \dot{f}(s) \quad , \quad \dot{} \equiv \dfrac{d}{ds} \\ (A u_x + B u_y + Cu)|_{\Gamma} = D(X(s), Y(s)) \end{cases}$$

I can fix $u_x|_{\Gamma}$ and $u_y|_{\Gamma}$ unless $\boxed{B\dot{X} - A\dot{Y} = 0}$ → Determinant with the system.

$$\frac{dx}{A} = \frac{dy}{B} \quad , \quad \frac{dy}{dx} = \frac{B}{A}$$

General solution: $F(x,y) = C_1$

Use $\eta = F(x,y) \rightarrow$ The general sol. depends on an arbitrary function of η.

$$D = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} \Longrightarrow \text{Equation} : Du + Cu = E \Rightarrow DF = 0 \,!!$$

(η is a first integral)

$A = Dx$

$B = Dy$

$$(x,y) \longrightarrow (\xi, \eta) \Rightarrow \boxed{D = g(\xi,\eta)\frac{\partial}{\partial \xi}} \xrightarrow{eq.} g\frac{\partial u}{\partial \xi} + C(\xi,\eta)u = E(\xi,\eta)$$

$\Downarrow$

Solution depends on $f(\eta)$

Example:

$$x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$$

$$D = \underbrace{x}_{A}\frac{\partial}{\partial x}\underbrace{-2y}_{B}\frac{\partial}{\partial y} \qquad \frac{dy}{-2y} = \frac{dx}{x} \rightarrow x^2y = C_1$$

$\eta = x^2y \quad , \xi = x$

$$Du = x\frac{\partial u}{\partial \xi} = 0 \rightarrow \boxed{u = f(x^2y), f \text{ arbitrary}}$$

Initial data:

$\Gamma : x = 1$

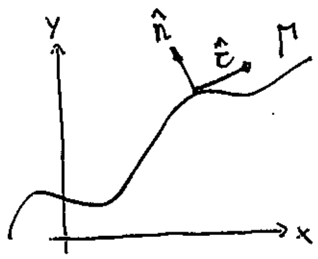
$u|_\Gamma = 2y + 1$ $\rightarrow u|_\Gamma = 2y + 1 = f(1^2 \cdot y) = f(y)$

$$\boxed{u = 2x^2y + 1}$$

2^{nd} ORDER PDEs

$$A(x,y)\,u_{xx} + B(x,y)\,u_{xy} + C(x,y)\,u_{yy} = F(x,y,u,u_x,u_y)$$

Linear on 2^{nd} derivatives



$$\Gamma : \begin{cases} x = X(s) \\ y = Y(s) \end{cases}$$

$\left.\begin{array}{l} u|_\Gamma = f(s) \\ \hat{n} \cdot \nabla u|_\Gamma = g(s) \end{array}\right\}$ Known

$\vec{\tau} = \left(\frac{dX}{ds}, \frac{dY}{ds}\right)$

$\vec{n} = \left(-\frac{dY}{ds}, \frac{dX}{ds}\right)$

$$\frac{d}{ds}\left(u|_\Gamma\right) = u_x\dot{X} + u_y\dot{Y} = \dot{f}(s)$$

$$-u_x\dot{Y} + u_y\dot{X} = g(s)$$

$$u_x\big|_\Gamma = -g(s)\,\dot{X} + f(s)\,\dot{X} \qquad (*)$$

$$u_y\big|_\Gamma = g(s)\,\dot{X} + f(s)\,\dot{Y} \qquad (**)$$

$$\frac{d}{ds}\left(u_x\big|_\Gamma\right) = u_{xx}\dot{X} + u_{xy}\dot{Y} = \frac{d}{ds}(*)$$

$$\frac{d}{ds}\left(u_y\big|_\Gamma\right) = u_{yx}\dot{X} + u_{yy}\dot{Y} = \frac{d}{ds}(**)$$

$$\left(A u_{xx} + B u_{xy} + C u_{yy}\right)\Big|_\Gamma = F\big|_\Gamma$$

↳ Will have a sol. except when the det. vanishes.

$$\begin{vmatrix} \dot{X} & \dot{Y} & 0 \\ 0 & \dot{X} & \dot{Y} \\ A & B & C \end{vmatrix} = 0 \Rightarrow \text{CHARACTERISTICS}$$

$$A\dot{Y}^2 - B\dot{X}\dot{Y} + C\dot{X}^2 = 0$$

$$A\,dy^2 - B\,dx\,dy + C\,dx^2 = 0$$

$$(*)\quad A\left(\frac{dy}{dx}\right)^2 - B\frac{dy}{dx} + C = 0 \;\rightarrow\; \text{Discriminant: } B^2-4AC \begin{cases} >0 \rightarrow 2 \text{ real sols} & \text{HYPERBOLIC} \\ =0 \rightarrow 1 \text{ real sol.} & \text{PARABOLIC} \\ <0 \rightarrow \text{complex sols.} & \text{ELLIPTIC} \end{cases}$$

depends on the point.
There might be regions in which the sign changes

1) HYPERBOLIC

Two sols.: $\xi = \varphi(x,y)$
$\eta = \psi(x,y)$

Canonical form

Change indep. vars.: $\{x,y\} \longrightarrow \{\xi,\eta\} \Rightarrow$ Rewrite eq. $(*)$ in the new vars. $\Rightarrow \dfrac{\partial^2 u}{\partial\xi\,\partial\eta} = \tilde{F}(\xi,\eta,u,u_\xi,u_\eta)$

Wave eq. in 2D is an example

2) Parabolic

One solution : $\xi = \varphi(x,y)$ $\Longrightarrow$ $\dfrac{\partial^2 u}{\partial \eta^2} = f(\xi, \eta, u, u_\xi, u_\eta)$ ← Canonical form
$\eta = x$

For example, heat eq.

3) Elliptic

No real solutions, two complex conjugate solutions : $\varphi(x,y)$, $\varphi(x,y)$ $(=\bar{\varphi})$

$$\xi = \frac{1}{2}(\varphi + \varphi) = Re(\varphi)$$

$$\eta = \frac{1}{2i}(\varphi - \varphi) = Im(\varphi)$$

$\Rightarrow \dfrac{\partial^2 u}{\partial \xi^2} + \dfrac{\partial^2 u}{\partial \eta^2} = f(\xi, \eta, u, u_\xi, u_\eta)$ ← Canonical form

For example, Laplace eq. $(f=0)$

	Boundary		
Hyperbolic	Open	Cauchy (initial) conds.	
Parabolic	Open	Dirichlet (u on boundary)	Neumann (normal ∇u on B)
Elliptic	Closed	"	" ⊕ Regularity ∞

Wave equation

$$u_{xx} = \frac{1}{c^2} u_{tt}$$

Compute characteristics $\left(A=1,\ B=0,\ C=-\frac{1}{c^2}\right)$

We get two sols :

$x - ct = \text{ctant.} = \eta$
$x + ct = \text{ctant.} = \xi$ (Hyperbolic)

$\{x, t\} \longrightarrow \{\xi, \eta\} \Rightarrow \ldots \Rightarrow \dfrac{\partial^2 u}{\partial \xi \partial \eta} = 0 \rightarrow$ General sol. : $u(\xi, \eta) = p(\xi) + q(\eta)$

p, q, arbitrary functions

Initial conds. : $\begin{cases} u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases}$ → data

$$u(t,x) = p(x+ct) + q(x-ct)$$

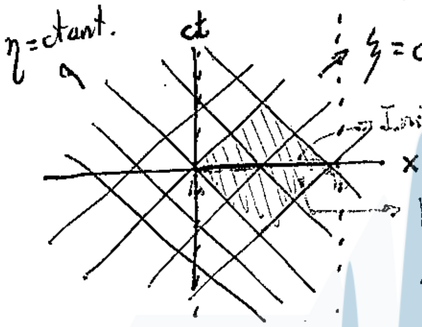
$$u(0,x) = p(x) + q(x) = f(x)$$

$$u_t(0,x) = \big(p'(x) - q'(x)\big)c = g(x)$$

$$p(x) - q(x) = \frac{1}{c}\int_a^x g(\tilde{x})\,d\tilde{x}$$

$$\left.\begin{aligned} p(\xi) &= \frac{1}{2}f(\xi) + \frac{1}{2c}\int_a^{\xi} g(\tilde{x})\,d\tilde{x} \\ q(\eta) &= \frac{1}{2}f(\eta) - \frac{1}{2c}\int_a^{\eta} g(\tilde{x})\,d\tilde{x} \end{aligned}\right\} \quad u(t,x) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tilde{x})\,d\tilde{x}$$

D'ALEMBERT SOLUTION



→ Initial values given on A, B.

→ Domain of dependence; region where we can fully solve the problem.

To know the solution for all t in [A,B], one has to supply B.C.

DEF. 1

A: event

$$\lim_{N \to \infty} \frac{N_A}{N} = P(A)$$ → probabity of A happening

DEF. 2

$n \equiv$ # outcomes

If all n outcomes have the same probabity of occurring (one of them event A), then

$$P(A) = \frac{n_A}{n}$$

Example:

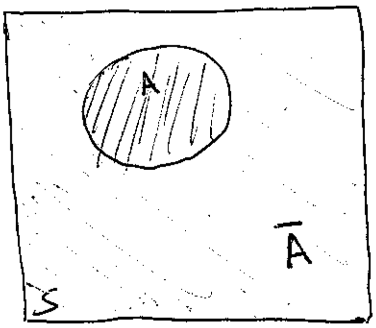
COIN → $n=2$, $n_{tails} = \frac{1}{2}$

DIE → $n=6$, $A \equiv$ getting an even number (2,4,6) → $n_A = 3$ → $P(A) = \frac{3}{6} = 1/2$

VENN DIAGRAMS

DEFINITIONS

- TRIAL ≡ experiment
- OUTCOME ≡ result of an experiment
- EVENT ≡ Set of outcomes (A, B, C...)
- SAMPLE SPACE ≡ set of all possible outcomes (S)
- EMPTY SET ≡ event with no outcomes

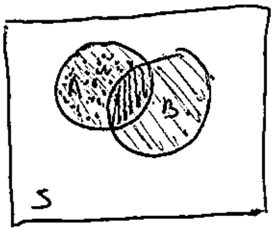


Complement of A ≡ $\bar{A}$ = Not A

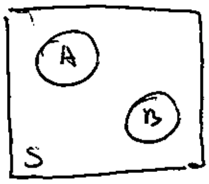
DIE ⇒ $S = \{1,2,3,4,5,6\}$

$A = \{1,2,3\} \to \bar{A} = \{4,5,6\}$

Given 2 events A, B:



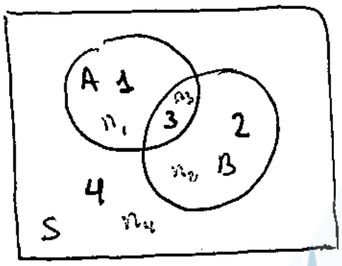
* UNION EVENT : $A \cup B \equiv$ Either A or B happens
* INTERSECTION EVENT : $A \cap B \equiv$ Both A & B happen
* DIFFERENCE : $A - B \equiv A \cap \overline{B}$



Exclusive A, B.

EXCLUSIVE EVENTS : $A \cap B = \emptyset$

PROBABILITY CALCULATION



elements of $S = n$

PROBABILITIES :

* A but not B (1) $\to n_1$
* B " " A (2) $\to n_2$
* Both A and B (3) $\to n_3$
* None A or B (4) $\to n_4$

Properties :

a) $P \in [0,1]$, $P_\emptyset = 0$, $P_S = 1$

b) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \dfrac{n_1+n_2+n_3}{n}$

$$P(A) = \frac{n_1+n_3}{n}, \quad P(B) = \frac{n_2+n_3}{n}, \quad P(A \cap B) = \frac{n_3}{n}$$

Example:

2 packs of cards (each $n=48$). Prob. of drawing 2 aces?

Event = Drawing at least 1 ace

A = Draw one ace from pack 1. $\to n_A = 4 \to P(A) = \frac{4}{48} = P(B)$ $\frac{4}{48} = 1/12$

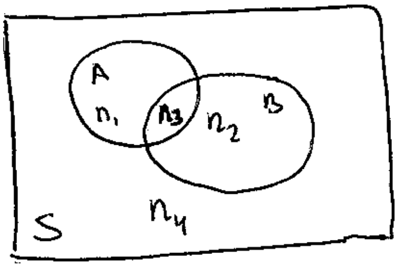
B = " " " " " 2.

$$P(A \cap B) = P(A) \cdot P(B) = \frac{1}{12} \cdot \frac{1}{12} = \frac{1}{144}$$

Event = $A \cup B$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{12} + \frac{1}{12} - \frac{1}{144}$$

CONDITIONAL PROBABILITIES



Prob. of A, given that B happened : $P(A|B)$

" " B, " " A " : $P(B|A)$

$$P(A|B) = \frac{n_3}{n_B} = \frac{n_3}{n_2+n_3} = \frac{P(A\cap B) \rightarrow \frac{n_3}{n}}{P(B) \rightarrow \frac{n_2+n_3}{n}}$$

$$P(B|A) = \frac{P(A\cap B)}{P(A)}$$

BAYES THEOREM : $\boxed{P(A|B) = P(B|A)\cdot\frac{P(A)}{P(B)}}$

Two events are statistically independent if $\begin{cases} P(A|B) = P(A) \\ P(B|A) = P(B) \end{cases}$

A & B INDEP. : $P(B|A) = P(B) = \frac{P(A\cap B)}{P(A)} \longrightarrow \underline{P(A\cap B) = P(A)\cdot P(B)}$

PROPERTIES OF PROBABILITY II

a) $A \cup \emptyset = A$

b) $A \cup S = S$

c) $A \cup \bar{A} = S$ $\Rightarrow 1 = P(S) = P(A\cup\bar{A}) = P(A) + P(\bar{A}) - \cancel{P(A\cap\bar{A})} \rightarrow$

$\rightarrow \underline{P(\bar{A}) = 1 - P(A)}$

d) $A\cup(B\cup C) = (A\cup B)\cup C$

$A\cap(B\cap C) = (A\cap B)\cap C$

e) $A\cup B = B\cup A$

$A\cap B = B\cap A$

f) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

g) $A \cap (B \cup C) = (A \cap B) \cup (B \cap C)$

h) $A \cap A = A$
$A \cup A = A$

$$* P(A \cup (B \cup C)) = P(A) + P(B \cup C) - P(A \cap (B \cup C)) =$$
$$= P(A) + P(B) + P(C) - P(B \cap C) - P((A \cap B) \cup (A \cap C)) =$$
$$= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P((A \cap B) \cap (A \cap C))$$
$$= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

DECOMPOTION THEOREM

$$* \boxed{P(B) = \underbrace{P(B|A)}_{\frac{P(B \cap A)}{P(A)}} P(A) + \underbrace{P(B|\bar{A})}_{\frac{P(B \cap \bar{A})}{P(\bar{A})}} P(\bar{A})} = P(B \cap A) + P(B \cap \bar{A}) =$$

$$= P((B \cap A) \cup (B \cap \bar{A})) + \cancel{P((B \cap A) \cap (B \cap \bar{A}))}^{\,0} = P(B \cap (A \cup \bar{A})) =$$
$$= P(B \cap S) = P(B)$$

Exercise 1:

- Pack of cards (48 cards) → P(Drawing 2 aces) — I) Put card back / II) Not replacing the card

$$P(2 \text{ aces}) = \underbrace{P(1 \text{ ace})}_{\frac{1}{12}} \cdot \underbrace{P(2^{nd} \text{ ace} | 1^{st} \text{ ace})}_{\frac{4}{48} \,//\, \frac{3}{47}}$$

Exercise 2:

- Pack of cards (48 cards) → Probability of drawing 1 ace OR 1 spade OR 1 figure (S/C/R)

$\quad$ 1 ace → A, 1 spade → B, 1 figure (S/C/R) → C

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) =$$
$$= \frac{4}{48} + \frac{12}{48} + \frac{12}{48} - \frac{1}{48} - 0 - \frac{3}{48} + 0 =$$

Exercise 3:

Die (6 sides). Prob. for each side is $\underset{1}{\frac{1}{2}p}, \underset{2}{p}, \underset{3}{p}, \underset{4}{p}, \underset{5}{p}, \underset{6}{2p}$. What is the value of p?

$$1 = \frac{1}{2}p + p + p + p + p + 2p = \frac{13}{2}p \rightarrow p = 2/13$$

DECOMPOSITION THEOREM II

$A \cap B = \emptyset \Rightarrow$ Mutually exclusive events

$A \cap A = A$

$A_1, A_2, \ldots, A_i, \ldots, A_N \quad , i = 1, 2, \ldots, N$

These are mutually exclusive events if

$$A_i \cap A_j \begin{cases} \emptyset, & i \neq j \\ A_i, & i = j \end{cases} \Rightarrow \text{SET OF MUTUALLY EXCLUSIVE EVENTS}$$

$$S = A_1 \cup A_2 \cup \ldots \cup A_i \cup A_{i+1} \cup \ldots \cup A_N \Rightarrow \text{COMPLETE SET OF EVENTS}$$

$$\parallel$$

$$\bigcup_{i=1}^{N} A_i$$

$$P(A_i | A_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$\parallel$$

$$\frac{P(A_i \cap A_j)}{P(A_j)}$$

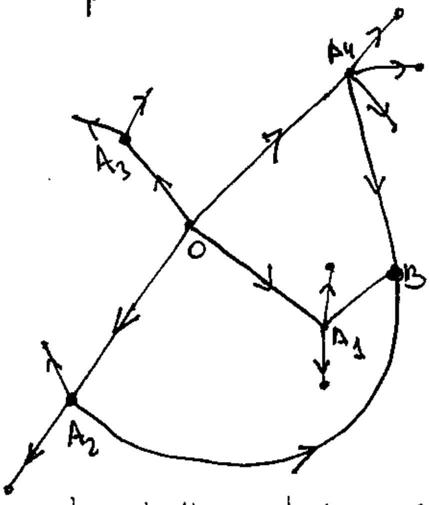
$$P(S) = P\left(\bigcup_{i=1}^{N} A_i\right) = \boxed{\sum_{i=1}^{N} P(A_i) = 1}$$

$P((B \cap A_1) \cap (B \cap A_2)) = 0$

$$P(B) = P(B \cap S) = P\left(B \cap \left[\bigcup_{i=1}^{N} A_i\right]\right) = P\left(\bigcup_{i=1}^{N} [B \cap A_i]\right) =$$

$$= \sum_{i=1}^{N} P(B \cap A_i) = \boxed{\sum_{i=1}^{N} P(B|A_i) \cdot P(A_i) = P(B)}$$

Example:



START AT O. → P(B)?

$P(A_i) = \frac{1}{4} \quad i = 1,2,3,4$

$P(B|A_4) = \frac{1}{4}$

$P(B|A_1) = \frac{1}{3}$

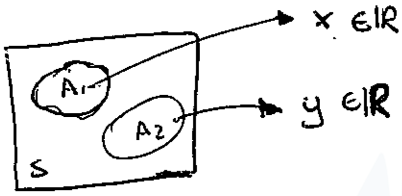
$P(B|A_2) = \frac{1}{3}$

$P(B|A_3) = 0$

↳ Mutually exclusive events, they form a complete set

$$P(B) = \sum_{i=1}^{4} P(B|A_i) \cdot P(A_i) = \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{3}\cdot\frac{1}{4} + 0 + \frac{1}{4}\cdot\frac{1}{4} = \frac{1}{12} + \frac{1}{12} + \frac{1}{16}$$

RANDOM VARIABLE



Example:

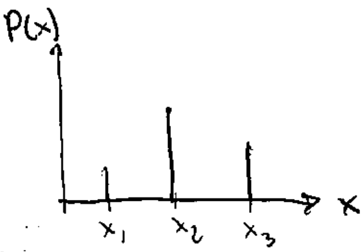
coin	X
heads	0
tails	1

There are two types of random variable:

- DISCRETE → Concrete values → $x \in \{x_1, x_2, \dots, x_n\}$
- CONTINUOUS → x can be any number in an interval → $x \in [a, b]$

Discrete X

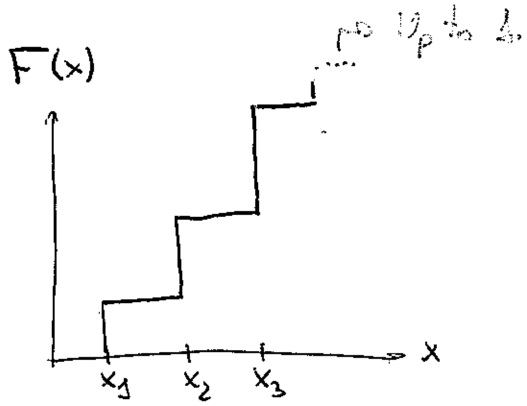
$$P(x) = \begin{cases} x \in \{x_1, \dots, x_n\} & P(x) = P(A) \\ x \notin \{x_1, \dots, x_n\} & P(x) = 0 \end{cases}$$



$$\sum_i P(x_i) = \sum_i P(A_i) = 1 = P(S)$$

Cumulative Probability Function (CPD)

$$F(x) = P(X \leq x) = \sum_{i=0}^{n} P(x_i)$$



$X = \{x_1, \dots, x_n\}$

$F(x_n) = 1$

$$P(x_{i_1} \leq X \leq x_{i_2}) = \sum_{i=i_1}^{i_2} P(x_i) = F(x_{i_2}) - F(x_{i_1})$$

Example:

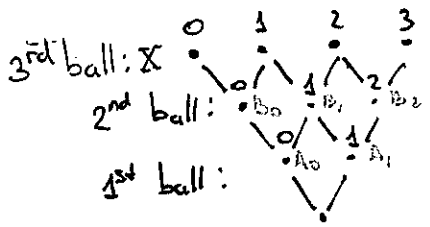
BAG $\longrightarrow$ 10 balls $\begin{cases} 7 \text{ red} \\ 3 \text{ white} \end{cases}$

Probability function of # red balls after taking 3 balls.

X = # red balls in my hand.

After taking 3 balls, I can have taken 0, 1, 2, or 3 red balls.

$X = \{0, 1, 2, 3\}$



$A_0 \equiv$ I have 0 red balls after taking 1 ball

$A_1 \equiv$ " " 1 " ball " " 1 ".

$B_1 \equiv$ I have 1 " " " " 2 balls.

$P(A_0) = \frac{3}{10}$

$P(A_1) = \frac{7}{10}$

$P(B_0) = \cancel{P(B_0|A_1)}^{=0}\,\cancel{P(A_1)} + P(B_0|A_0) \cdot P(A_0) = \frac{2}{9} \cdot \frac{3}{10}$

$P(B_1) = P(B_1|A_0) \cdot P(A_0) + P(B_1|A_1) P(A_1) = \frac{7}{9} \cdot \frac{3}{10} + \frac{3}{9} \cdot \frac{7}{10} =$

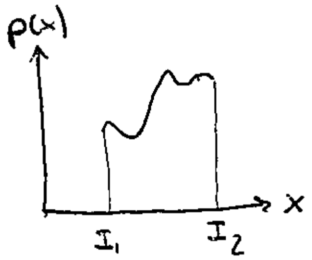
$P(B_2) =$

$$P(X=0) = P(X=0|B_0)P(B_0) + P(X=0|B_1)P(B_1) + P(X=0|B_2)P(B_2)$$

Continuous Random Variables

$X \in [I_1, I_2] \subseteq \mathbb{R}$

$p(x) = P(x \leqslant X \leqslant x + dx) \longrightarrow$ Probability Density Function (PDF)



$$F(x) = P(X \leqslant x) = \int_{I_1}^{x} p(x)\,dx \qquad (\text{CPF})$$

$$P(a_1 \leqslant X \leqslant a_2) = \int_{a_1}^{a_2} p(x)\,dx = \int_{I_1}^{a_2} p(x)\,dx - \int_{I_1}^{a_1} p(x)\,dx = F(a_2) - F(a_1)$$

$$F(I_2) = 1 \iff \boxed{\int_{I_1}^{I_2} p(x)\,dx = 1}$$

Example:

$p(x) = Ae^{-x}, \quad x \in [1, 2]$

$$A \int_{1}^{2} e^{-x} = 1$$

$$\boxed{\begin{array}{l}\text{Discrete distrib} \rightsquigarrow \text{Continuous} \\ \displaystyle f(x) = \sum_{i=0}^{n} p(x_i)\,\delta(x - x_i)\end{array}}$$

X, Y are indep.

$\Updownarrow \quad \Updownarrow$

A, B are indep.

$$\Rightarrow \quad P_x(x) = \begin{cases} P(A_i), & x = x_i \\ 0, & x \neq x_i \end{cases} \qquad P_y(y) = \begin{cases} P(B_i), & y = y_i \\ 0, & y \neq y_i \end{cases} \quad \Rightarrow \quad P(x, y) = P_x(x) \cdot P_y(y)$$

Properties of Distributions

$X \equiv$ random var.

$p(x) \equiv$ Distribution

* Expectation value of $g(x)$:

$$E[g(x)] = \langle g(x) \rangle \overset{\text{CONT.}}{\equiv} \int_{I_1}^{I_2} g(x)\, p(x)\, dx$$

$$\overset{\text{DISC.}}{\equiv} \sum_i g(x_i)\, p(x_i)$$

Properties of expectation values:

1) $\langle a \rangle = a$

$$\langle a \rangle = \int_{I_1}^{I_2} a\, p(x)\, dx = a \int_{I_1}^{I_2} p(x)\, dx = a$$

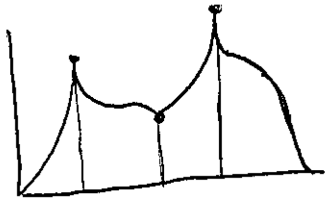
2) $\langle a\, g(x) + b\, h(x) \rangle = a \langle g(x) \rangle + b \langle h(x) \rangle$

* Mean:

$$\langle x \rangle \begin{cases} \equiv \displaystyle\int_{I_1}^{I_2} x\, p(x)\, dx \\ \equiv \displaystyle\sum_i x_i\, p(x_i) \end{cases}$$

* Mode: Extremses the distribution

$$\underline{\partial_x P(x_m) = 0}$$



* Median:

$$F(x) = \int_{I_1}^{x} p(x)\, dx = P(X \le x)$$

$$F(x_{1/2}) = \frac{1}{2}$$

$\downarrow$

Median

* Percentiles

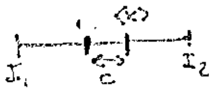
n^{th} percentile: X_n $\qquad F(X_n) \equiv \frac{n}{100}$

* Variance

$$V = \langle (x - \langle x \rangle)^2 \rangle \equiv \sigma^2$$

$$\boxed{P(|x - \langle x \rangle| \geq c) \leq \frac{\sigma^2}{c^2}}$$

BIENAYMÉ – CHEVYSHEV INEQUALITY



<u>Proof</u>: $\sigma^2 \equiv \langle (x - \langle x \rangle)^2 \rangle = \int_{I_1}^{I_2} (x - \langle x \rangle)^2 p(x)\,dx \geq \int_{|x-\langle x \rangle| \leq c} (x - \langle x \rangle)^2 p(x)\,dx \geq$

$$\geq c^2 \int_{|x-\langle x \rangle| \leq c} p(x)\,dx = c^2 P(|x - \langle x \rangle| \leq c)$$

* Moments

$$K\text{-th moment} \equiv \langle x^K \rangle = \begin{cases} \sum_i x_i^K p(x_i) & \text{DISCRETE CASE} \\ \int_{I_1}^{I_2} x^K p(x)\,dx & \text{CONTINUOUS CASE} \end{cases}$$

$K=0 \rightarrow$ Mean

$K=2 \sim$ Variance

$$\sigma^2 \equiv \int (x - \langle x \rangle)^2 p(x)\,dx = \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 - 2x\langle x \rangle + \langle x \rangle^2 \rangle =$$

$$= \langle x^2 \rangle - \underbrace{\langle 2x \langle x \rangle \rangle}_{= -2\langle x \rangle \langle x \rangle = -2\langle x \rangle^2} + \langle x^2 \rangle = \boxed{\langle x^2 \rangle - \langle x \rangle^2 = \sigma^2}$$

Example:

6 SIDED BIASED DIE

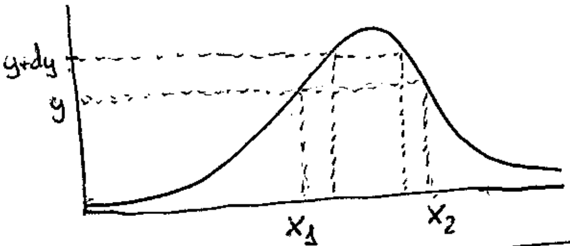
P(x)	P/2	P	P	P	P	2P
x	1	2	3	4	5	6

i) Mean

ii) 2^{nd} moment

iii) Variance

- $y(x) \longrightarrow x(y)$ multivalued

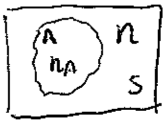


$$P_y\,dy = P_r(y \leq Y \leq y+dy) =$$

$$= P_r(\underbrace{x_1}_{x_1(y)} \leq X \leq \underbrace{x_1+dx}_{x_1(y+dy)}) + P_r(\underbrace{x_2}_{x_2(y)} \leq X \leq \underbrace{x_2+dx}_{x_2(y+dy)}).$$

$$\boxed{P_y(y)\,dy = P_x(x_1(y))\left|\frac{dx}{dy}\right|_{x_1(y)} + P_x(x_2(y))\left|\frac{dx}{dy}\right|_{x_2(y)}}$$

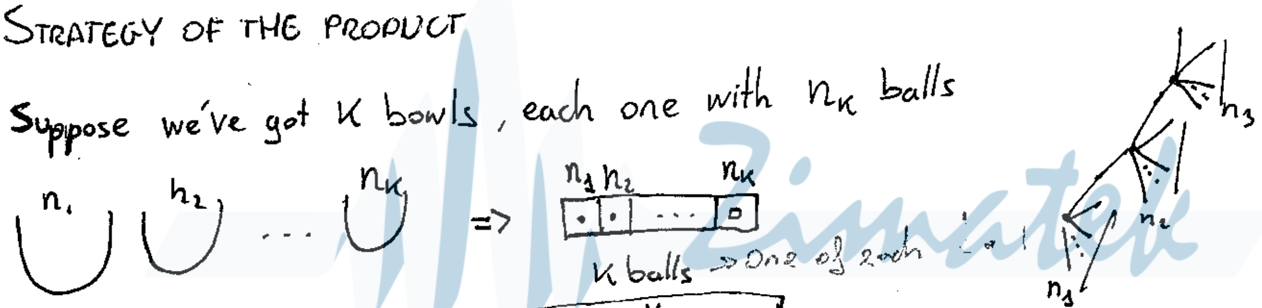
PERMUTATIONS, VARIATIONS & COMBINATIONS



$$P(A) = \frac{n_A}{n}$$

- STRATEGY OF THE PRODUCT

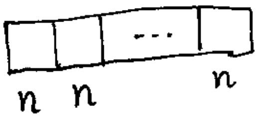
Ex.: Suppose we've got K bowls, each one with n_K balls



TOTAL Nº OF POSSIBILITIES: $\boxed{\eta = \prod_{i=1}^{K} n_K}$

- VARIATIONS WITH REPETITIONS

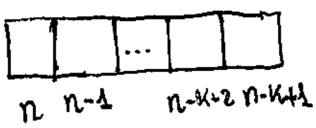
Ex.: One box with n balls. I pick K of these balls (putting them back each time).



$$\Longrightarrow \boxed{\eta = n^K}$$

- REGULAR VARIATIONS → # of ways n objects can be arranged in a group of K elements ($n > K$).

Ex.: Same as before, without putting the balls back in the box.



$$\Rightarrow \boxed{\eta = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-K+2)(n-K+1) = \frac{n!}{(n-K)!}} \equiv V_K^n$$

PERMUTATION if $K=n$ → # of ways n objects can be arranged → $n!$

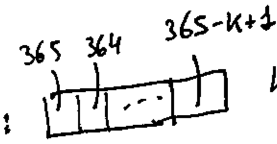
Problem:

A class of K people. What's the prob. of, at least, 2 of them having the same birthday?

A: Two people share their birthday.

$P(A) = 1 - P(\bar{A})$

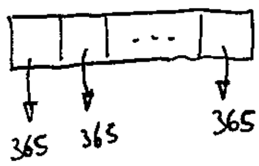
$$n_{\bar{A}} = \frac{n!}{(n-K)!} = \frac{365!}{(365-K)!} \rightarrow \text{\# of outcomes for } \bar{A} \Rightarrow \text{Nobody shares birthday}:$$



K people

$$\Rightarrow P(A) = 1 - \frac{365!}{(365-K)! \cdot 365^K}$$

$N = 365^K \rightarrow$



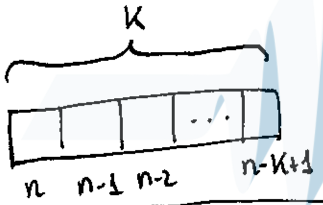
K people

$\downarrow$

Total possible events

- COMBINATIONS

Ej.: One box with n balls. We pick K of them, but we don't care about the order of appearance.



There are $K!$ ways to order the balls. The same result appears $K!$ times.

$$\boxed{C_K^n = \frac{V_p^n}{K!} = \frac{n!}{(n-K)!\,K!}}$$

Problem:

I throw n coins. What's the probability of getting K heads? $\equiv A$

$$P(A) = \frac{n_A}{N}$$

$$\Rightarrow P(A) = \frac{n!}{(n-K)!\,K!\,2^n}$$

$N = 2^n$

$$n_A = \frac{V_K^n}{K!} = C_K^n = \frac{n!}{(n-K)!\,K!}$$

BINOMIAL DISTRIBUTION

- Experiments with 2 outcomes: Success $\to p$, Fail $\to (1-p)$
- We repeat the experiment n times.
- Probability of K successes?

$\underbrace{h\,t\,h\,h}_{k \text{ heads}}\,\underbrace{t\,t\ldots}_{n-k \text{ tails}}$

$P_{1} = p^{k} \cdot (1-p)^{n-k}$

$$\boxed{P(k) = \frac{n!}{(n-k)!\,k!}\, p^{k}(1-p)^{n-k}}$$

PROPERTIES:

- $\sum_{k=1}^{n} P(k) = 1$

$$\sum_{k=1}^{n} C_k^n\, x^k\, y^{n-k} = (x+y)^n$$

$$\sum_{k=1}^{n} C_k^n\, p^k (1-p)^{n-k} = (p+1-p)^n = 1^n = 1$$

- $\langle K \rangle = n \cdot p$

For a fair coin, $p = 1/2 \to \langle K \rangle = n/2$

- $\sigma^2 = np(1-p)$

$\sigma = \sqrt{np(1-p)}$

Example:

We throw a die 5 times. What's the probability of getting 3 sixes.

$n = 5$

$K = 3$ $\to P(3) = \frac{5!}{2!\,3!}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 = \frac{125}{486} \approx 0{,}26$

$p = \frac{1}{6}$

$1-p = \frac{5}{6}$

POISSON DISTRIBUTION

- Experiment with 2 outcomes $\begin{cases} p & \text{success} \\ 1-p & \text{fail} \end{cases}$

- The experiment is repeated 'n' times, $n \gg 1$. ↗ Or in a continuum (over time)

- The probability of success is very little, $p \ll 1$, $\Rightarrow \langle k \rangle = np \to \lambda$

What is the probability of getting K successes?

$$\boxed{P_{POISSON}(k) = \frac{\lambda^k e^{-\lambda}}{k!}}$$

Proof:

$$P_{BIN.}(k) = \frac{n!}{(n-k)!\,k!}\, p^k (1-p)^{n-k}$$

$$\frac{n!}{(n-k)!} = \overbrace{n(n-1)(n-2)\cdot\ldots\cdot(n-(k-1))}^{k \text{ factors}} \xrightarrow{n \gg 1} n^k$$

$$(1-p)^{n-k} \longrightarrow \left(1-\frac{\lambda}{n}\right)^n \simeq e^{-\lambda}$$

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

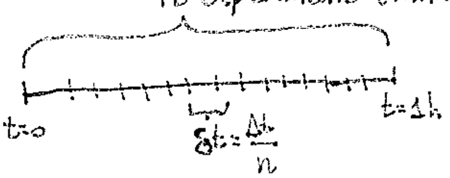
$$P(k) = \frac{\cancel{n^k}\, e^{-\lambda} \lambda^k}{\cancel{n^k}\, k!} = \frac{e^{-\lambda}\lambda^k}{k!}$$

Example:

A person receives an email every half an hour. Assuming that these are sent randomly, what's the probability that they receive $x = 0,1,2,3,4$ emails in an hour?

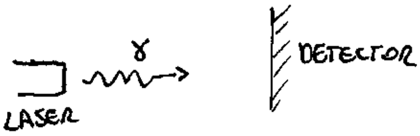
In one hour, $\lambda = 2$.

$$P_{POISSON} = \frac{2^k e^{-2}}{k!}$$



↓ Can be made as big as we want
↓ Prob. goes down.

Example: Low intensity laser

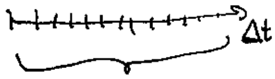


How many fotons are detected in Δt?

$$\langle n_{\Delta t} \rangle = I \cdot \Delta t = \lambda$$

$$I = \left\langle \frac{dn}{dt} \right\rangle$$

$$P(k) = \frac{(I\Delta t)^k e^{-I\Delta t}}{k!}$$



PROPERTIES:

- $\langle k \rangle = \lambda$
- $\sigma^2 = \lambda \rightarrow \sigma^2_{bin.} = np(1-p) \xrightarrow[p \to 0]{np \to \lambda}$
- $\sigma = \sqrt{\lambda}$

Example:

- 200 bulbs
- 0,5% prob. of being defective

$P(0, 1, 2, \ldots)$?
$\downarrow$
defective

FIND λ $\rightarrow \langle k \rangle = 200 \cdot p = 1 = \lambda$

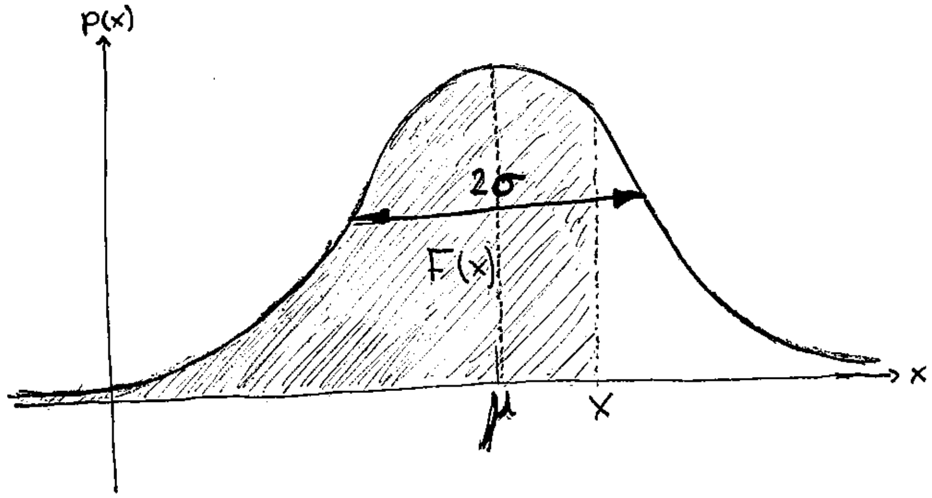
$$P_{poisson}(k) = \frac{1^k e^{-1}}{k!} = \frac{1}{k!\, e}$$

GAUSSIAN DISTRIBUTION

P.D.F. $\rightarrow$ $\boxed{p(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]}$, $x \in (-\infty, \infty)$

PROPERTIES:

- Mean: $\langle x \rangle = \mu$
- Variance: $\sigma^2 = \langle (x - \langle x \rangle)^2 \rangle$



$\sigma \downarrow \Rightarrow$ Narrow

$\sigma \uparrow \Rightarrow$ Wide

C.P.F. $\longrightarrow F(x) = P_r(X < x) = \int_{I_1}^{x} p(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$

cumulative *
prob. function

THE STANDARD GAUSSIAN:

$$p(z)\,dz = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right] \quad (\mu = 0,\ \sigma = 1)$$

z : standard variable

$$z \equiv \frac{x-\mu}{\sigma}$$

- $\Phi^*(z) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left[-\frac{z^2}{2}\right]$ $\rightarrow$ Prob. $(Z_1 < z)$

- $1 - \Phi(z) = \text{Prob}\,(Z_1 \geq z)$

- $\Phi(z_2) - \Phi(z_1) = \text{Prob}\,(z_1 \leq Z_1 \leq z_2)$

$$\boxed{F(x) = \text{Prob}\,(X < x) = \Phi\left(\frac{x-\mu}{\sigma}\right)}$$

$\text{Prob}\,(\mu - \sigma \leq x \leq \mu + \sigma) \approx 0{,}68$

$\text{Prob}\,(\mu - 2\sigma \leq x \leq \mu + 2\sigma) \approx 0{,}95$

$\text{Prob}\,(\mu - 3\sigma \leq x \leq \mu + 3\sigma) \approx 0{,}997$

Central Limit Theorem

- n random variables $\{x_1, \dots, x_n\}$, all independent, which have their own distributions $\{P_1(x), \dots, P_n(x)\}$
- The mean and variance of $P_i(x_i)$ is μ_i and σ_i^2.
- We define the following random variable: $y = \dfrac{\sum_i x_i}{n}$.

i) $E[y] = \langle y \rangle = \sum_i \dfrac{\mu_i}{n}$

ii) $\sigma_y^2 = V[y] = \sum_i \dfrac{\sigma_i^2}{n^2}$

iii) If $n \to \infty$
$P(y) \to$ Gaussian

Example: Measuring a quantity

$P_i(x) = P(x)$

$\mu_i = \mu$ $\quad y = \sum_i \dfrac{x_i}{n}$, $\quad P(y) = \dfrac{1}{\sqrt{2\pi}(\sigma/\sqrt{n})} \exp\left[-\dfrac{(y-\mu)^2}{2(\sigma/\sqrt{n})^2}\right]$

$\sigma_i = \sigma$

$\hookrightarrow \sigma_y^2 = \dfrac{n\sigma^2}{n^2} = \dfrac{\sigma^2}{n}$

If $n \to \infty$

$\dfrac{\sigma}{\sqrt{n}} = \sigma_y \to 0$

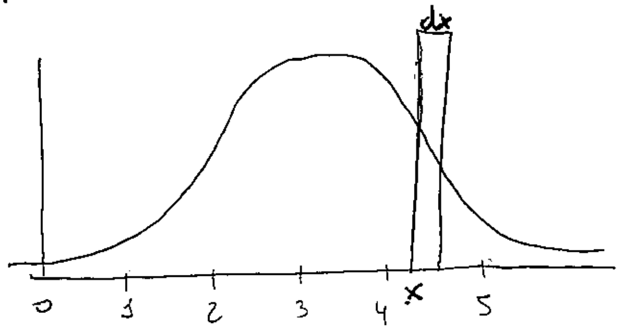
There are situations in which the Poisson and binomial distribs. can be approximated to the Gaussian.

GAUSSIAN APPROXIMATION OF THE BINOMIAL DISTRIB.

$$\text{Bin}(n,p) \underset{\substack{n\to\infty \\ p \to \text{fixed}}}{\sim} \text{Normal}\left(\mu = np,\ \sigma^2 = np(1-p)\right)$$

$\underline{n \geq 10}$

$p(x)\,dx = \text{Prob}(x \leqslant X \leqslant x + dx)$



$$P_{BIN}(k) \approx \int_{x-1/2}^{x+1/2} p_{gaussian}(x) \cdot dx$$

$$P_{BIN}(k < k_1) = \sum_{k=0}^{k=k_1} P_{bin.}(k) \approx \int_{0-1/2}^{k_1 - 1/2} p_g(x)\,dx$$

$$P_{BIN}(k_1 \leqslant k \leqslant k_2) = \int_{k_1-1/2}^{k_2+1/2} P_g(x)\,dx.$$

Example:

200 chips
10% defective

$\Big\{$ Prob (15 or more defective)

↳ Can't use Poisson, as p is large.

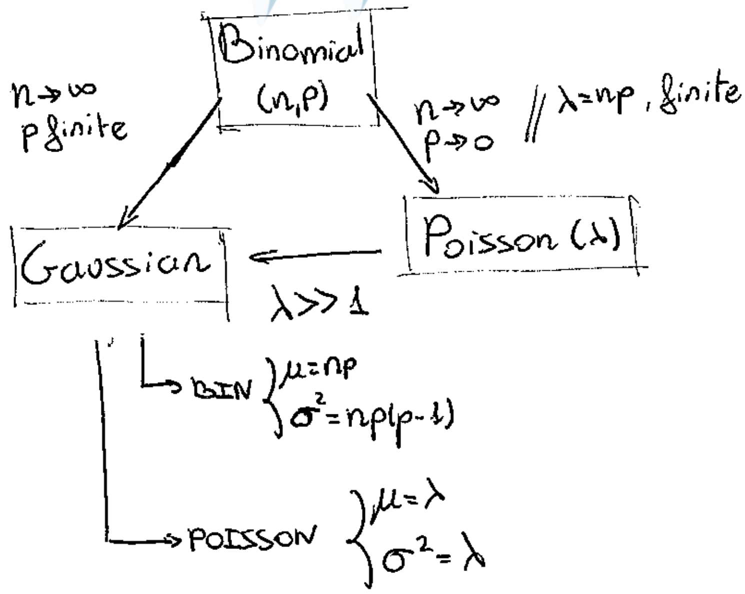
$$P = \sum_{k=15}^{200} P_{bin}(x) \approx \int_{15-1/2}^{200-1/2} P_g(x)\,dx$$

GAUSSIAN APPROX. OF POISSON DISTRIB.

$$\text{Poisson}(\lambda) \sim \text{Normal}(\mu = \lambda,\ \sigma^2 = \lambda)$$

$\lambda \to \infty$

$\underline{\lambda \geqslant 10}$



Problem:

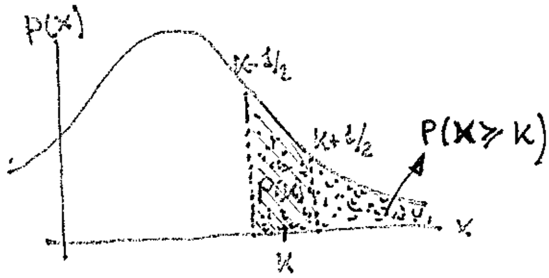
A person receives 1 email/hour. Prob. of receiving ≥ 24 emails in one day?

$\lambda = 24$

$$P(\lambda \geq 24) = 1 - \sum_{K=1}^{23} P_{POIS.}(K)$$

We can use the Gaussian approximation:

$$P(x) = \frac{1}{\sqrt{2\pi}\sqrt{\lambda}} \exp\left[-\frac{(x-\lambda)^2}{2\lambda}\right]$$



$$P(x \geq 24) = \int_{24-1/2}^{\infty} \frac{1}{\sqrt{2\pi\lambda}} \exp\left[-\frac{(x-\lambda)^2}{2\lambda}\right] dx$$

$Z = X + Y$, X & Y independent

random variables

$X \sim P_x(x) \rightarrow \mu_x, \sigma_x^2$

$Y \sim P_y(x) \rightarrow \mu_y, \sigma_y^2$

$$\langle Z \rangle = \langle X \rangle + \langle Y \rangle = \mu_x + \mu_y$$

$$\sigma_z^2 = \langle (z - \langle z \rangle)^2 \rangle = \langle (X + Y - \mu_x - \mu_y)^2 \rangle =$$

$$= \langle (X - \mu_x)^2 + (Y - \mu_y)^2 + 2(X - \mu_x)(Y - \mu_y) \rangle =$$

$$= \underbrace{\langle (X - \mu_x)^2 \rangle}_{= \sigma_x^2} + \underbrace{\langle (Y - \mu_y)^2 \rangle}_{= \sigma_y^2} + 2\langle (X - \mu_x)(Y - \mu_y) \rangle$$

$= \Rightarrow X, Y$ indep.

$$2 \underbrace{\langle (X - \mu_x) \rangle}_{=0} \underbrace{\langle (Y - \mu_y) \rangle}_{=0} = 0$$

$$\sigma_z^2 = \sigma_x^2 + \sigma_y^2$$

Suppose K_1, K_2 indep. random vars.

* $K_1 \overset{\text{comes from}}{\sim} P_{bin.}(n_1, p)$

$K_2 \sim P_{bin}(n_2, P)$

$K_3 \equiv K_1 + K_2$

* n_1 coins in the one hand, n_2 in the other.

Doesn't matter if a throw them separately or altogether.

$$\boxed{K_3 \sim P_{bin}(n_1+n_2, p)}$$

* $K_1 \sim P_{pois.}(\lambda_1)$

$K_2 \sim P_{pois}(\lambda_2)$

$K_3 \equiv K_1 + K_2$

$\langle K_3 \rangle = \langle K_1 + K_2 \rangle = \langle K_1 \rangle + \langle K_2 \rangle = np_1 + np_2 = n(p_1 + p_2)$

$$\boxed{K_3 \sim P_{pois.}(\lambda = \lambda_1 + \lambda_2)}$$

* GAUSSIAN COMPOSITION

$X \sim P_{Gauss}(\mu_1, \sigma_1)$

$Y \sim P_{Gauss}(\mu_2, \sigma_2)$

$Z \equiv X + Y$

$$Z \sim P_{Gauss}\left(\mu = \mu_1 + \mu_2, \sigma_3 = \sqrt{\sigma_1^2 + \sigma_2^2}\right)$$

FUNCTIONS OF RANDOM VARIABLES (II)

$z = f(x, y)$, x, y indep.

$X \sim p_x(x) < \begin{matrix} \mu_x \\ \sigma_x^2 \end{matrix}$

μ_z, σ_z^2 ?

$Y \sim P_y(y) < \begin{matrix} \mu_y \\ \sigma_y^2 \end{matrix}$

$$z = f(\mu_x, \mu_y) + \left.\frac{\partial f}{\partial x}\right|_{\mu_x} (x - \mu_x) + \left.\frac{\partial f}{\partial y}\right|_{\mu_y} (y - \mu_y) + O(2)$$

$$\langle z \rangle \overset{\text{linearity}}{=} f(\mu_x, \mu_y) + \cancel{\left.\frac{\partial f}{\partial x}\right|_{\mu_x} \overset{0}{\langle x - \mu_x \rangle}} + \cancel{\left.\frac{\partial f}{\partial y}\right|_{\mu_y} \overset{0}{\langle y - \mu_y \rangle}} + \ldots = f(\mu_x, \mu_y)$$

$$\sigma_z^2 = \left(\frac{\partial f}{\partial x}\right)^2\Big|_{\mu_x}\sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2\Big|_{\mu_y}\sigma_y^2$$

$$\|$$

$$\langle (z - \langle z\rangle)^2\rangle = \langle (f(x,y) - f(\mu_x,\mu_y))^2\rangle = \left\langle \left(\frac{\partial f}{\partial x}\Big|_{\mu_x}(x-\mu_x) + \frac{\partial f}{\partial y}\Big|_{\mu_y}(y-\mu_y)\right)^2\right\rangle$$

BIVARIATE DISTRIBUTIONS

- Discrete distributions:

$$X \in \{x_1, \dots, x_{n_x}\}, \quad Y \in \{y_1, \dots, y_{n_y}\}$$

$$p(X,Y) = \begin{cases} \text{Prob}(X_i \cap Y_i), & \text{if } X = x_i \text{ and } Y = y_i \\ 0 & \text{, otherwise} \end{cases}$$

$$\sum_{\forall i,j} p(x_i, y_i) = 1$$

$$F(x_j, y_k) = \text{Prob}(X \leq x_j, Y \leq y_k) = \sum_{i=1}^{j}\sum_{l=1}^{k} p(x_i, y_l)$$

If X and Y are independent: $p(x_i, y_j) = p(x_i)\,p(y_j)$

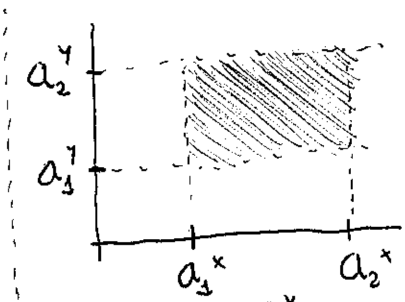
- Continuous distributions:

$$X \in [I_1^x, I_2^x] \quad , \quad Y \in [I_1^y, I_2^y]$$

$$p(x,y)\,dx\,dy = \text{Prob}(x \leq X \leq x+dx \,,\; y \leq Y \leq y+dy)$$

$$\int_{I_1^x}^{I_2^x} dx \int_{I_1^y}^{I_2^y} dy\, p(x,y) = 1$$

$$F(x,y) = \int_{I_1^x}^{x} d\tilde{x} \int_{I_1^y}^{y} d\tilde{y}\, p(\tilde{x},\tilde{y})$$



$$P = \int_{a_1^x}^{a_2^x} d\tilde{x} \int_{a_1^y}^{a_2^y} d\tilde{y}\, p(\tilde{x},\tilde{y})$$

$$\langle \underbrace{x}_{g(x,y)} y \rangle = \int_{I_1^x}^{I_2^x} dx \int_{I_1^y}^{I_2^y} dy \, \underbrace{x}_{g(x,y)} \, p(x,y)$$

Marginal Distributions

$$P_x(x)dx \equiv \text{Prob}(x \leq X \leq x+dx) = \int_{I_1^y}^{I_2^y} d\tilde{y} \, P(x,\tilde{y})$$

Conditional Distributions

$$y = y_0 \longrightarrow g(x) = \frac{P(x,y_0)}{P_y(y_0)}$$

Covariance and Correlation

$$X,Y \begin{cases} \text{Continuous} \rightarrow p(x,y)dx\,dy \\ \text{Discrete} \rightarrow p(x,y) \end{cases}$$

If they're independent: $p(x,y)dx\,dy = P_x(x)\,P_y(y)\,dx\,dy$

- Covariance

$$\text{Cov}[x,y] = \langle (X-\mu_x)(Y-\mu_y) \rangle$$

$$\overset{\text{CONT}}{=} \int_{I_1^x}^{I_2^x} dx \int_{I_1^y}^{I_2^y} dy \, p(x,y)(x-\mu_x)(y-\mu_y)$$

If X, Y are indep. $\rightarrow$ $\text{Cov}[x,y] = \left[\int_{I_1^x}^{I_2^x} dx \, P_x(x)(x-\mu_x)\right]\left[\int_{I_1^y}^{I_2^y} dy \, P_y(y)(y-\mu_y)\right] =$

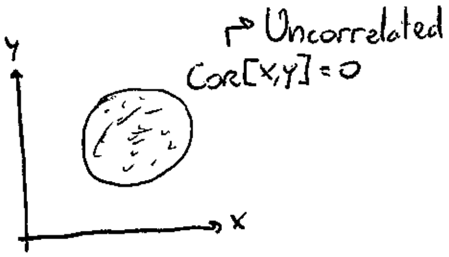
$$= \langle X-\mu_x \rangle \langle Y-\mu_y \rangle = 0$$

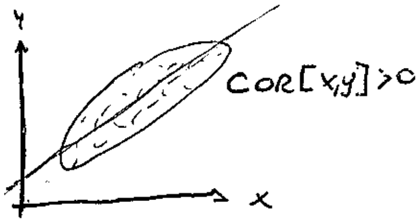
- Correlation

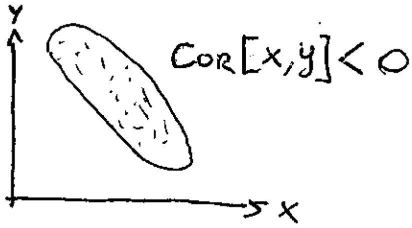
$$\text{Cor}[x,y] = \frac{\text{Cov}[x,y]}{\sigma_x \cdot \sigma_y} \in [-1,1]$$

$\sigma_x^2 = \langle (x-\mu_x)^2 \rangle$

$\sigma_y^2 = \langle (Y-\mu_y)^2 \rangle$







Function of correlated random variables

$$Z = f(x,y) \qquad \mu_z, \sigma_z$$

$$f(\mu_x, \mu_y) = \mu_z$$

$$f(x,y) \approx f(\mu_x, \mu_y) + \left.\frac{\partial f}{\partial x}\right|_{\mu_x}(x-\mu_x) + \left.\frac{\partial f}{\partial y}\right|_{\mu_y}(y-\mu_y) + O(2)$$

$$\sigma_z^2 = \langle (z - \langle z \rangle)^2 \rangle = \left\langle \left(\left.\frac{\partial f}{\partial x}\right|_{\mu_x}(x-\mu_x) + \left.\frac{\partial f}{\partial y}\right|_{\mu_y}(y-\mu_y) \right)^2 \right\rangle =$$

$$\doteq \left(\left.\frac{\partial f}{\partial x}\right|_{\mu_x}\right)^2 \langle (x-\mu_x)^2 \rangle + \left(\left.\frac{\partial f}{\partial y}\right|_{\mu_y}\right)^2 \langle (y-\mu_y)^2 \rangle + 2\left(\left.\frac{\partial f}{\partial x}\right|_{\mu_x} \left.\frac{\partial f}{\partial y}\right|_{\mu_y}\right) \langle (x-\mu_x)(y-\mu_y) \rangle =$$

$$= \left[\left.\frac{\partial f}{\partial x}\right|_{\mu_x}\right]^2 \sigma_x^2 + \left[\left.\frac{\partial f}{\partial y}\right|_{\mu_y}\right]^2 \sigma_y^2 + 2\left(\left.\frac{\partial f}{\partial x}\right|_{\mu_x} \left.\frac{\partial f}{\partial y}\right|_{\mu_y}\right) \mathrm{Cov}[x,y]$$

$\vec{X} = \{x_1, \ldots, x_N\}$: results of N experiments

SAMPLE

POPULATION → $p(x_1, \ldots, x_i, \ldots, x_n)$; joint prob. distribution for the sample.

If x_i are obtained under the same conditions, then,

$$p(x_1, \ldots, x_n) = p(x_1) \cdot p(x_2) \cdot \ldots \cdot p(x_n)$$

In general, the population is written as

$$p(\vec{x} | \vec{a}) : \begin{cases} \vec{x} = \{x_1, \ldots, x_n\} \\ \vec{a} = \{a_1, \ldots, a_m\} \end{cases}$$

STATISTICS (FUNCTIONS OF THE DATA)

- MEAN : $\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i$
  - Geometric mean : $\bar{x}_g = \left(\prod_{i=1}^{N} x_i \right)^{1/N}$
  - Harmonic mean : $\bar{x}_h = \frac{N}{\sum_{i=1}^{N} \frac{1}{x_i}}$
- VARIANCE : $s^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2$
  - Standard deviation : s
- MOMENTS
  - r-th moment : $m_r = \frac{1}{N} \sum_{i=1}^{N} x_i^r$
  - r-th central moment : $n_r = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^r$

• COVARIANCE / CORRELATION

$x \longrightarrow \vec{X} = (x_1, \dots, x_n)$ (x, measured n times)

$y \longrightarrow \vec{Y} = (y_1, \dots, y_n)$ (")

$$V_{xy} = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y}) \rightarrow \text{COVARIANCE}$$

$$\varphi_{xy} = \frac{V_{xy}}{S_x S_y} \rightarrow \text{CORRELATION}$$

ESTIMATORS

Example:

One random. var. (x) which follows the Gaussian distrib.:

$$p(x)dx = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$\vec{a} = (\mu, \sigma) \rightarrow \text{estimator}$$

$x \longrightarrow \vec{x} = (x_1, \dots, x_N)$: SAMPLE (we've repeated the exp. N times)

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

$$p(\bar{x} \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma/\sqrt{N}} \exp\left[-\frac{(\bar{x}-\mu)^2}{2\,\sigma^2/N}\right]$$ CENTRAL LIMIT THEOREM

(CONT.)

Repeat an exp. N times: $\vec{x} = (x_1, \dots, x_n)$ SAMPLE

$P(\vec{x}) \rightarrow$ Population $\rightarrow$ Depends on parameters $(a_1, \dots, a_m) = \vec{a}$: $P(\vec{x} \mid \vec{a})$

Estimator: $\hat{a}(\vec{x})$

$P(\vec{x} \mid \vec{a}) = p(x_1 \mid \vec{a}) \cdot \dots \cdot p(x_n \mid \vec{a}) \rightarrow$ N indep. and identical experiments

$\hat{a}(\vec{x})$ is a random variable $\longrightarrow P(\hat{a} \mid \vec{a})$: SAMPLING DISTRIBUTION

$$P(\hat{a} \mid \vec{a})\,d\hat{a} = P(\vec{x} \mid \vec{a}) \underbrace{d^N \vec{x}}_{\hat{a}(\vec{x}) \in [\hat{a},\, \hat{a} + d\hat{a}]}$$

(CONT.)

$$P(\vec{x}|\vec{a})\, d^N x = \left(P(x_1|\sigma,\mu)\, dx_1\right)\left(P(x_2|\sigma,\mu)\, dx_2\right)\cdots\left(P(x_N|\sigma,\mu)\, dx_N\right) =$$

$$= \frac{1}{(\sqrt{2\pi}\,\sigma)^N}\cdot \exp\left[-\frac{\sum_{i=1}^{N}(x_i-\mu)^N}{2\sigma^2}\right] d^N x$$

$$\underbrace{\hat{a}(x) = \frac{1}{N}\sum_{i=1}^{N} x_i = \bar{X}}_{\text{FUNCTION OF THE DATA}}$$

(Should give an expected value of x)
(μ is the theoretical exp. value)

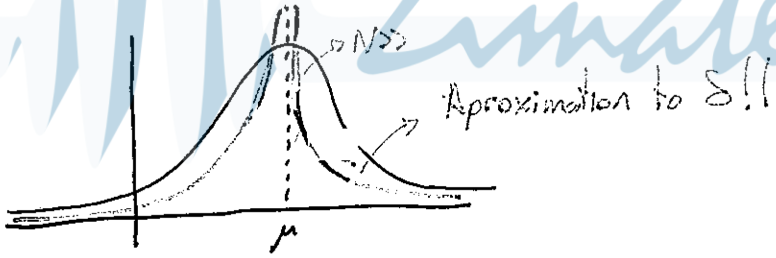
Sampling distribution : $P(\hat{a}|\vec{a})\, d\hat{a} \longrightarrow P(\bar{x}|\mu,\sigma)\, d\bar{x} =$

$$= \frac{1}{\sqrt{2\pi}\,\sigma/\sqrt{N}}\exp\left[-\frac{(\bar{x}-\mu)^2}{2\sigma^2/N}\right] dx$$

PROPERTIES OF ESTIMATORS:

- Consistency

$$\lim_{N\to\infty} \hat{a}(x) = a \Rightarrow P(\hat{a}|\vec{a})\, d\hat{a} \xrightarrow{N\to\infty} \delta(\hat{a}-a)\, d\hat{a}$$



- Bias

$$b(\hat{a}) = \langle \hat{a} \rangle - a$$

$$\langle \bar{x} \rangle = \int_{-\infty}^{\infty} d\bar{x}\, \frac{1}{\sqrt{2\pi}\,\sigma/_N}\exp\left[-\frac{(\bar{x}-\mu)^2}{2\sigma^2/N}\right] = \mu \rightarrow \underline{\text{Unbiased}}$$

- Efficiency

Variance of $\hat{a}$: $\langle (\hat{a}-\langle\hat{a}\rangle)^2 \rangle = \int_{-\infty}^{\infty} (\hat{a}-\langle\hat{a}\rangle)\, P(\vec{x}|\vec{a})\, d^N x$

FISHER'S INEQUALITY : $\langle (\hat{a}-\langle\hat{a}\rangle)^2 \rangle \geq V_{min} \equiv \dfrac{\left(1+\frac{\partial b}{\partial a}\right)^2}{\left\langle -\frac{\partial^2 \ln P}{\partial a^2}\right\rangle}$ → population

Efficiency: $e \equiv \dfrac{V_{min}}{\langle(\hat{a}-\langle a\rangle)^2\rangle} \leq 1$ $\begin{cases} = 1 \rightarrow \text{efficient} \\ < 1 \rightarrow \text{inefficient} \end{cases}$

Ex.: Efficiency of $\bar{x}$ in a Gaussian distr.

Sample dist.: $P(\bar{x}|\sigma,\mu)\,d\bar{x} = \frac{1}{\sqrt{2\pi}\,\sigma/\sqrt{N}} \exp\left[\frac{-(\bar{x}-\mu)^2}{2\sigma^2/N}\right] d\bar{x}$

$\langle(\bar{x}-\langle\bar{x}\rangle)^2\rangle = \frac{\sigma^2}{N}$ $\quad\|\quad$ $P(\vec{x}|\sigma,\mu) = \prod_{i=1}^{N}\left[\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)\right]$

$$\left.\begin{aligned} V_{min} &= \frac{1}{\langle\frac{N}{\sigma^2}\rangle} \overset{b=0}{=} \frac{\sigma^2}{N} \\ \text{Variance}\,(\bar{x}) &= \frac{\sigma^2}{N} \end{aligned}\right\} e = 1$$

$\hookrightarrow \ln P = \sum_{i=1}^{N}\left[-\ln(2\pi\sigma) - \frac{(x_i-\mu)^2}{2\sigma^2}\right]$

$$\frac{\partial^2 \ln P}{\partial\mu^2} = -\frac{N}{\sigma^2}$$

CONFIDENCE LIMITS / CONFIDENCE LEVELS

$\hat{a} \to$ Function of data

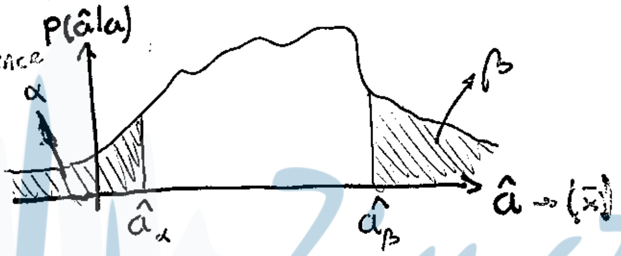
$a \to$ Theoretical value

$\hat{a}(x) \longrightarrow P(\hat{a}|\underline{a})\,d\hat{a}$

Given two reals $\alpha, \beta < 1$, we define:

$$\left\{\begin{aligned} P(\hat{a} < \hat{a}_\alpha) &= \alpha \\ P(\hat{a} > \hat{a}_\beta) &= \beta \end{aligned}\right.$$

confidence levels



$\downarrow$

$P(\hat{a}_\alpha < \hat{a} < \hat{a}_\beta) = 1-\alpha-\beta$

$$P(\hat{a} < \hat{a}_\alpha) = \int_{-\infty}^{\hat{a}_\alpha(a)} d\hat{a}\, p(\hat{a}|a) = \alpha$$

$$P(\hat{a}_\beta > \hat{a}) = \int_{\hat{a}_\beta(a)}^{\infty} d\hat{a}\, p(\hat{a}|a) = \beta$$

$$\int_{-\infty}^{\hat{a}_{obs}} P(\hat{a}|a_+)\,d\hat{a} = \alpha$$

$$\int_{\hat{a}_{obs}}^{\infty} P(\hat{a}|a_-)\,d\hat{a} = \beta$$

$$\vec{x} \Rightarrow \underset{\text{observed}}{\hat{a}_{obs}(\vec{x})} \longrightarrow \left\{\begin{aligned} \hat{a}_\alpha(a_+) &\equiv \hat{a}_{obs} \\ \hat{a}_\beta(a_-) &\equiv \hat{a}_{obs} \end{aligned}\right.$$

$$\alpha = P(\hat{a}_{obs} < \hat{a}_\alpha(a)) = P(\hat{a}_\alpha^{-1}(\hat{a}_{obs}) < a) = P(a_+ < a)$$

$$\beta = P(\hat{a}_{obs} > \hat{a}_\beta(a)) = P(\hat{a}_\beta^{-1}(\hat{a}_{obs}) > a) = P(a_- > a)$$

$$P(a_- < a < a_+) = 1-\alpha-\beta$$

$$\boxed{a = \hat{a}_{obs}{}^{+d}_{-c} \Rightarrow \begin{cases} d \equiv a_+ - \hat{a}_{obs} \\ c \equiv \hat{a}_{obs} - a_- \end{cases}}$$

$\alpha = \beta \Leftrightarrow$ central confidence interval

$a \in [a_-, a_+]$

$P(a_- \leq a \leq a_+) = 1 - \alpha - \beta$

Example: Sampling distr.: Gaussian

$$P(\hat{a}|a)\,d\hat{a} = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(\hat{a}-a)^2}{2\sigma^2}\right]$$

$$P(\hat{a} < a_\alpha(a)) = \alpha \rightarrow P(\hat{a}_{obs} < a_\alpha(a_+)) = \alpha = \int_{-\infty}^{\hat{a}_{obs}} d\hat{a}\, \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(\hat{a}-a_+)^2}{2\sigma^2}\right] = \Phi\left(\frac{\hat{a}_{obs}-a_+}{\sigma}\right)$$

$$P(\hat{a} > a_\beta(a)) = \beta \rightarrow P(\hat{a}_{obs} > a_\beta(a_-)) = \beta = \int_{\hat{a}_{obs}}^{\infty} d\hat{a}\, \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(\hat{a}-a_-)^2}{2\sigma^2}\right] = 1 - \Phi\left(\frac{\hat{a}_{obs}-a_-}{\sigma}\right)$$

$$\hat{a}_{obs} - a_+ = \sigma\,\Phi^{-1}(\alpha) \rightarrow \boxed{a_+ = \hat{a}_{obs} - \sigma\,\Phi^{-1}(\alpha)}$$

$$\hat{a}_{obs} - a_- = \sigma\,\Phi^{-1}(1-\beta) \rightarrow \boxed{a_- = \hat{a}_{obs} - \sigma\,\Phi^{-1}(1-\beta)}$$

STANDARD ERROR OF AN ESTIMATOR

$$\hat{a}(\vec{x}) \rightarrow P(\hat{a}|a)\,d\hat{a} \rightarrow Var[\hat{a}] = \langle(\hat{a} - \langle\hat{a}\rangle)^2\rangle \rightarrow \sigma_{\hat{a}} = \sqrt{Var[\hat{a}]}$$

If $\hat{a}(\vec{x})$ is unbiased $\rightarrow$ std. error $\equiv \sigma_{\hat{a}} \rightarrow \hat{a} = \hat{a}_{obs} \pm \sigma_{\hat{a}}$

If $\hat{a}(\vec{x})$ is biased $\rightarrow$ $\varepsilon_{\hat{a}}^2 = \langle(\hat{a}-a)^2\rangle =$

$$= \langle(\hat{a} - \langle\hat{a}\rangle + \langle\hat{a}\rangle - a)^2\rangle =$$

$$= \langle(\hat{a} - \langle\hat{a}\rangle)^2\rangle + \langle(\langle a\rangle - a)^2\rangle = \boxed{\sigma_{\hat{a}}^2 + b^2(a)}$$

Ex.: • DATA:

X_i ($i=1,\dots,10$)	2,22	2,56	1,07	0,24	0,18	0,95	0,73	-0,79	2,09	1,81

• $X_i \xrightarrow{\text{GAUSSIAN}} \sigma = 1,\ \mu?$

$$\bar{X} = \frac{1}{N}\sum_{n=1}^{10} x_n = 1,11$$

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{N}}$$

ESTIMATORS OF THE VARIANCE

$$\begin{cases} P(\hat{a}\,|\,\vec{a}) \\ \vec{a}=(\mu,\sigma,\dots) \end{cases}$$

When μ is Known:

$$\hat{\sigma}_a^2 = \frac{1}{N}\cdot\sum_{i=1}^{N}(x_i-\mu)^2$$

When μ is UNKnown:

sample variance

$$\hat{\sigma}_a^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i-\bar{x})^2 = \frac{N}{N-1}S^2$$

BIVARIATE DISTRIBUTIONS : COVARIANCE & CORRELATION ESTIMATORS

$$p(x,y) \rightarrow Cov[x,y] = \frac{1}{N}\sum_{i=1}^{N}(x_i-\bar{x})(y_i-\bar{y}) \equiv V_{xy}$$

$$\widehat{Cov[x,y]} = \frac{1}{N-1}\sum_{i=1}^{N}(x_i-\bar{x})(y_i-\bar{y}) \rightarrow \text{Population covariance} \left(=\frac{N}{N-1}V_{xy}\right)$$

$$Corr[x,y] = \frac{cov[x,y]}{S_x S_y} = r_{xy}$$

$$\widehat{Corr[x,y]} = \frac{N}{N-1} r_{xy}$$

$$Cov[x,y] = \frac{1}{N}\sum_{i=1}^{N}(x_i y_i - x_i\bar{y} - \bar{x}y_i + \bar{x}\bar{y}) = \langle xy\rangle - \langle x\rangle\langle y\rangle$$

Example: DATA: {same as previous ex.} = $\vec{X} = \{x_1, \dots, x_i, \dots, x_{10}\}$

$$X_i \sim \text{GAUSSIAN}[\underbrace{\mu, \sigma}_{\vec{a}=(\mu,\sigma)}] \Rightarrow P(x_i) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right]$$

Estimator for μ : $\bar{X} = \frac{1}{N}\sum_{i=1}^{10} x_i = 1{,}11 \rightarrow \mu = \bar{X} \pm \sigma_{\bar{X}} = 1{,}11 \pm 0{,}36$

$$\Downarrow$$

$$P(\vec{x}|\mu,\sigma) = \prod_{i=1}^{10} G[\mu,\sigma](x_i)$$

$$\hat{\sigma}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2 = (1{,}06)^2 \rightarrow \hat{\sigma} = 1{,}06$$

$$P(\bar{x}|\mu,\sigma_{\bar{x}}) \overset{N\to\infty}{\sim} \text{GAUSSIAN}\left[\mu, \frac{\sigma}{\sqrt{N}}\right]$$

$$\sigma_{\bar{X}} = \frac{1{,}06}{\sqrt{10}} = 0{,}36$$

• 90% central confidence limits?

$$\mu \in [\mu_-, \mu_+] \Leftrightarrow P(\mu_- \leq \mu \leq \mu_+) = 0{,}9 = 1-\alpha-\beta = 1-2\alpha \rightarrow \alpha = 0{,}05 = \beta$$

$[a_-, a_+]$

$$\alpha = \int_{-\infty}^{\hat{a}_{obs}} d\hat{a}\, P(\hat{a}|a_+) = \int_{-\infty}^{\bar{x}} du \frac{1}{\sqrt{2\pi}\,\sigma_{\bar{x}}} \exp\left[-\frac{(\bar{x}-\mu_+)^2}{2\sigma_{\bar{x}}^2}\right] du = F(\bar{X}|\mu_+, \sigma_{\bar{x}}) =$$

$$= \Phi\left(\frac{\bar{x}-\mu_+}{\sigma_{\bar{x}}}\right) = \alpha \rightarrow \mu_+ = \underset{1{,}11}{\bar{X}} - \underset{0{,}36}{\sigma_{\bar{x}}}\, \Phi^{-1}(\underset{0{,}05}{\alpha}) = 1{,}64$$

$$\Phi^{-1}(1-\alpha) = -\Phi^{-1}(\alpha)$$

$$\alpha = \beta = \int_{\bar{x}}^{\infty} du \frac{1}{\sqrt{2\pi}\,\sigma_{\bar{x}}} \exp\left[-\frac{(u-\mu_-)^2}{2\sigma_{\bar{x}}}\right] = 1 - \int_{-\infty}^{\bar{x}} (\dots) = 1 - F(\bar{x}|\mu,\sigma_{\bar{x}}) =$$

$$= 1 - \Phi\left(\frac{\bar{x}-\mu_-}{\sigma_{\bar{x}}}\right) = \overset{\alpha}{\beta} \rightarrow \Phi^{-1}(1-\alpha) = \frac{\bar{x}-\mu_-}{\sigma_{\bar{x}}} \rightarrow \mu_- = \underset{1{,}11}{\bar{X}} - \Phi^{-1}(\underset{0{,}95}{1-\alpha})\,\underset{0{,}36}{\sigma_{\bar{x}}} = 0{,}58$$

$$\mu \in [0{,}58 \,,\, 1{,}64] \;;\; \text{Prob}(0{,}58 \leq \mu \leq 1{,}64) = 0{,}9$$

① $\downarrow P$ i) $a = \dfrac{mg - Kv}{m} \overset{\text{DOWNWARDS}}{=} -g - \dfrac{K}{m}\vec{V} \Rightarrow \dfrac{d^2x}{dt^2} = -g - \dfrac{K}{m}\dfrac{dx}{dt} \Rightarrow y'' = g - \dfrac{K}{m}y'$

OPPOSED TO VELOCITY

ii) $a = \dfrac{mg - Kv^2}{m} \Rightarrow \dfrac{d^2x}{dt^2} = -g - \dfrac{K}{m}\left(\dfrac{dx}{dt}\right)^2 \Rightarrow y'' = -g - \dfrac{k}{m}(y')^2$

② $y^2 - 2y = x^2 - x - 1$ sol. for $2y' = \dfrac{2x-1}{y-1}$

$\Downarrow$

$g(x,y) = y^2 - 2y - x^2 + x + 1 = 0 \Rightarrow y' = -\dfrac{\partial g/\partial x}{\partial g/\partial y} = -\dfrac{-2x+1}{2y-2} = \dfrac{2x-1}{2y-2} \rightarrow 2y' = \dfrac{2x-1}{y-1}$

$\rightarrow y = 1 \pm \sqrt{x(x-1)} \Rightarrow$ Def. interval. $\Rightarrow (-\infty, 0] \cup [1, \infty)$

$\hookrightarrow y^2 - 2y - (x^2 - x - 1) = 0 \longrightarrow y = \dfrac{2 \pm \sqrt{4 + 4(x^2 - x - 1)}}{2}$

③ $y' = \dfrac{xy}{x^2 - 1}$

Is $x^2 + Cy^2 = 1$ a sol.?

$g(x,y) = x^2 + Cy^2 - 1 = 0 \Rightarrow C = \dfrac{1 - x^2}{y^2}$

$y' = -\dfrac{2x}{2Cy} = -\dfrac{x}{Cy} = -\dfrac{y^2 x}{(1-x^2)y} = -\dfrac{xy}{1-x^2}$

④ Consider $x + yy' = 0$

Is $y = \begin{cases} \sqrt{c^2 - x^2} & x < 0 \\ -\sqrt{c^2 - x^2} & x \geq 0 \end{cases}$ a solution?

$y' = \begin{cases} \dfrac{-2x}{2\sqrt{c^2 - x^2}} = -\dfrac{x}{y} \\ -\dfrac{-2x}{2\sqrt{c^2 - x^2}} = \dfrac{x}{-y} \end{cases} \Rightarrow$ It might be, but it's not continuous at $x = 0$.

$\lim_{x \to 0} y = \begin{cases} c \\ -c \end{cases}$

As y appears in the eq., it must be continuous according to our expression for it to be admissible as a solution.

⑥ $y = \frac{1-Ce^{2x}}{1+Ce^{2x}}$ $\quad y' = y^2 - 1$

$$y' = \frac{-2Ce^{2x}(1+Ce^{2x}) - 2Ce^{2x}(1-Ce^{2x})}{(1+Ce^{2x})} = -\frac{4c^2e^{2x}}{(1+Ce^{2x})^2}$$

$$-\frac{4c^2e^{2x}}{(1+Ce^{2x})} = \underbrace{\frac{(1-Ce^{2x})^2}{(1+Ce^{2x})^2}}_{y^2} - 1 \Rightarrow \checkmark$$

Two easy sols. $\Rightarrow$ $y = a,\ y' = 0 \to 0 = a^2 - 1 \to a = \pm 1.$

We should ask ourselves whether $y \equiv \pm 1$ is included in the general solution. So, are there values of C which give us either $y = 1$ or $y = -1$?

$y = 1 \longrightarrow C = 0$
$y = -1 \longrightarrow C = \infty$
$\Rightarrow$ They're particular solutions!

⑦ $xy' = y$
i) $y(0) = 0$ $\Rightarrow$ $0 \cdot y' = y(0) \to y(0) = 0$ ✓

ii) Remember the normal form to discuss uniqueness:

$y' = f(x,y)$

In our case, $y' = \frac{y}{x} \to$ Not continuous at $(0,0)$, which is the reason of the non-uniqueness.

iii) $y = \begin{cases} 0 & x \leq 0 \\ x & x \geq 0 \end{cases}$

$y' = \begin{cases} 0 & x \leq 0 \\ 1 & x \geq 0 \end{cases}$ $\Rightarrow$ y' is not continuous at $x = 0$, so it's not admissible.

⑧ $y' = 2\sqrt{y}$, $y(0) = 0$ admits as a sol. $y = \begin{cases} 0, & x \leq c \\ (x-c)^2, & x \geq c \end{cases}$

$y' = \begin{cases} 0 & x \leq c \\ 2(x-c) & x \geq c \end{cases} \rightarrow \sqrt{y} = \begin{cases} 0 & x \leq c \\ \sqrt{2(x-c)} & x \geq c \end{cases}$ both y & y' are continuous ✓

PROBLEMS → 1, 2, 3, 6, 7, 8, 9, 10, 11, 17, 20, 23

(11) $y' - x^2 y = x^5$ → Linear

$A = -x^2$

$B = x^5$

(6) $(1-x^2)y' = 1-y^2$ → Separable

$$\frac{dy}{1-y^2} = \frac{dx}{1-x^2}$$

(8) $y' = \frac{2(x^3y) - (y^4)}{(x^4) - 2(xy^3)}$ → Homogeneous (also admits an integrating factor of the sort $\mu(xy)$)

4th power 4th power ! 4th power 4th power

(23) $y' = \left(\frac{x-y+3}{x-y+1}\right)^2$ ⇒ Parallel lines

(20)

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(2) $\frac{(1+y^2)dx + (1+x^2)dy}{(1-xy)^2} = 0 = \frac{1+y^2}{(1-xy)^2}dx + \frac{1+x^2}{(1-xy)^2}dy$

P Q

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} = \frac{2x+2y-2x^2y-2xy^2}{(1-xy)^2}$$

SOL. → $\int P\,dx + \int \left\{ Q - \int \frac{\partial P}{\partial y} dx \right\} dy = 0$

$$\int \frac{1+y^2}{(1-xy)^2}dx = (1+y^2)\int \frac{dx}{(1-xy)^2} = (1+y^2)\cdot\frac{-1}{(-y)(1-xy)} = \frac{1+y^2}{y(1-xy)}$$

$$\cancel{\int \frac{\partial P}{\partial y}dx = \int \frac{2x+2y-2x^2y-2xy^2}{(1-xy)^2}dx = 2\int\frac{x\cdot dx}{(1-xy)^2} + 2y\int\frac{dx}{(1-xy)^2} - 2y\int\frac{x^2\,dx}{(1-xy)^2} -}$$

$$\cancel{-2y^2\int\frac{x\,dx}{(1-xy)^2} = 2\cdot\left(\frac{1}{(-y)^2(1-xy)} + \frac{1}{(-y)^2}\ln(1-xy)\right) + 2y}$$

(2) $\dfrac{(1+y^2)dx + (1+x^2)dy}{(1-xy)^2} = 0$

Multiplying both sides by $\mu(x,y) = (1-xy)^2$ we obtain

$\mu = \dfrac{(1-xy)^2}{(1+y^2)(1+x^2)}$

$(1+y^2)dx + (1+x^2)dy = 0$

This is a separable eq., so:

$$\frac{(1+y^2)}{(1+x^2)(1+y^2)}dx + \frac{(1+x^2)}{(1+x^2)(1+y^2)}dy = 0 = \frac{dx}{(1+x^2)} = -\frac{dy}{(1+y^2)}$$

$\text{arctg}\, x = -\text{arctg}\, y + C \to$ Gen.

(1) a) $y = Kx$, $y' = K$ $\Big\{\; y = y'x$

b) $y = C\cdot\sin 2x \to C = \dfrac{y}{\sin 2x}$

$y' = 2C\cdot\cos 2x \longrightarrow y' = 2\cdot\dfrac{y}{\sin 2x}\cdot\cos 2x = 2y\,\text{cotg}\,2x$

(3) $(ye^{xy} + 2x)dx + (x\cdot e^{xy} - 2y)dy = 0$

EXACT

$\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} = e^{xy} + xye^{xy}$

(...)

$u = e^{xy} + x^2 - y^2 + C$

⑥ $(1-x^2)y' = 1-y^2 \Rightarrow$ Separable

$$\frac{dy}{1-y^2} = \frac{dx}{1-x^2} \Rightarrow \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + \frac{1}{2}\ln C$$

$$\frac{1+y}{1-y} = \frac{1+x}{1-x}\cdot C \rightarrow y = \frac{C(1+x)+x-1}{C(1+x)-x+1}$$

$y(1)=1 \rightarrow$ for all C

$y(1)=-1 \rightarrow C=0$

⑦ What's the condition that the eq. $Pdx + Qdy$ has to satisfy for it to admit $\mu(xy)$?

$$\frac{\mu'}{\mu} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q\frac{\partial h}{\partial x} - P\frac{\partial h}{\partial y}} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Qy - Px} \underset{\substack{\uparrow \\ \text{in ex.} \\ 2.8}}{=} \frac{6y^3-6x^3}{3xy(x^3-y^3)} = -\frac{2}{xy}$$

In particular

$\Downarrow$

Must be a pure function of $h = xy$

$\mu = \frac{1}{(xy)^2}$ is an int. factor for ex. 2.8

⑩ $2xy\,dx + (1-x^2-y^2)\,dy = 0$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

$$\frac{\partial}{\partial y}(\mu(y)P) = \frac{\partial}{\partial x}(\mu(y)Q)$$

$$\mu' 2xy + \mu(y)2x = \mu\cdot(-2x) \rightarrow \frac{\mu'(y)}{\mu(y)} = (\ln \mu(y))' = -\frac{2}{y} \Rightarrow \mu = y^{-2} = \frac{1}{y^2}$$

$$\frac{\partial u}{\partial x} = \frac{2x}{y} \Rightarrow u = \frac{x^2}{y} + h(y)$$

$$\frac{\partial u}{\partial y} = \frac{1}{y^2} - \frac{x^2}{y^2} - 1 = -\frac{x^2}{y^2} + h'(y) \rightarrow h(y) = -\frac{1}{y} - y + C$$

$$u = \frac{x^2}{y} - y - \frac{1}{y} + C$$

(11) $y' - x^2 y = x^5$ Linear inhomogeneous

$$\int A\cdot dx = -\int x^2\, dx = -\frac{x^3}{3}$$

$$y = e^{-\int A dx} \cdot \int B\cdot e^{\int A dx}\, dx + e^{-\int A dx}\cdot C =$$

$$= e^{\frac{x^3}{3}} \cdot \underbrace{\int x^5 \cdot e^{-\frac{x^3}{3}}\, dx}_{I} + e^{\frac{x^3}{3}}\cdot C$$

$$I = -\int -x^5 \cdot e^{-\frac{x^3}{3}}\, dx = -x^3 \cdot e^{-\frac{x^3}{3}} - \int e^{-\frac{x^3}{3}} \cdot 3x^2 \cdot dx = -x^3 e^{-\frac{x^3}{3}} + 3\cdot e^{-\frac{x^3}{3}}$$

$u = x^3$

$dv = -x^2 \cdot e^{-\frac{x^3}{3}} dx$

(20) $(6x+4y+3)dx + (3x+2y+2)dy = 0$

$$y' = \frac{-6x-4y-3}{3x+2y+2} \Rightarrow \text{Parallel lines}$$

$u = -6x - 4y$

$u' = -6 - 4y'$

$$\Rightarrow y' = \frac{u-3}{-\frac{u}{2}+2}$$

$$u' = -6 - \frac{4u-12}{-\frac{u+4}{2}} = -6 - \frac{8u-24}{-u+4} \Rightarrow \int \frac{du}{-6-\frac{8u-24}{-u+4}} = \int dx + C \Rightarrow$$

$$\Rightarrow 2\cdot \ln(-6x-4y) - \frac{1}{2}(-6x-4y) + x = C$$

(23) $y' = \left(\dfrac{x-y+3}{x-y+1}\right)^2$

$u = x-y$
$u' = 1-y'$ $\rightarrow y' = \left(\dfrac{u-3}{u-1}\right)^2$

$\hookrightarrow u' = 1-\left(\dfrac{u-3}{u-1}\right)^2 \rightarrow \displaystyle\int \frac{du}{1-\left(\frac{u-3}{u-1}\right)^2} = \int dx + C$

$$x + C = \int \frac{du}{1-\left(\frac{u-3}{u-1}\right)^2} = \int \frac{u^2+2u+1}{-4u-8}\,du \rightarrow \frac{(x-y)^2}{2} + \ln(x-y+2) = 4x + C$$

(17) $(y^2-1)dx + (3x^2-2xy)dy = 0$

$\dfrac{\partial P}{\partial x} - \dfrac{\partial Q}{\partial y} = 2y - 6x + 2y = 4y - 6x = 2(2y-3x) = -\dfrac{2}{x}(3x^2-2xy) = -\dfrac{2}{x}Q$

Not exact ($\neq 0$)

$\boxed{\mu = x^{-2}}$

Gen sol. $\rightarrow x = \dfrac{y^2-1}{3y+C}$

$\{1, 2, 5, 7, 8, 12-15, 18-21, 23, 24, 39\}$

① $(1+y'^2)y''' = 3y'(y'')^2$

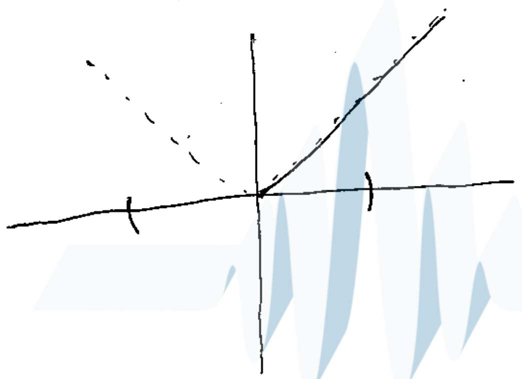
② $y^{V} - \frac{1}{x}y^{IV} = 0 \xrightarrow{u = y^{IV}} u' - \frac{1}{x}u = 0 \rightarrow (\dots) \rightarrow u = C_1 x = y^{IV}$

$$y = C_1 x^5 + C_2 x^3 + C_3 x^2 + C_4 x + C_5$$

⑤ $y_1 = x$

$y_2 = |x|$

a) Linearly indep. on $(-1,1)$?



$C_1 x + C_2 |x| = 0$

$\downarrow$

Let us evaluate at $x = -1$ & $x = 1$:

$x = 1 \rightarrow C_1 + C_2 = 0$

$x = -1 \rightarrow -C_1 + C_2 = 0$

$\rightarrow C_1 = C_2 = 0$

$\hookrightarrow$ Linearly indep.

b) Wronskian?

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & |x| \\ 1 & \frac{|x|}{x} \end{vmatrix} = 0$$

$$\frac{|x|}{x} \hookrightarrow \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

But,

linear dependence $\Longrightarrow W = 0$

$W = 0 \not\Longrightarrow$ linear dep.

⑦ a) $x, x^2 \to W\{x, x^2, y\} = 0$

$$0 = W = \begin{vmatrix} x & x^2 & y \\ 1 & 2x & y' \\ 0 & 2 & y'' \end{vmatrix} = x \begin{vmatrix} 2x & y' \\ 2 & y'' \end{vmatrix} - 1 \begin{vmatrix} x^2 & y \\ 2 & y'' \end{vmatrix} = x^2 y'' - x \cdot 2y' + 2y = 0$$

$\hookrightarrow$ y is a lin. comb. of x, x^2

b) $x+1, x^2+1 \to W\{x+1, x^2+1, y\} = 0$

$$0 = \begin{vmatrix} x+1 & x^2+1 & y \\ 1 & 2x & y' \\ 0 & 2 & y'' \end{vmatrix} = (\dots) = (x^2+2x+1)y'' - 2(x+1)y' + 2y = 0$$

c) $x, e^x \to W\{x, e^{2x}, y\} = 0$

$$0 = \begin{vmatrix} x & e^{2x} & y \\ 1 & 2e^{2x} & y' \\ 0 & 4e^{2x} & y'' \end{vmatrix} = (\dots) = y''(2xe^{2x} - e^{2x}) - 4xe^{2x}y' + 4e^{2x}y = 0$$

⑧ $(x+1)y'' + xy' - y = (x+1)^2 \to$ LINEAR INHOMOGENEOUS

Hom. part:

$$(x+1)y'' + xy' - y = 0$$

$$y'' + \frac{x}{x+1}y' - \frac{y}{x+1} = 0$$

Part. sol. $\to y_1 = x$

Applying d'Alembert's method, we solve the homogeneous eq.:

$$y_{hom} = C_1(-e^{-x}) + C_2 x$$

Applying the variation of parameters method, we find that a particular sol. of the inhomogeneous eq. is:

$$y_{part} = x^2 + 1$$

Then, the sol of the inh. eq. is:

$$y_{inh} = y_{hom} + y_{part} = C_1(-e^{-x}) + C_2 x + x^2 + 1$$

---

(12) $y'' - y = xe^x$

We solve the hom. part first:

$$y'' - y = 0$$

$$\downarrow$$

$$K^2 - 1 = 0 = (K+1)(K-1)$$

$$y_{hom} = C_1 \cdot e^x + C_2 \cdot e^{-x}$$

Now, we'll do the same as in the previous ex. to get y_p:

$$\begin{cases} g'e^x + h'e^{-x} = 0 & (1) \\ g'e^x - h'e^{-x} = xe^x & (2) \end{cases}$$

$(1) \to h' = -\frac{1}{2}xe^{2x} \to h = -\frac{e^{2x}}{4}\left(x - \frac{1}{2}\right)$

$(1) + (2) \to g' = \frac{x}{2} \to g = \frac{x^2}{4}$

$$y_{inhom.} = C_1 e^x + C_2 e^{-x} + \frac{x^2}{4}e^x - \frac{e^x}{4}\left(x - \frac{1}{2}\right)$$

(14) $(D^3+D)y = 1 + e^{2x} + \cos x$

$\underbrace{D(D^2+1)y}$ $= 1 + e^{2x} + \cos x$

hom. eq. $D(D^2+1)=0$

$K(K^2+1)=0 \rightarrow \begin{cases} K=0 \\ K=\pm i \end{cases} \rightarrow y_h = A + B\cos x + C\cdot\sin x$

$y_p = y_{p1} + y_{p2} + y_{p3}$

$y_{p1} = (ax+b)$

$y'''_{p1} + y'_{p1} = 1.$

$\left.\begin{array}{l} y'_{p1} = a \\ y'''_{p1} = 0 \end{array}\right\} y'''_{p1} + y'_{p1} = a = 1 \rightarrow y_{p1} = x + b \rightarrow$ As I need a single part. sol., I can choose $b=0$ because I prefer it.

$\boxed{y_{p1} = x}$

$y_{p2} = a\cdot e^{2x}$

$y'''_{p2} + y'_{p2} = e^{2x}$

$\left.\begin{array}{l} y'_{p2} = 2ae^{2x} \\ y''_{p2} = 4ae^{2x} \\ y'''_{p2} = 8ae^{2x} \end{array}\right\} y'''_{p2} + y'_{p2} = 8ae^{2x} + 2ae^{2x} = 10ae^{2x} = e^{2x}$

$a = \frac{1}{10}$

$\boxed{y_{p2} = \frac{1}{10}e^{2x}}$

$y_{p3} = (ax+b)\cos x + (cx+d)\sin x$

$y'''_{p3} + y'_{p3} = \cos x$

$(\ldots) \rightarrow y_{p3} = -\frac{x}{2}\cos x$

$y_{inh} = y_{hom} + y_{p1} + y_{p2} + y_{p3}$

(15) $(D+1)^3 y = e^{-x} + x^2$

For the hom. sol.:

$(D+1)^3 y = 0$

$(K+1)^3 = 0 \Rightarrow K = -1 \ (3)$

$y_{hom} = Ae^{-x} + Bxe^{-x} + Cx^2e^{-x}$

Now we'll go for the particular sols.:

$(D+1)^3 y = (D^3 + 3D^2 + 3D + 1) y = e^{-x} + x^2$

$y'''_{p1} + 3y''_{p1} + 3y'_{p1} + y_{p1} = e^{-x}$

Let's try $y_{p1} = (ax^3 + bx^2 + cx + d)e^{-x}$ → We've raisen the order as Ae^{-x} is included in the y_{hom} as well as Bxe^{-x} & Cx^2e^{-x}.

$y'_{p1} =$

$y''_{p1} =$

$y'''_{p1} =$

For the y_{p2}, we'll try: $y_{p2} = ax^2 + bx + c$ → It's not included in the gen. sol., so we won't raise the order.

$y'''_{p2} + 3y''_{p2} + 3y'_{p2} + y_{p2} = x^2$

$y_{p2}' = 2ax + b$
$y_{p2}'' = 2a$
$y_{p3}''' = 0$

$\rightarrow 0 + 3 \cdot 2a + 3 \cdot (2ax + b) + ax^2 + bx + c = x^2$

$a = 1$
$c = -6a - 3b \rightarrow c = -6 + 18 = 12$
$6a = -b \rightarrow b = -6$

(20) $y'' + 10y' + 25y = 2^x + xe^{-5x}$

$$(D+5)^2 = e^{x\ln 2} + x \cdot e^{-5x}$$

5 is a double root, and we have an x in front of e^{-5x}.

Try $y = (ax^3 + bx^2 + cx + d)e^{-5x}$

① Solve:

$$\begin{cases} \ddot{x} = y \\ \ddot{y} = x \end{cases}$$

1^{st} option: make it a single eq. $\rightarrow \ddot{x} = y \rightarrow \dddot{x} = \dot{y} \rightarrow x^{(IV)} = \ddot{y}$

$$x^{(IV)} - x = 0$$

2^{nd} option: work with the system $\rightarrow$ Write a system of 1^{st} order eqs. $\rightarrow$ 2^{nd} order derivatives $\times$ 2 vars. = 4 new vars

$$\begin{cases} x_1 = x \longrightarrow \boxed{\dot{x}_1 = \dot{x} = x^3} \\ x_2 = y \longrightarrow \boxed{\dot{x}_2 = \dot{y} = x_4} \\ x_3 = \dot{x} \longrightarrow \dot{x}_3 = \ddot{x} = y = x_2 \\ x_4 = \dot{y} \longrightarrow \dot{x}_4 = \ddot{y} = x = x_1 \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_4 \\ \dot{x}_3 = x_2 \\ \dot{x}_4 = x_1 \end{cases}$$

1^{st} option

$$x^{(IV)} - x = 0$$

$\downarrow$ linear, ctant. coef., hom.

$$K^4 - 1 = 0 \xrightarrow{K^2 = r} r^2 = \pm 1 \longrightarrow K = \{1, -1, i, -i\}$$

$$x = Ae^t + Be^{-t} + C\cos t + D\sin t$$

$$y = \ddot{x} = Ae^t + Be^{-t} - C\cos t - D\sin t$$

2^{nd} option

$$\mathbb{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \rightarrow |\mathbb{A} - K\mathbb{I}| = \begin{vmatrix} -K & 0 & 1 & 0 \\ 0 & -K & 0 & 1 \\ 0 & 1 & -K & 0 \\ 1 & 0 & 0 & -K \end{vmatrix} = K^4 - 1 = 0$$

---

⑯

homogeneous part ($\mathbb{A}$)

$$\begin{cases} \dot{x} = x + y + z \\ \dot{y} = -2y + t \\ \dot{z} = 2z + \sin t \end{cases}$$

inhomogeneous part ($\vec{b}$)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \mathbb{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \vec{b}$$

$$\mathbb{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} 0 \\ t \\ \sin t \end{pmatrix}$$

$$|\mathbb{A}-K\mathbb{I}| = \begin{vmatrix} 1-K & 1 & 1 \\ 0 & -2-K & 0 \\ 0 & 0 & 2-K \end{vmatrix} = (2-K)(-2-K)(1-K) = 0 \rightarrow \begin{cases} K=1 \\ K=-2 \\ K=2 \end{cases}$$

$$\vec{X} = A\vec{V}_1 e^{t} + B\vec{V}_2 e^{-2t} + C\vec{V}_3 e^{2t}$$

$\vec{X}$ ↑ gen. sol of the hom.

eigenvector associated with the eigenvalue.

I can use any possible sol.

$$\mathbb{A}\vec{V}_1 = \underset{\substack{\downarrow \\ K=1}}{\lambda_1}\vec{V}_1 \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{cases} x+y+z = x \rightarrow x=x \\ -2y = y \rightarrow y=0 \\ 2z = z \rightarrow z=0 \end{cases} \rightarrow \vec{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbb{A}\vec{V}_2 = \lambda_2\vec{V}_2 \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{cases} x+y+\cancel{z} = -2x \rightarrow y=-3x \\ -2y = -2y \rightarrow y=y \\ 2z = -2z \rightarrow z=0 \end{cases} \rightarrow \vec{V}_2 = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$$

$$\mathbb{A}\vec{V}_3 = \lambda_3\vec{V}_3 \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{cases} x+y+z = 2x \rightarrow x=z \\ -2y = 2y \rightarrow y=0 \\ 2z = 2z \rightarrow z=z \end{cases} \rightarrow \vec{V}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{X}_h = A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}e^{t} + B\begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}e^{-2t} + C\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}e^{2t}$$

$$\vec{X}_h = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

hom. sol $\begin{cases} x = Ae^{t} + Be^{-2t} + Ce^{2t} \\ y = -3Be^{-2t} \\ z = Ce^{2t} \end{cases}$ and the fundamental matrix $\mathbb{F} = \begin{pmatrix} e^{t} & e^{-2t} & e^{2t} \\ 0 & -3e^{-2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}$

$$\vec{X}_p = \mathbb{F}\int \mathbb{F}^{-1}\vec{b}\,dt$$

$$\vec{X}_p = \begin{pmatrix} \frac{1}{3}\left(\frac{1}{4}-\frac{t}{2}\right) - \frac{1}{5}(\cos t + 2\sin t) + \frac{1}{6}\left(-2(1+t) + 3(\cos t + \sin t)\right) \\ -\frac{1}{4} + \frac{t}{2} \\ -\frac{1}{5}(\cos t + \sin 2t) \end{pmatrix}$$

⑤ Are these vectors linearly independent?

$\vec{V}_1 = \begin{pmatrix} 1 \\ t \end{pmatrix}$ $\quad A\vec{V}_1 + B\vec{V}_2 = 0 \Leftrightarrow A = B = 0$?

$\vec{V}_2 = \begin{pmatrix} t^2 \\ t^3 \end{pmatrix}$ $\quad A\begin{pmatrix} 1 \\ t \end{pmatrix} + B\begin{pmatrix} t^2 \\ t^3 \end{pmatrix} = \begin{pmatrix} A + Bt^2 \\ At + Bt^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{cases} A + Bt^2 = 0 \\ At + Bt^3 = 0 \rightarrow t(A+Bt^2) = 0 \xrightarrow{\forall t} A + Bt^2 = 0 \rightarrow \underline{\underline{A = B = 0}} \end{cases}$$

But, what happens to their Wronskian? Explain the result.

$$W = \begin{vmatrix} 1 & t^2 \\ t & t^3 \end{vmatrix} = t^3 - t^3 = 0$$

They are linearly indep., but they cannot be the two sols. of a 2D system.

㉕ $\begin{cases} \dot{x} - y + z = 0 \\ \dot{y} - x - y = t \\ \dot{z} - x - z = t \end{cases}$ $\quad \vec{X} = \vec{X}_h + \vec{X}_p$

Let us try the inhom. part first by "inspection."

$$\begin{cases} x = a + bt \\ y = c + dt \\ z = e + ft \end{cases}$$

$$\begin{cases} b - c - dt + e + ft = 0 \rightarrow (b - c + e) + (-d + f)t = 0 \\ d - a - bt - c - dt = t \rightarrow (d - a - c) + (-b - d)t = t \\ f - a - bt - e - ft = t \rightarrow (f - a - e) + (-b - f)t = t \end{cases} \rightarrow \begin{cases} b - c - e = 0 \\ -d + f = 0 \\ d - a - c = 0 \\ -b - d = 1 \\ f - a - e = 0 \\ -b - f = 1 \end{cases}$$

<u>Solution</u> $\rightarrow a = b = 0, \; c = d = e = f = -1$

For the homogeneous part;

$K = 0, 1$ double

$\quad$ | needed sols.

$A, \; Be^t, \; Cte^t$

(1) $\ddot{x} + x = \begin{cases} e^t \sin t & 0 < t < \pi/2 \\ 0 & t \geq \pi/2 \end{cases} = e^t \sin t \left(\theta(t) - \theta(t - \pi/2)\right)$

$x(0) = \dot{x}(0) = 0$

$$\mathcal{L}[\ddot{x} + x] = s^2 X(s) + X(s) = (s^2+1) X$$

$$\mathcal{L}\left[e^t \sin t \,\theta(t)\right] = \frac{1}{(s-1)^2+1}$$

$$\mathcal{L}\left[e^t \sin t \cdot \theta(t-\pi/2)\right] = e^{\pi/2} \mathcal{L}\left[e^{t-\pi/2} \cdot \cos(t - \pi/2)\,\theta(t-\pi/2)\right] =$$

$$= e^{\pi/2}\left[e^{-\pi/2\, s} \frac{s-1}{(s-1)^2+1}\right]$$

$$X = \frac{1}{s^2+1} \cdot \frac{1}{(s-1)^2+1} + \frac{1}{s^2+1} \cdot \frac{e^{-\pi/2\, s}(s-1)}{(s-1)^2+1} e^{\pi/2}$$

$$\frac{1}{(s^2+1)\left[(s-1)^2+1\right]} = \frac{A_1 s + A_2}{s^2+1} + \frac{B_1 s + B_2}{(s-1)^2+1} = (\ldots)$$

$$X = \frac{1}{5}\left(\frac{2s+1}{s^2+1} + \boxed{\frac{-2s+3}{(s-1)^2+1}}\right) - \frac{e^{\frac{\pi}{2}(1-s)}}{5}\left(-\frac{s+3}{s^2+1} + \frac{s+1}{(s-1)^2+1}\right)$$

*

$$x(t) = \frac{1}{5}\left[2\cos t + \sin t - 2e^t \cos t + e^t \sin t\right]\theta(t) -$$

$$- \frac{e^{\frac{\pi}{2}}}{5}\left[-\left(\cos(t-\tfrac{\pi}{2}) + 3\sin(t-\tfrac{\pi}{2})\right) + e^{(t-\frac{\pi}{2})}\cos\left(t-\tfrac{\pi}{2}\right) + 2e^{t-\frac{\pi}{2}}\sin\left(t-\tfrac{\pi}{2}\right)\right]\theta\left(t-\tfrac{\pi}{2}\right)$$

* $\dfrac{-2s+3}{(s-1)^2+1} = -2\dfrac{s-1}{(s-1)^2+1} + \dfrac{1}{(s-1)^2+1}$

②

$$\begin{cases} \dot{x} - x + y = \theta(t) \\ \dot{y} - 2x - 4y = \theta(t-3) \end{cases} \xrightarrow{[\mathcal{L}]} \begin{cases} (s-1)X + Y - 7 = \dfrac{1}{s} & (1) \\ (s-4)Y - 2X + 7 = \dfrac{e^{-3s}}{s} & (2) \end{cases}$$

$x(0) = 7, \; y(0) = -7$

$$(-s+4)(1) + (2) \to -(2 + (s-1)(s-4))X = -7(s-4) - \frac{s-4}{s} + \frac{e^{-3s}}{s} - 7$$

$$-[(s-3)(s-2)]X = -7(s-4) + \frac{4}{s} + \frac{e^{-3s}}{s} - 8$$

$$X = \frac{7(s-4)}{(s-3)(s-2)} - 4\frac{1}{s(s-3)(s-2)} + \frac{8}{(s-3)(s-2)} - \frac{e^{-3s}}{s(s-3)(s-2)} =$$

$$= (\cdots) = 7\left[-\frac{1}{s-3} + \frac{2}{s-2}\right] - 4\left[\frac{1/6}{s} + \frac{1/3}{s-3} - \frac{1/2}{s-2}\right] + 8\left[\frac{1}{s-3} - \frac{1}{s-2}\right] -$$

$$- e^{-3s}\left[\frac{1/6}{s} + \frac{1/3}{s-3} - \frac{1/2}{s-2}\right] =$$

$$= -\frac{1/3}{s-3} + \frac{8}{s-2} - \frac{2/3}{s} - e^{-3s}\left[\frac{1/6}{s} + \frac{1/3}{s-3} - \frac{1/2}{s-2}\right] =$$

$$x(t) = \left[-\frac{1}{3}e^{3t} + 8e^{2t} - \frac{2}{3}\right]\theta(t) - \left[\frac{1}{6} + \frac{1}{3}e^{3(t-3)} - \frac{1}{2}e^{2(t-3)}\right]\theta(t-3)$$

$$Y = -(s-1)X + 7 + \frac{1}{s}$$

① $(x-1)^2 y'' + (x-1)y' + (x^2-2x)y = 0$

We'll solve the eq. around $x=1$.

$$\tilde{x} = x-1 \to \tilde{x}^2 y'' + \tilde{x} y' + (\tilde{x}^2 - 1)y = 0$$

$$\hookrightarrow \boxed{\nu = 1}$$

$$y = AJ_1(\tilde{x}) + BN_1(\tilde{x}) =$$

$$= AJ_1(x-1) + BN_1(x-1)$$

⑧ $xy'' - 2(x-1)y' + (x-2)y = 0$

$$x^2y'' - 2x(x-1)y' + x(x-2)y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+\lambda}$$

$$x^2y'' = \sum_{n=0}^{\infty} (n+\lambda)(n+\lambda-1)a_n x^{n+\lambda}$$

$$-2x^2y' = -2\sum_{n=0}^{\infty} a_n(n+\lambda)x^{n+\lambda+1} = -2\sum_{n=1}^{\infty} a_{n-1}(n+\lambda-1)x^{n+\lambda}$$

$$2xy' = 2\sum_{n=0}^{\infty} (n+\lambda)a_n x^{n+\lambda}$$

$$x^2 y = \sum_{n=0}^{\infty} a_n x^{n+\lambda+2} = \sum_{n=2}^{\infty} a_{n-2}x^{n+\lambda}$$

$$-2xy = -2\sum_{n=1}^{\infty} a_{n-1}x^{n+\lambda}$$

$$n=0 \to \lambda(\lambda-1)+2\lambda=0 \to \lambda(\lambda+1)=0 \to \lambda=-1,0$$

$$\lambda=0 \to n=1 \to 2a_1 - 2a_0 \overset{=0}{\to} a_1 = a_0$$

$$n=2 \to 2a_2 - 2a_1 + 4a_2 + a_0 - 2a_1 = 0 \to a_2 = \frac{1}{2}a_0$$

$$n=3 \to 6a_3 - 4a_2 + 6a_3 + \cancel{a_1 - 2a_2} = 0 \to a_3 = \frac{1}{6}a_0$$

...

$$a_n = \frac{a_0}{n!}$$

$$\boxed{y_1 = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x}$$

⑤ $2x^2y'' - x(4x+1)y' + (1+x+2x^2)y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^{n+\lambda}$$

$$xy = \sum_{n=1}^{\infty} a_{n-1} x^{n+\lambda}$$

$$2x^2y = \sum_{n=2}^{\infty} 2a_{n-2} x^{n+\lambda}$$

$$-xy' = -\sum_{n=0}^{\infty} a_n(n+\lambda) x^{n+\lambda}$$

$$-4x^2y' = -\sum_{n=1}^{\infty} 4a_{n-1}(n+\lambda-1)x^{n+\lambda}$$

$$2x^2y'' = \sum_{n=0}^{\infty} 2a_n(n+\lambda)(n+\lambda-1)x^{n+\lambda}$$

$n=0 \rightarrow (2\lambda^2 - 3\lambda + 1)a_0 = 0 \rightarrow \lambda = 1, 1/2$

$\boxed{\lambda = 1}$

$n=1 \rightarrow 3a_1 - 3a_0 = 0 \rightarrow a_1 = a_0$

$$\boxed{\begin{array}{l} y_1 = xe^x \\ y_2 = e^x\sqrt{x} \end{array}}$$

....

② $(x^2+1)y'' - 2xy' + 2y = x^3 + 3x$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$x^2y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^n$$

$$y'' = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} x^n$$

$$-2xy' = -2\sum_{n=0}^{\infty} n a_n x^n$$

$$2y = 2\sum_{n=0}^{\infty} a_n x^n$$

We'll solve the hom. part. first.

$(\dots)$

$$a_{n+2} = \frac{-n(n-1)+2n-2}{(n+1)(n+2)} a_n$$

$$y = a_0 + a_1 x + \underset{-a_0}{a_2 x^2} + \underset{0}{a_3 x^3} + \underset{0}{\dots}$$

$$= a_0 + a_1 x - a_0 x^2$$

$$n=0 \to a_2 = -\frac{2}{2} a_0 = -a_0$$

$$n=1 \to a_3 = 0 \to a_{2k+1} = 0, \quad \forall k = 1,2,3,\dots$$

$$n=2 \to a_4 = \frac{\sim}{\sim} = 0 \to a_{2k} = 0, \quad \forall k \geq 2$$

$$\boxed{y_{hom} = a_0(1-x^2) + a_1 x}$$

⑨ $x^2 y'' + xy' - (x^2 + \frac{1}{4})y = 0$

$$x^2 y'' = \sum_{n=0}^{\infty} (n+\lambda)(n+\lambda-1) x^{n+\lambda} a_n$$

$$xy' = \sum_{n=0}^{\infty} (n+\lambda) x^{n+\lambda} a_n$$

$$-x^2 y = \sum_{n=2}^{\infty} a_{n-2} x^{n+\lambda}$$

$$-\frac{1}{4} y = \sum_{n=0}^{\infty} -\frac{1}{4} a_n x^{n+\lambda}$$

$$n=0 \to \left[\lambda(\lambda-1) + \lambda - \frac{1}{4}\right] a_0 = 0 \to \lambda = \pm 1/2$$

$$n=1 \to \left[\lambda(\lambda+1) + (\lambda+1) - \frac{1}{4}\right] a_1 = 0 \to \underline{a_1 = 0}$$

$$a_n = \frac{a_{n-2}}{(n+\lambda)^2 - \frac{1}{4}} = \frac{a_{n-2}}{n(n+2\lambda)} \rightarrow a_{2k} = \frac{a_{2(k-1)}}{4k(k+\lambda)} = \frac{a_0}{2^{2k} k! (k+\lambda)(k+\lambda-1)\dots(\lambda+1)}$$

$$= a_0 \frac{\Gamma(\lambda+1)}{2^{2k} k! \Gamma(\lambda+k+1)}$$

$$y = a_0 2^k \Gamma(\lambda+1) \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(\lambda+k+1)} \left(\frac{x}{2}\right)^{2k+\lambda}$$

$$J_\lambda(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\lambda+k+1)} \left(\frac{x}{2}\right)^{2k+\lambda}$$

$$y_1 = c_1 J_{1/2}(ix) \rightarrow y_2 = c_2 J_{-1/2}(ix)$$

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$$(x^3 - 4x^2)y'' - (x^3 - 4x^2 + 4x)y' + (2x^2 - 10x + 16)y = 0$$

$$x^3 y'' = \sum_{n=1}^{\infty} a_{n-1}(n+\lambda-1)(n+\lambda)x^{n+\lambda}$$

$$-4x^2 y'' = \sum_{n=0}^{\infty} -4(n+\lambda)(n+\lambda-1)x^{n+\lambda} a_n$$

$$-x^3 y' = \sum_{n=2}^{\infty} -(n+\lambda-2)x^{n+\lambda} a_{n-2}$$

$$4x^2 y' = \sum_{n=1}^{\infty} 4(n+\lambda-1)x^{n+\lambda} a_{n-1}$$

$$-4xy' = \sum_{n=0}^{\infty} -4(n+\lambda)x^{n+\lambda} a_n$$

$$2x^2 y = \sum_{n=2}^{\infty} 2a_{n-2}x^{n+\lambda}$$

$$-10xy = \sum_{n=1}^{\infty} -10a_{n-1}x^{n+\lambda}$$

$$16y = \sum_{n=0}^{\infty} 16a_n x^{n+\lambda}$$

$n=0 \rightarrow [-4\lambda(\lambda-1)-4\lambda+16]a_0=0 \rightarrow \lambda=\pm 2 \rightarrow \underline{\lambda=2}$

$n=1 \rightarrow [a_0\cdot\lambda(\lambda-1)-4\lambda(\lambda+1)a_1+4\lambda a_0-4\lambda a_1-10a_0+16a_1]=0$

$a_1=0$

$\rightarrow a_n=0 \quad \forall n\geq 0$

$n=2 \rightarrow a_2=0$

$$y_1=\sum_{n=0}^{\infty} a_n x^{n+\lambda} = a_0 x^2 \xrightarrow{\text{D'ALEMBERT}} y_2=\ldots$$

(16) $xy''+xy'+y=0$.

$x^2y''+x^2y'+xy=0$

$$x^2y''=\sum_{n=0}^{\infty}(n+\lambda)(n+\lambda-1)\,a_n x^{n+\lambda}$$

$$x^2y'=\sum_{n=1}^{\infty}(n+\lambda-1)\,a_{n-1}x^{n+\lambda}$$

$$xy=\sum_{n=1}^{\infty}a_{n-1}x^{n+\lambda}$$

$n=0 \rightarrow \lambda(\lambda-1)=0 \rightarrow \lambda=0,1$

$$a_n=-\frac{\cancel{n+\lambda}}{\cancel{(n+\lambda)}(n+\lambda-1)}a_{n-1} \xrightarrow{\lambda=1} a_n=\frac{-a_{n-1}}{n} \rightarrow a_n=\frac{(-1)^n}{n!}$$

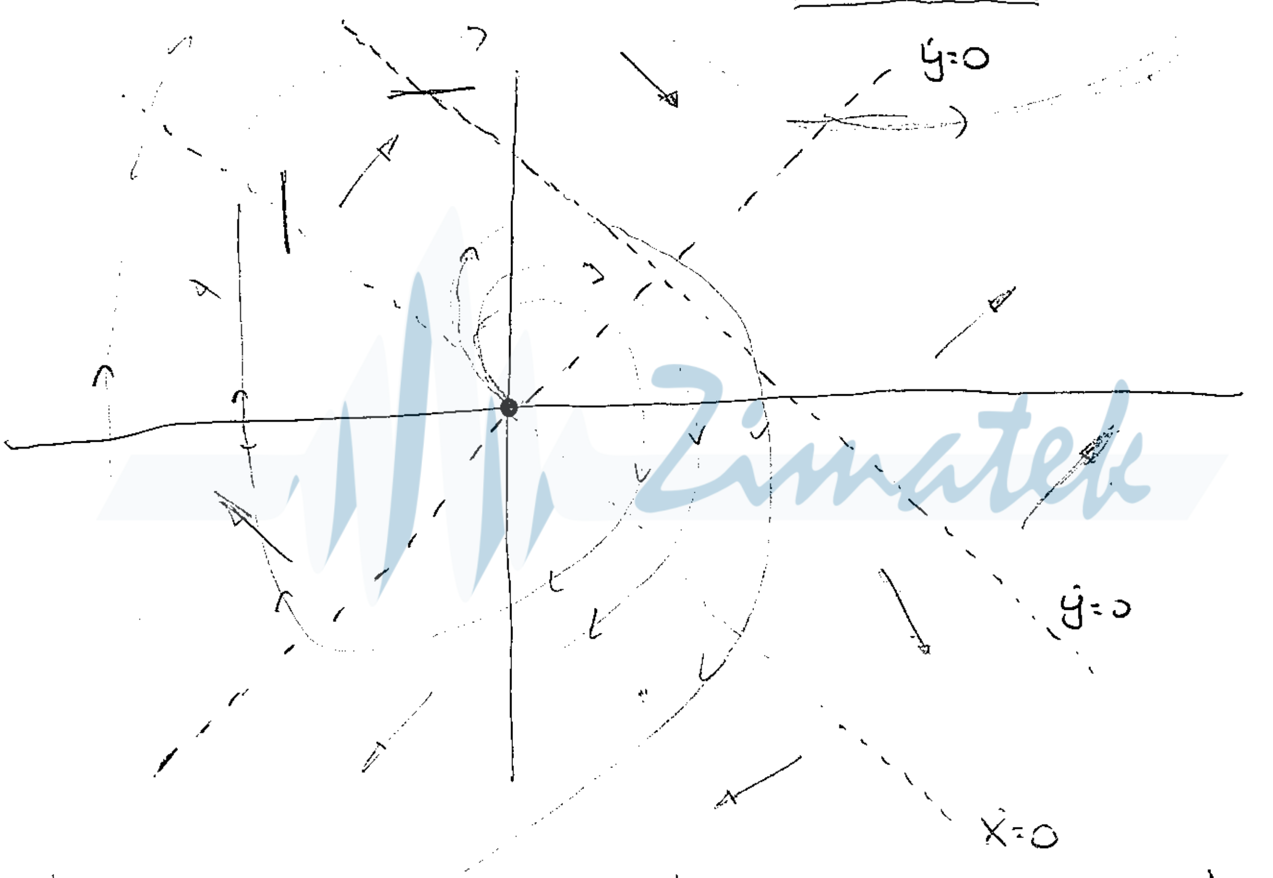
$y_1=x\cdot e^{-x}$

$$\circledast \quad \dot{x} = x+y$$
$$\dot{y} = (x-y)(x+y-1) \longrightarrow \begin{cases} \dot{x}=0 \to y=-x \\ \dot{y}=0 \to \begin{cases} y=1-x \\ y=x \end{cases} \end{cases}$$

Fixed points:

$\dot{x}=0 \to y=-x$

$\dot{y}=0 = (x+x)(x-x-1) = -2x \Rightarrow \boxed{\begin{array}{c} x=0 \to y=0 \\ P(0,0) \end{array}}$



Slope:

If $y=0$ & x very big:

$\dot{x}>0$

$\dot{y} = x(x-1)>0$

$$A = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix}_{(0,0)} = \begin{pmatrix} 1 & 1 \\ (x+y-1)+ & -(x+y-1)+ \\ +(x+y) & +(x+y) \end{pmatrix}_{(0,0)} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda+1 & 1 \\ -1 & -\lambda+1 \end{vmatrix} = (-\lambda+1)^2 + 1 = \lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i \to \text{Spiral}$$

(*) $\dot{x} = y$

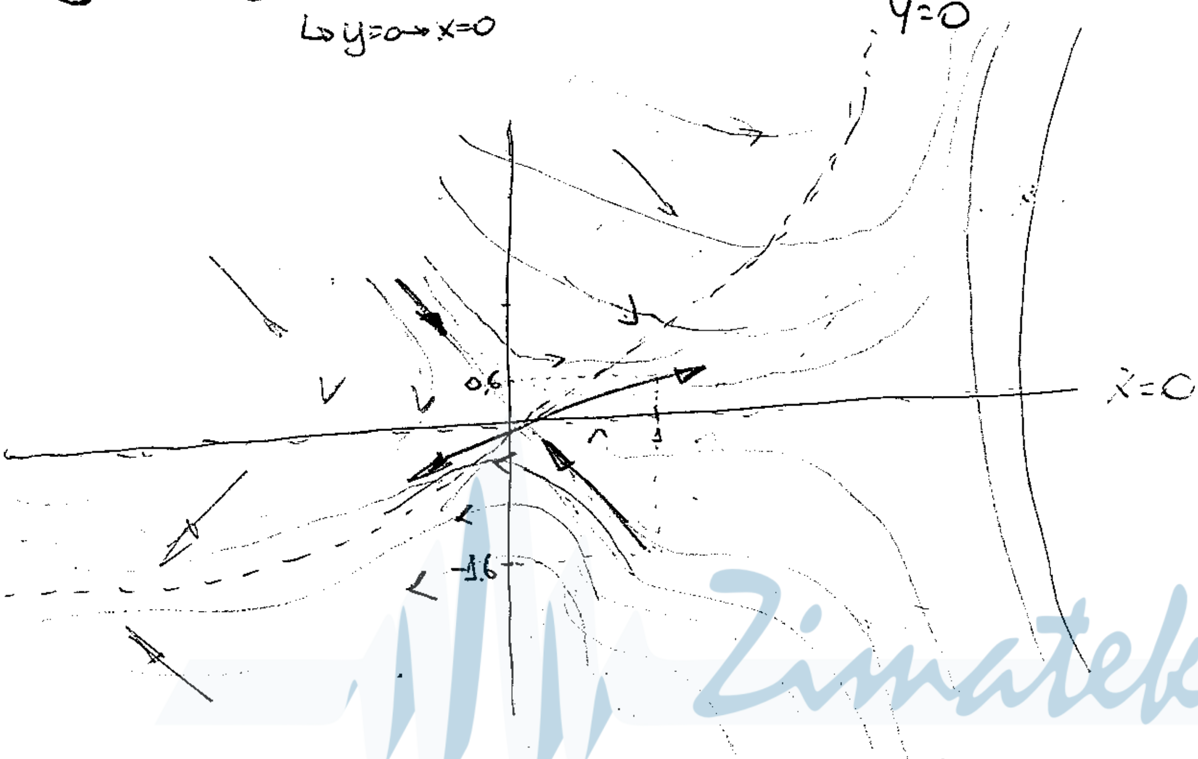
$\dot{y} = e^{x} - (y+1)$

Fixed points

$\dot{x}=0 \rightarrow y=0$ $\rightarrow P(0,0)$

$\dot{y}=0 \rightarrow y=e^{x}-1 \rightarrow y' = e^{x} \xrightarrow{\text{at } (0,0)} y'=1$ (slope)

$\hookrightarrow y=0 \rightarrow x=0$



$$A = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix}_{(0,0)} = \begin{pmatrix} 0 & 1 \\ e^{x} & -1 \end{pmatrix}_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

SADDLE POINT

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = \lambda + \lambda^2 - 1 = \lambda^2 + \lambda - 1 = 0 \rightarrow \lambda = \frac{-1 \pm \sqrt{1+4}}{2} = -\frac{1 \pm \sqrt{5}}{2} \begin{cases} 0.6 \\ -1.6 \end{cases}$$

Eigenvectors $\rightarrow A\vec{v} = \lambda \vec{v} \Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad \begin{pmatrix} 1 \\ 0.6 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ -1.6 \end{pmatrix}$

$$\begin{cases} y = \lambda x \\ x - y = \lambda y \rightarrow x - \lambda x = \lambda^2 x \rightarrow \underbrace{(\lambda^2 + \lambda - 1)}_{0} x = 0 \rightarrow \end{cases}$$ We can choose x as we want

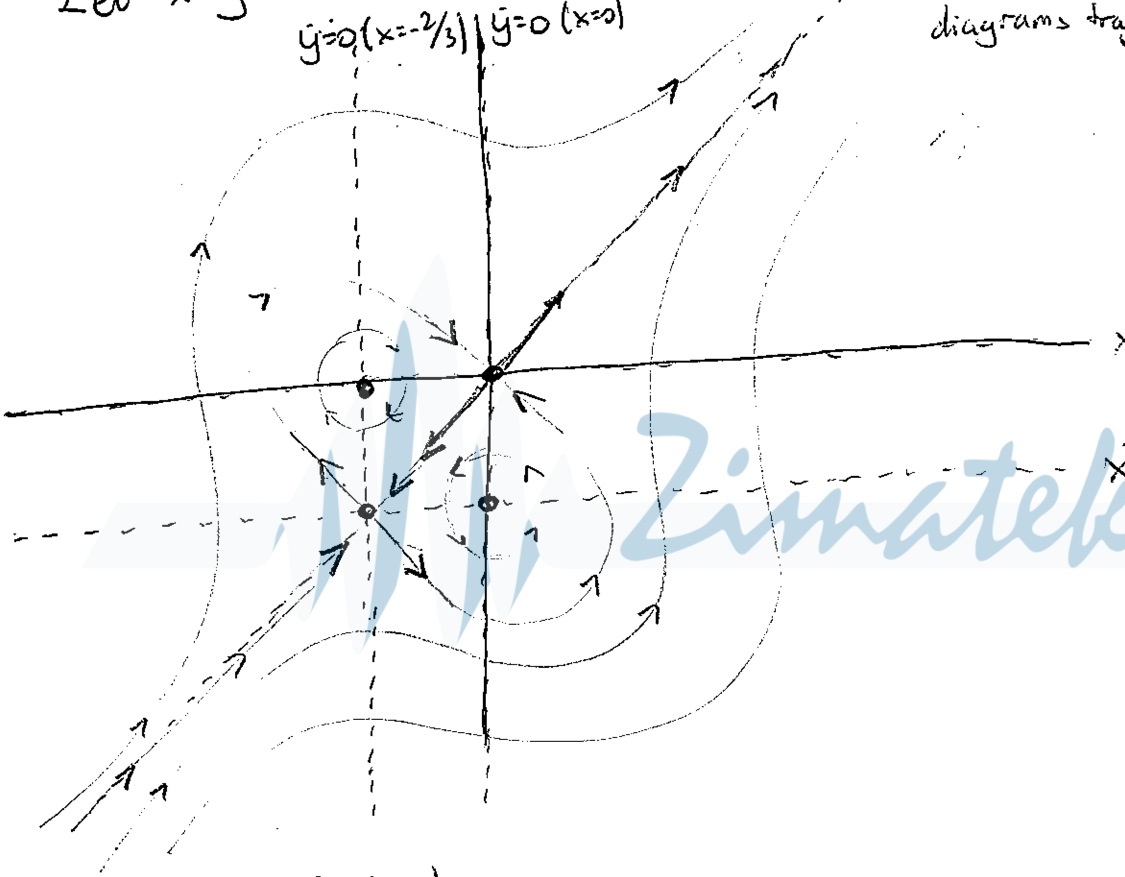
More hints about the figure:

$x \to \infty \Rightarrow \dot{y} \to \infty$

$x \to -\infty \Rightarrow$ If $y = 1 \to \dot{y} = 0$

$$\circledast \begin{cases} \dot{x} = 2y + 3y^2 = y(2+3y) \to \dot{x} = 0 \Rightarrow y = \{0, -\frac{2}{3}\} \\ \dot{y} = 2x + 3x^2 = x(2+3x) \to \dot{y} = 0 \Rightarrow x = \{0, -\frac{2}{3}\} \end{cases}$$

Let $x = y \Rightarrow \dot{x} = 2x + 3x^2$



Fixed points $\to \begin{cases} P_1(0,0) \\ P_2(0,-2/3) \\ P_3(-2/3,0) \\ P_4(-2/3,-2/3) \end{cases}$

$$\mathbb{A} = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 2+6y \\ 2+6x & 0 \end{pmatrix}$$

$$\mathbb{A}_{(0,0)} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \quad \mathbb{A}_{(0,-2/3)} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \quad \mathbb{A}_{(-2/3,0)} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \quad \mathbb{A}_{(-2/3,-2/3)} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$

$x \leftrightarrow y$

For $A_{(0,0)}$:

$$\begin{vmatrix} -\lambda & 2 \\ 2 & \lambda \end{vmatrix} = \lambda^2 - 4 = 0 \to \lambda = \pm 2$$

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \pm 2 \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{cases} 2y = \pm 2x \\ 2x = \pm 2y \end{cases} \to x = 1 \to y = \pm 1$$

$$\lambda = 2 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \lambda = -2 \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lim_{x\to\infty} \dot{x} = 3y^2$$

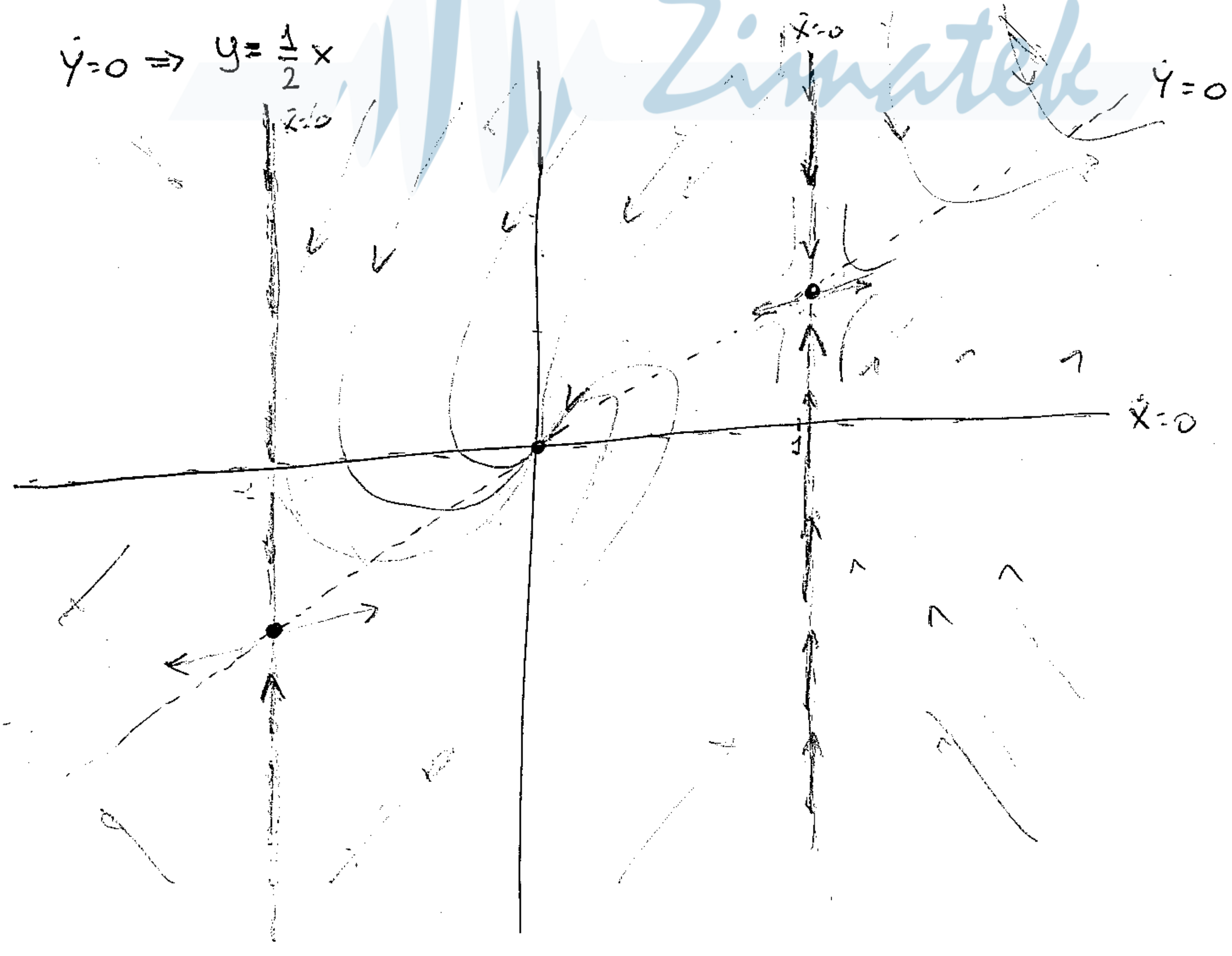
$$\lim_{y\to\infty} \dot{y} = 3x^2$$

SYMETRY: $x \to -x$, $y \to -y$

(10) $\begin{cases} \dot{x} = -y + x^2 y = y(x^2-1) \\ \dot{y} = x - 2y \end{cases}$

$$\dot{x} = 0 \Rightarrow \begin{cases} y = 0 \\ x = \pm 1 \end{cases}$$

$$\dot{y} = 0 \Rightarrow y = \frac{1}{2} x$$



$P_1(1, 1/2)$

$P_2(-1, -1/2)$

$P_3(0,0)$

$$\mathbb{A} = \begin{pmatrix} 2xy & -1+x^2 \\ 1 & -2 \end{pmatrix}$$

$$\mathbb{A}_{P_1} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \quad \mathbb{A}_{P_2} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \quad \mathbb{A}_{P_3} = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$$

For P_1 & P_2:

$$\begin{vmatrix} -\lambda+1 & 0 \\ 1 & -\lambda-2 \end{vmatrix} = -(1-\lambda)(2+\lambda) = 0 \rightarrow \lambda = \{1, -2\}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad \begin{matrix} x = x \\ x - 2y = y \end{matrix}$$

For P_3:

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -2-\lambda \end{vmatrix} = \lambda(2+\lambda) + 1 = \lambda^2 + 2\lambda + 1 = 0 \rightarrow \lambda = \frac{-2 \pm \sqrt{4-4}}{2} = -1$$

$$\begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{cases} -y = -x \\ x - 2y = -y \rightarrow x = y \end{cases} \longrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Particular lines:

$$\begin{cases} \dot{x} = y(x^2-1) \\ \dot{y} = x - 2y \end{cases}$$

Particular lines

$$\text{If } x = ct \rightarrow \dot{x} = 0 \rightarrow \boxed{x = \pm 1} \longrightarrow \underline{\dot{y} = \pm 1 - 2y}$$

(I1) For $n \neq \pm 1$ it is true

For $n = 1 \rightarrow \int_0^{2\pi} \cos^2 x \, dx = \pi$

For $n = -1 \rightarrow \int_0^{2\pi} \cos(x) \cos(-x) \, dx = \pi$

(I2) False.

$$y(0) = 3 \cdot \underbrace{y_1(0)}_{= 1} + 2 \cdot \underbrace{y_1(0)}_{\neq 1} = 5 \neq 1$$

(I3) $\dfrac{\partial^2 u}{\partial y^2} + 4x^2 u = x e^y$

Hom. part : $\dfrac{\partial^2 u}{\partial y^2} + 4x^2 u = 0 \rightarrow u = A(x) \sin(2x^2 y) + B(x) \cos(2x^2 y)$

Gen. sol. : $\boxed{u = \dfrac{x}{1+x^4} e^y + A(x) \sin(2x^2 y) + B(x) \cos(2x^2 y)}$

b) $u(x,y,z) \Longrightarrow A(x) \rightarrow A(x,z)$

$B(x) \rightarrow B(x,z)$

c) $u_{xy} = 0 \rightarrow \dfrac{\partial^2 u}{\partial x \partial y} = 0$

$$\frac{\partial}{\partial y}\Big(\underbrace{\frac{\partial u}{\partial x}}_{\text{unknown}}\Big) = 0 \rightarrow \frac{\partial u}{\partial x} = f(x) \rightarrow \boxed{u = F(x) + f(y)}$$

I4

$$\int_{-1}^{1}\frac{dt}{t^2} = \int_{-1}^{0}\frac{dt}{t^2} + \int_{0}^{1}\frac{dt}{t^2} = \left[-\frac{1}{t}\right]_{-1}^{0} + \left[-\frac{1}{t}\right]_{0}^{1} =$$

$$= \lim_{\varepsilon\to 0}\left[-\frac{1}{t}\right]_{-1}^{\varepsilon} + \lim_{\varepsilon\to 0}\left[-\frac{1}{t}\right]_{\varepsilon}^{1} = \lim_{\varepsilon\to 0}\left[-\frac{1}{\varepsilon}-1\right] + \lim_{\varepsilon\to 0}\left[-1+\frac{1}{\varepsilon}\right] = \text{Indet.}$$

False.

I5

$$\phi = xy^2z + 3y - z^2 + C$$

$$\nabla\phi = \vec{n}$$

↳ normal of the surface

I6

$$I_{2n}(a) = \int_0^\infty dx\, x^{2n} e^{-ax^2}$$

$$I_0 = \int_0^\infty dx\, e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\frac{dI_0(a)}{dx} = \int_0^\infty dx\, e^{-ax^2}(-x^2) = I_1 \qquad \Big\{ I_2(a) = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$

$$\parallel \quad \frac{1}{2}\sqrt{\pi}\, a^{-3/2}\left(-\frac{1}{2}\right)$$

By induction, the result is proven.

b) $I_1(a) = \int_0^\infty e^{-ax}\, x \cdot dx \;\underset{=}{\overset{x^2=z}{}}$

I7

a) $f(x) = \underbrace{\boxed{\dfrac{f(x)+f(-x)}{2}}}_{E_f(x)} + \underbrace{\boxed{\dfrac{f(x)-f(-x)}{2}}}_{O_f(x)}$

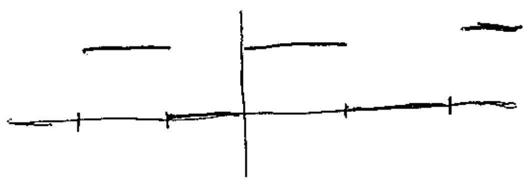
$E_f(-x) = E_f(x)$

$O_f(-x) = -O_f(x)$

$$e^x \to E(x) = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$O(x) = \frac{e^x - e^{-x}}{2} = \sinh x$$

(18) a)



$T = 2$

b) $T = 4$

c) $T = 2\pi$

d) Not periodic

(19)

$$I = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx \to \begin{matrix} e^{-ax} = 1 - ax + \dots \\ e^{-bx} = 1 - bx + \dots \end{matrix} \to e^{-ax} - e^{-bx} = (b-a) + \dots$$

Integrable if $a, b > 0$

must fix the const.

$$\left\{ \begin{aligned} \frac{\partial I}{\partial a} &= \int_0^\infty \frac{(-x) e^{-ax}}{x}\,dx = -\frac{1}{a} \\ \frac{\partial I}{\partial b} &= \dots = \frac{1}{b} \end{aligned} \right.$$

$$\to I(a,b) = \log b/a + C$$

$$I(a,a) = 0 \to \underline{C = 0}$$

b) They're not integrable.

① b) $P \Rightarrow \begin{cases} e^{-x^2} & \text{(Hermite)} \\ x & \text{(Bessel)} \\ e^{-x} x & \dots \\ \sqrt{1-x^2} & \dots \end{cases}$

② $\underline{L_2(0,\infty)_\omega}$, $\omega(x) = e^{-x^2}$

$$\langle f | g \rangle = \int_0^\infty \bar{f} \cdot g \cdot \omega \, dx = \int_0^\infty 2 \cdot 3x \cdot e^{-x^2} dx = 6 \int_0^\infty x e^{-x^2} = \dots$$

b) $\begin{cases} \langle \alpha f + \beta g | f \rangle = 0 & \text{(i)} \\ \langle \alpha f + \beta g | \alpha f + \beta g \rangle = 1 & \text{(ii)} \end{cases} \rightarrow \underline{\alpha, \beta}$

→ real functions.

(i) $\int_0^\infty (2\alpha + 3\beta x) \cdot 2 \cdot e^{-x^2} dx = \dots$

⑤ $y'' + \lambda y = 0$, $y(0) = 0$, $y(1) - 2y'(1) = 0$

→ Homogeneous & separated

↳ Regular S-L problem

$\lambda > 0 \rightarrow y = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$

$\lambda = 0 \rightarrow y = Ax + B$

$\lambda < 0 \rightarrow y = Ae^{x} + Be^{-x} \rightarrow$ <u>No solution</u>

$\underline{\lambda > 0}$

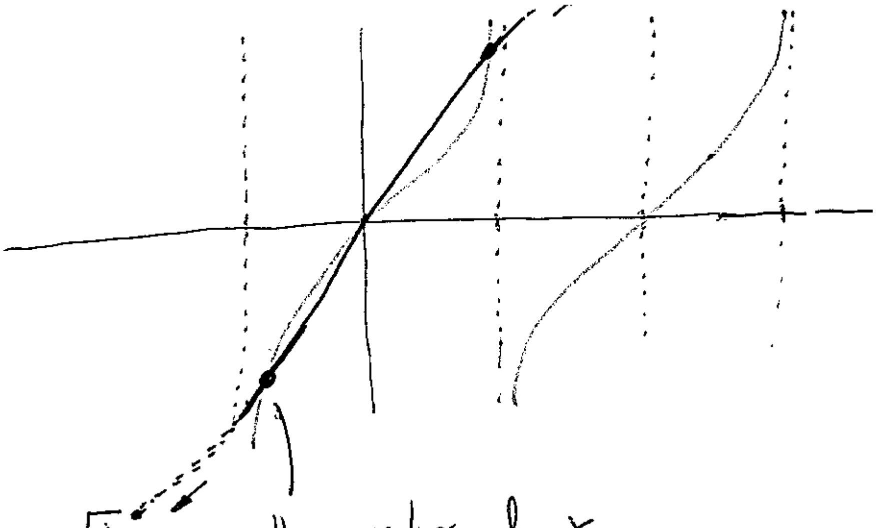
$y(0) = A = 0$

$y(1) - 2y'(1) = B\left[\sin(\sqrt{\lambda}) - 2\sqrt{\lambda}\cos(\sqrt{\lambda})\right]$

$\boxed{\sin\sqrt{\lambda} = 2\sqrt{\lambda}\cos\sqrt{\lambda}} \rightarrow$ secular eq.

↳ trascendental eq. → Not able to solve it analytically

$\operatorname{tg}\sqrt{\lambda} = 2\sqrt{\lambda}$



$\sqrt{\lambda_n}$ are these values of x.

⑦ $y'' + a\,\delta(x)\,y + \lambda y = 0\ ,\quad y(\pi) = y(-\pi) = 0$

General solution : $\lambda > 0 \rightarrow y = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x) - aA\,\dfrac{\sin(\sqrt{\lambda}\,x)}{\sqrt{\lambda}}\,\theta(x)$

$$\boxed{f(x)\,\delta(x) = f(0)\,\delta(x)} \quad (*)$$

" $\delta(x) = \begin{cases} 0 & \forall x \neq 0 \\ \infty & x = 0 \end{cases}$ "

Basic idea for distributions:

$$D : \mathcal{D}(\mathbb{R}) \xrightarrow[\text{linear}]{} \mathbb{R}$$
$$\uparrow$$
$$\varphi$$

A function is always a distribution:

$$F(x) : \mathcal{D}(\mathbb{R}) \longrightarrow \mathbb{R}$$

$$\int_{-\infty}^{\infty} F(x)\,\varphi(x)\,dx = \text{number}$$

$$\delta : \begin{array}{l} \mathcal{D}(\mathbb{R}) \longrightarrow \mathbb{R} \\ \varphi \longrightarrow \varphi(0) \end{array} \Rightarrow \int_{-\infty}^{\infty} \delta(x)\,\varphi(x)\,dx = \varphi(0)$$

$$D'(\varphi) \equiv -D(\varphi') \rightarrow \int_{-\infty}^{\infty} F'\varphi\,dx = \cancel{F\varphi\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} F\varphi'\,dx = -F(\varphi')$$

$$\theta'(\varphi) = -\theta(\varphi') = -\int_{-\infty}^{\infty} \theta(x)\varphi'(x)\,dx = -\int_{0}^{\infty} \varphi'(x)\,dx = -\varphi\Big|_{0}^{\infty} = \varphi(0) = \delta(\varphi)$$

$\mathcal{D}(\mathbb{R}) : \{\varphi : \mathbb{R} \to \mathbb{R}\}$ compact support

- - - - - - - - - - -

$x<0 \to y = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$

$x \geq 0 \to y = C\cos(\sqrt{\lambda}\,x) + D\sin(\sqrt{\lambda}\,x)$

$$y = [A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)]\theta(-x) + [C\cos(\sqrt{\lambda}\,x) + D\sin(\sqrt{\lambda}\,x)]\theta(x)$$

$$y' = (-A\sin(\sqrt{\lambda}\,x) + B\cos(\sqrt{\lambda}\,x))\sqrt{\lambda}\,\theta(-x) + [\ ]\delta(x) + (-C\sin(\sqrt{\lambda}\,x) + D\cos(\sqrt{\lambda}\,x))\sqrt{\lambda}\,\theta(x) + [\ ]\delta(x)$$

(*) $-A\delta + C\delta \overset{?}{=} 0 \to A = C$ ← so "y" is continuous

$$y'' = \underbrace{-\lambda(\ )\theta(-x) - \lambda(\ )\theta(x)}_{-\lambda y} + \underset{\displaystyle -a\delta y = -a(y_0)\delta}{\underset{\shortparallel}{\sqrt{\lambda}(D-B)\delta}}$$

$\sqrt{\lambda}(D-B) = -a(y_0)$

- - - - - - -

From now on, we will choose $\theta(0) = 1/2$ so

$\theta(x) + \theta(-x) = 1$ everywhere

$$\theta(x) = \begin{cases} 1, & x>0 \\ 1/2, & x=0 \\ 0, & x<0 \end{cases}$$

- - - - - -

$\sqrt{\lambda}(D-B) = -aA$

$D = B - \frac{a}{\sqrt{\lambda}}A \to$ the gen. sol. is obtained

$y(-\pi)=0 \rightarrow A\cos(\sqrt{\lambda}\pi) - B\sin(\sqrt{\lambda}\pi)=0$

$y(\pi)=0 \rightarrow A\cos(\sqrt{\lambda}\pi) + B\sin(\sqrt{\lambda}\pi) - \frac{aA}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\pi)=0$

$$A\left(\cos(\sqrt{\lambda}\pi) - \frac{a}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\pi)\right) + B\sin(\sqrt{\lambda}\pi)=0$$

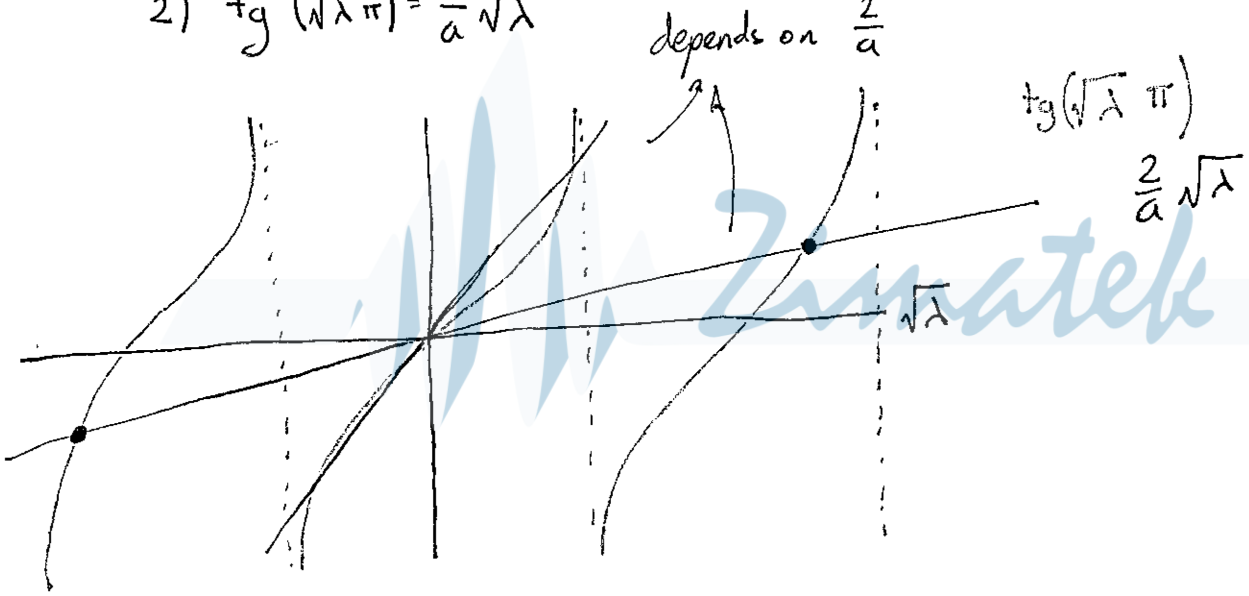
Two hom. eqs. $\rightarrow$ To get non-trivial sols. its determinant must be equal to 0. :

$$\sin(\sqrt{\lambda}\pi)\left[2\cos(\sqrt{\lambda}\pi) - \frac{a}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\pi)\right]=0$$

$\rightarrow$ Substitute in system $\rightarrow$ $\underline{A=0}$

1) $\sqrt{\lambda}\pi = n\pi \rightarrow \lambda_n = n^2$, $n=1,2,3,\dots \rightarrow \underline{y_n = \sin(nx)}$

2) $tg(\sqrt{\lambda}\pi) = \frac{2}{a}\sqrt{\lambda}$



$\{\lambda_n\}$ all real and $\lambda_n > 0$, $n=1,2,3,\dots$

From the eqs, $B = \frac{A}{tg(\sqrt{\lambda}\pi)} = \frac{A}{\frac{2}{a}\sqrt{\lambda}} = \frac{Aa}{2\sqrt{\lambda}}$

$$y_n = \cos(\sqrt{\lambda_n}x) + \frac{a}{2\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}x) - \frac{a}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}x)\,\theta(x) =$$

$$= \cos(\sqrt{\lambda_n}x) + \frac{a}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}x)\left[\frac{1}{2} - \theta(x)\right]$$

$\lambda<0$

Gen. sol.: $y = A\cosh(\sqrt{-\lambda}\,x) + B\sinh(\sqrt{-\lambda}\,x) - aA\,\dfrac{\sinh(\sqrt{-\lambda}\,x)}{\sqrt{-\lambda}}\,\theta(x)$

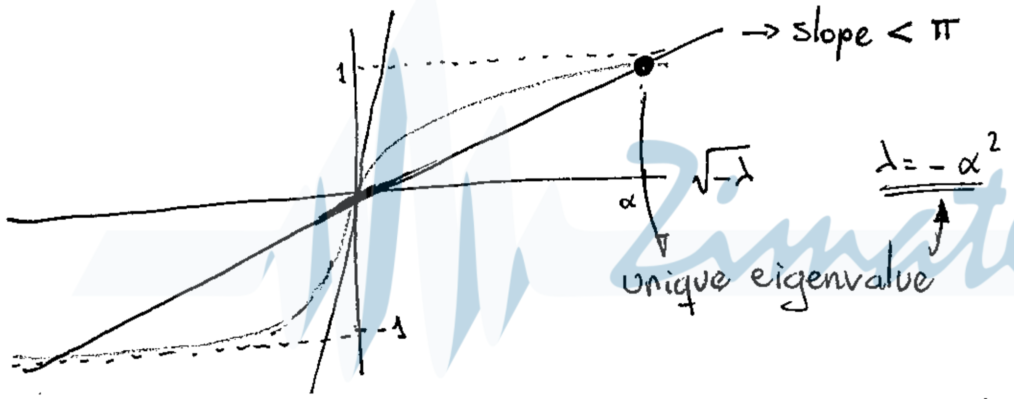
$$\begin{cases} y(-\pi)=0 \to A\cosh(\sqrt{-\lambda}\,\pi) + B\sinh(\sqrt{-\lambda}\,\pi)=0 \\ y(\pi)=0 \to A\left(\cosh(\sqrt{-\lambda}\,\pi) - \dfrac{a}{\sqrt{-\lambda}}\sinh(\sqrt{-\lambda}\,\pi)\right) + B\sinh(\sqrt{-\lambda}\,\pi)=0 \end{cases}$$

Det. $=0$

$\hookrightarrow \sinh(\sqrt{-\lambda}\,\pi)\left[2\cosh(\sqrt{-\lambda}\,\pi) - \dfrac{a}{\sqrt{-\lambda}}\sinh(\sqrt{-\lambda}\,\pi)\right]=0$

$\neq 0$ — Only at $\lambda=0$ (not included!)

$\tanh(\sqrt{-\lambda}\,\pi) = \dfrac{2}{a}\sqrt{-\lambda}$



slope of $\left(\frac{2}{a}\sqrt{-\lambda}\right) <$ slope of $\left(\tanh(\sqrt{-\lambda}\,\pi)\right)$ at the origin

$$\frac{2}{a} < \pi \to \boxed{a > \frac{2}{\pi}} \quad \text{(Necessary cond.)}$$

$\lambda=0$

Gen. sol.: $y = A + Bx - aAx\,\theta(x)$

$$y(-\pi)=y(\pi)=0 \to \begin{cases} A = B\pi \\ B\pi(2-a\pi)=0 \end{cases} \Rightarrow a=\frac{2}{\pi}$$

Only if $a=\frac{2}{\pi}$, $\lambda=0$ is the unique eigenvalue.

$$y_0 = \pi + x - 2\pi x\,\theta(x)$$

Schrödinger's eq.: $-\psi'' + V(x)\psi = E\psi$

$$\begin{cases} \psi \equiv y \\ \lambda \equiv E \\ V(x) \equiv -a\,\delta(x) \end{cases}$$

(8) $L = \frac{1}{4}(1+x^2)^2 \frac{d^2}{dx^2} + \frac{1}{2}x(1+x^2)\frac{d}{dx} + a$;

Weight $\to$ use formula $\to$ $\rho(x) = \dfrac{4c}{1+x^2}$ $\to$ choosing $c = \frac{1}{4}$ $\to$ $\rho(x) = \dfrac{1}{1+x^2}$

$x = \tan\left(\frac{\theta}{2}\right)$

$\to \dfrac{d}{d\theta} = (1+x^2)\dfrac{d}{dx}$

$\to \dfrac{d^2}{d\theta^2} = \dfrac{1}{4}(1+x^2)\dfrac{d^2}{dx^2} + \dfrac{1}{2}x(1+x^2)\dfrac{d}{dx}$ $\quad \to L = \dfrac{d^2}{d\theta^2} + a$

With the change of vars:

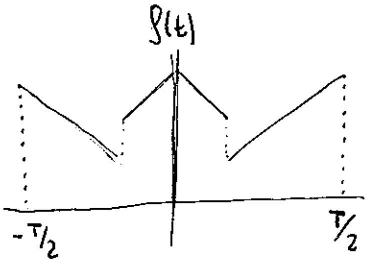
$$\frac{d^2y}{d\theta^2} + ay + \lambda y = 0$$

$x = \tan\left(\frac{\theta}{2}\right) \begin{cases} x = 1 \to \theta = \pi/2 \\ x = -1 \to \theta = -\pi/2 \end{cases} \to y(-\pi/2) = y(\pi/2) = 0$

$\lambda_n + a = n^2$.

$y_n = \sin\left(n(\theta + \pi/2)\right) = \sin\left(n\left(2\,\mathrm{arctg}\,x + \frac{\pi}{2}\right)\right)$

2



a) Correct.

b) Correct. The integral is within a period.

d) Correct. It's the original expression for the fourier expansion.

c)
$$c_n = \frac{1}{T}\int_0^{2T} f(x)\cos(n\Omega x)\,dx = \frac{1}{T}\int_0^{T}\cdots + \frac{1}{T}\int_T^{2T}\cdots \underset{x = x' + T}{=}$$

for the 2nd integral

$$= \frac{1}{T}\int_0^T f(x)\cos\left(n\frac{\pi}{T}x\right)dx +$$

$$+ \frac{1}{T}\int_0^T \underbrace{f(x')}_{f(x'+T)} \cos\left(n\frac{\pi}{T}x' + n\frac{\pi}{\cancel{T}}\cancel{T}\right)dx' =$$

$$\cos\left(n\frac{\pi}{T}x'\right)\underbrace{\cos(n\pi)}_{(-1)^n} - \cancel{\sin\left(n\frac{\pi}{T}x'\right)\sin(n\pi)}$$

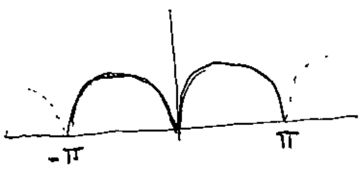
$$= \left[\frac{1}{T}\int_0^T f(x)\cos\left(n\frac{\pi}{T}x\right)dx\right]\left(1 + (-1)^n\right)$$

$n = 2K+1 \rightarrow 0$

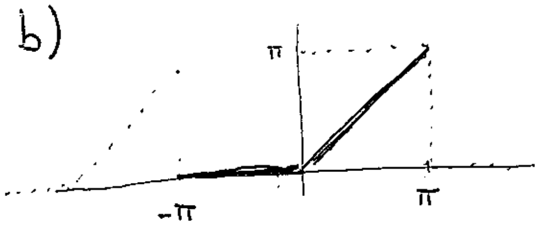
$n = 2K \rightarrow c_n$ as in a)

1 a) $f(x) = \sin^2 x$

$$f(x) = -\frac{1}{2} - \frac{1}{2}\cos(2x)$$



b)



$a_0 = \frac{\pi}{4}$

$$a_K = \frac{1}{\pi} \int_0^{\pi} x \cos(Kx)\, dx = \frac{1}{K^2 \pi} \left((-1)^n - 1 \right)$$

$a_{2n} = 0$

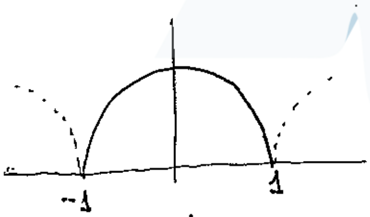
$$a_{2n+1} = -\frac{2}{(2n+1)^2 \pi}$$

$$b_K = \frac{1}{\pi} \int_0^{\pi} x \sin(Kx)\, dx = \dots = -\frac{(-1)^K}{K} = \frac{(-1)^{K+1}}{K}$$

$$\frac{\pi}{4} + \sum_{K=1}^{\infty} \frac{(-1)^{K+1}}{K} \sin(Kx) - \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos\left((2n+1)x\right) = f(x)$$

$x \in (-\pi, \pi)$

c) $f(x) = (1-x^2)$, $T = 2 \to \omega = \pi$

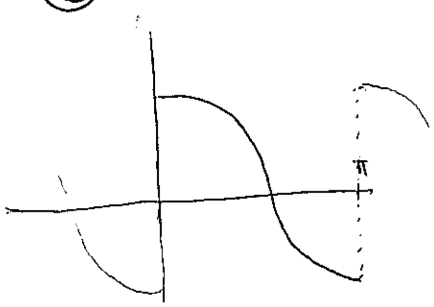


$b_K = 0$ since f is even

$a_0 = 2/3$

$$a_K = \int_{-1}^{1} (1-x^2) \cos(K\pi x)\, dx = (\dots) = \frac{4}{\pi^2} \frac{(-1)^{K+1}}{K^2}$$

3



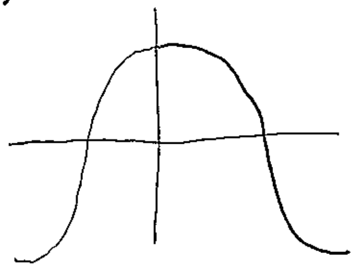
$T = \pi \to \omega = 2$

ODD $\to a_K = 0 \ \forall K$

$$b_K = \frac{2}{\pi} \int_0^{\pi} \cos x \cdot \sin(2Kx)\, dx = \frac{8K}{\pi(4K^2-1)}$$

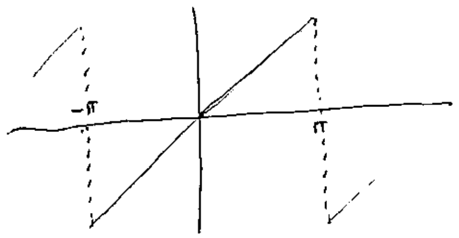
$$\cos x = \sum_{K=1}^{\infty} \frac{8K}{\pi(4K^2-1)} \sin(2Kx), \quad 0 < x < \pi$$

b)



c) In the first case, it will, as there are discontinuities at $x=0, \pi, \ldots$

⑤ $f(x)=x, \quad x\in(-\pi,\pi) \to T=2\pi$

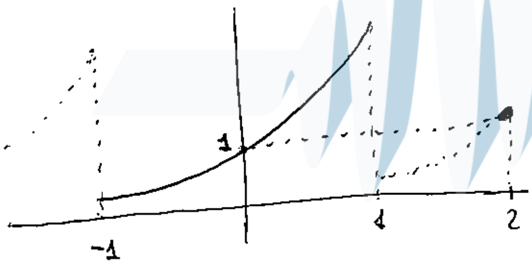


$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)\,dx=-\frac{2}{k}(-1)^k$$

$$x=-2\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\sin(kx), \quad x\in(-\pi,\pi)$$

$$x=\frac{\pi}{2}\to\frac{\pi}{2}=-2\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\sin\left(k\frac{\pi}{2}\right)$$

⑥ $f(x)=e^x, \quad x\in(-1,1)\to T=2, \ \omega=\pi$



$$a_0=\frac{1}{2}\int_{-1}^{1}e^x\,dx=\sinh(1)$$

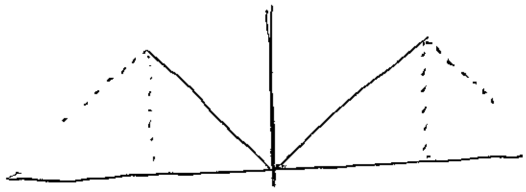
$$a_k=\int_{-1}^{1}e^x\cos(k\pi x)\,dx=\ldots=\frac{(-1)^k}{1+k^2\pi^2}2\sinh(1)$$

$$b_k=\int_{-1}^{1}e^x\sin(k\pi x)\,dx=\ldots=\frac{-k\pi(-1)^k}{1+k^2\pi^2}2\sinh(1)$$

$$e^x=\sinh(1)\left[1+2\sum_{k=1}^{\infty}\frac{(-1)^k}{1+k^2\pi^2}\left(\cos(k\pi x)-k\pi\sin(k\pi x)\right)\right], \quad x\in(-1,1)$$

At $x=2$, $\underline{f(x)=1}$ (As in $x=0$, since $f(x)$ is periodic)

(4) $f(x) = |x|$, $x \in (-\pi, \pi)$; $T = 2\pi \to \omega = 1$



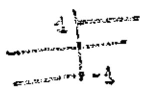
even $\to b_k = 0 \ \forall k$:

$a_0 = \frac{\pi}{2}$

$$a_k = \frac{2}{k^2\pi}\left((-1)^k - 1\right) \begin{cases} a_{2n} = 0 & ; n = 1, 2, \dots \\ a_{2n+1} = \dfrac{-4}{(2n+1)^2\pi} & n = 0, 1, 2, \dots \end{cases}$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos((2n+1)x)}{(2n+1)^2} \quad , \ x \in (-\pi, \pi)$$

$$\frac{x^2}{2}\,\mathrm{sign}(x) + C = \frac{\pi}{2}x - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin((2n+1)x)}{(2n+1)^3} \quad , \ x \in (-\pi, \pi)$$



$x = 0 \to C = 0$

$$x = \frac{\pi}{2} \to \frac{\pi^2}{8} = \frac{\pi^2}{4} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} \to \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}$$

(7)

$$e^x + C = \sinh(1)\left[x + 2\sum_{k=1}^{\infty}\frac{(-1)^k}{1+k^2\pi^2}\left(\frac{1}{k\pi}\sin(k\pi x) + \cos(k\pi x)\right)\right] \overset{(5)}{=}$$

$$= \sinh(1)\left[-\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\sin(k\pi x) + 2\sum_{k=1}^{\infty}\frac{(-1)^k}{1+k^2\pi^2}\left(\frac{1}{k\pi}\sin(k\pi x) + \cos(k\pi x)\right)\right] =$$

$$(\dots) = 2\sinh(1)\left[\sum_{n=1}^{\infty}\frac{(-1)^k}{1+k^2\pi^2}\left(\cos(k\pi x) - k\pi\sin(k\pi x)\right)\right] \overset{(6)}{=} e^x \underbrace{- \sinh(1)}_{C}$$

$$x=0 \longrightarrow \quad 1 = \sinh(1)\left[1 + 2\sum_{k=1}^{\infty} \frac{(-1)^k}{1+k^2\pi^2}\right]$$

$$(e^x)' = \sinh(1)\cdot 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{1+k^2\pi^2}\left(-k\pi \sin(k\pi x) - k^2\pi^2 \cos(k\pi x)\right) \neq e^x !!$$

↳ does not tend to 0!!

$\sum_{n=1}^{\infty}(|a_k|^2 + |b_k|^2)$ must be convergent.

$$a_k \to 0$$
$$b_k \to 0 \quad , k \to \infty$$

① $\left(\frac{1}{x^2}y'\right)' = 2(x+1) \quad \oplus \quad y(1)=y(2)=0$

HOMOGENEOUS PROBLEM

$$\left(\frac{1}{x^2}y'\right)'=0 \to \frac{1}{x^2}y'=A \to y'=Ax^2 \to y=Ax^3+B$$

$$\left.\begin{array}{l} y(1)=A+B=0 \\ y(2)=8A+B=0 \end{array}\right\} \underline{A=B=0} \to \text{Trivial solution} \to \text{There's } G$$

i) $\left(\frac{1}{x^2}y'\right)'+\lambda y=0 \oplus$ B.C. $\to$ Very tedious...

ii) $\left(\frac{1}{x^2}G'\right)'=\delta(x-\xi) \quad \oplus \quad G(1,\xi)=G(2,\xi)=0$

$$x<\xi \to G=Ax^3+B \to G(1)=A+B=0 \xrightarrow{B=-A} G=(x^3-1)A$$

$$x>\xi \to G=Cx^3+D \to G(2)=8C+D=0 \to G=C(x^3-8)$$

Applying continuity everywhere (including $x=\xi$):

$$A(\xi^3-1)=C(\xi^3-8) \to C=A\frac{\xi^3-1}{\xi^3-8}$$

$$G=A\left[(x^3-1)\,\theta(\xi-x)+\frac{\xi^3-1}{\xi^3-8}(x^3-8)\,\theta(x-\xi)\right]$$

i) Compute the derivatives, use the initial equation and obtain A.

ii) Use the following formula:

$$a_0y''+a_1y'+a_2=Ly \to \frac{1}{a_0(\xi)}=(f_2'-f_1')(\xi)$$

$$G=f_2\,\theta(x-\xi)+f_1\,\theta(\xi-x)$$

We'll apply method i):

$$G' = A\left[3x^2\theta(\xi-x) - \cancel{(x^3-1)\delta(x-\xi)} + \frac{\xi^3-1}{\xi^3-8}\,3x^2\theta(x-\xi) + \cancel{\frac{\xi^3-1}{\xi^3-8}(x^3-8)\delta(x-\xi)}\right]$$

$$\frac{G'}{x^2} = 3A\left[\theta(\xi-x) + \frac{\xi^3-1}{\xi^3-8}\theta(x-\xi)\right]$$

$$\left(\frac{G'}{x^2}\right)' = 3A\left(-1+\frac{\xi^3-1}{\xi^3-8}\right)\delta(x-\xi) = \frac{21A}{\xi^3-8}\delta(x-\xi) = \delta(x-\xi)$$

$$A = \frac{\xi^3-8}{21}$$

$$\boxed{G = \frac{(\xi^3-8)(x^3-1)}{21}\theta(\xi-x) + \frac{(\xi^3-1)(x^3-8)}{21}\theta(x-\xi)}$$

Applying method ii):

$$a_0 = \frac{1}{x^2} \rightarrow \frac{1}{a_0(\xi)} = \xi^2$$

$$G = A\left[\underbrace{(x^3-1)}_{g_1}\theta(\xi-x) + \underbrace{\frac{\xi^3-1}{\xi^3-8}(x^3-8)}_{g_2}\theta(x-\xi)\right]$$

$$\xi^2 = A\left(\frac{\xi^3-1}{\xi^3-8}3x^2 - 3x^2\right)\Bigg|_{x=\xi} = A\,3\xi^2\left(\frac{\xi^3-1-\xi^3+8}{\xi^3-1}\right) \rightarrow A = \frac{\xi^3-8}{21}$$

We'll use the Green function to solve the equation:

$$y(x) = \int_1^2 G(x,\xi)f(\xi)\,d\xi = \int_1^2 G(x,\xi)\,2(\xi+1)\,d\xi =$$

$$= \int_1^x 2(\xi+1)\frac{(\xi^3-1)}{21}(x^3-8)\,d\xi + \int_x^2 2(\xi+1)\frac{(\xi^3-8)}{21}(x^3-1)\,d\xi$$

(2) $\left(\frac{y'}{x+1}\right)' = f(x)$ ⊕ $y(0)=0,\ y(\sqrt{2}) \pm y'(\sqrt{2}) = 0$

HOMOGENEOUS PROBLEM $\rightarrow \left(\frac{y'}{x+1}\right)' = 0 \rightarrow \frac{y'}{x+1} = A \rightarrow y' = 2A(x+1) \rightarrow$

$\rightarrow y = Ax^2 + 2Ax + B$

$y(0) = B = 0$

$y(\sqrt{2}) \pm y'(\sqrt{2}) = 2A + 2\sqrt{2}A \pm 2A(\sqrt{2}+1) = 2A(1+\sqrt{2} \pm (\sqrt{2}+1))$ ⊕ No sol. → GREEN ; ⊖ Sol.

$\left(\frac{G'}{x+1}\right)' = \delta(x-\xi),\ G(0,\xi)=0,\ G(\sqrt{2},\xi) + G'(\sqrt{2},\xi) = 0$

(3)

$x^2y'' + xy' + \lambda y = 0 \quad , y(1)=0,\ y'(e^{\pi}) = 0$

$\rho(x) = \frac{1}{x}$

$\underline{\lambda = 0} \rightarrow y = A\ln x \rightarrow \begin{cases} y(1) = B = 0 \\ y'(e^{\pi}) = Ae^{-\pi} = 0 \rightarrow A = 0 \end{cases} \rightarrow \lambda = 0$ is not an eigenvalue.

$y' = A/x$

$\underline{\lambda < 0} \rightarrow x^{\alpha}$ is a sol. if $\alpha(\alpha-1) + \alpha + \lambda = 0 \rightarrow \alpha^2 + \lambda = 0$.

$\alpha = \pm\sqrt{-\lambda}$

$y = Ax^{\sqrt{-\lambda}} + Bx^{-\sqrt{-\lambda}}$ $\begin{cases} y(1) = A + B = 0 \\ y'(e^{\pi}) = A\sqrt{-\lambda}\,e^{-\pi}(e^{\pi})^{\sqrt{-\lambda}} - B\sqrt{-\lambda}\,e^{-\pi}(e^{\pi})^{-\sqrt{-\lambda}} = 0 \end{cases}$

No solution.

$\lambda > 0$

$$"x^{\pm i\sqrt{\lambda}}" = e^{\pm i\sqrt{\lambda}\ln x} = \cos(\sqrt{\lambda}\ln x) \pm i\sin(\sqrt{\lambda}\ln x)$$

$$y = A\cos(\sqrt{\lambda}\ln x) + B\sin(\sqrt{\lambda}\ln x)$$

$$y(1) = A = 0$$

$$y'(e^{\pi}) = B\sqrt{\lambda}\, e^{-\pi}\cos(\sqrt{\lambda}\underbrace{\ln \pi}_{\ln(e^{\pi})}) = 0$$

Secular equation: $\cos(\sqrt{\lambda}\,\pi) = 0 \rightarrow \underline{\lambda_n = \left(n+\frac{1}{2}\right)^2}$, $n = 0, 1, 2, \ldots$

$$\underline{y_n = B_n \sin\left(\left(n+\tfrac{1}{2}\right)\ln x\right)} = \sqrt{\frac{2}{\pi}}\sin\left(\left(n+\tfrac{1}{2}\right)\ln x\right)$$

$$\|y_n^2\| = \langle y_n | y_n \rangle = \int_1^{e^{\pi}} B_n^2 \sin^2\left(\left(n+\tfrac{1}{2}\right)\ln x\right)\frac{1}{x}\,dx \underset{x=e^t}{=}$$

$$= \int_0^{\pi} B_n^2 \sin^2\left(\left(n+\tfrac{1}{2}\right)t\right)dt = B_n^2\frac{\pi}{2} \underset{\text{normalization}}{=} 1 \rightarrow B_n = \sqrt{\frac{2}{\pi}}$$

Green function

i) $$G(x,\xi) = \frac{1}{\xi}\,\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{\mu-\left(n+\frac{1}{2}\right)^2}\sin\left[\left(n+\tfrac{1}{2}\right)\ln\xi\right]\sin\left[\left(n+\tfrac{1}{2}\right)\ln\xi\right]$$

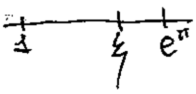
ii) $x^2G'' + xG' + \mu G = \delta(x-\xi)$ ⊕ $G(1,\xi) = 0$, $G'(e^{\pi},\xi) = 0$

$\underline{\mu \neq 0 \;\&\; \mu \neq \lambda_n}$

$\underline{x<\xi} \rightarrow G = \cancel{A\cos(\sqrt{\mu}\ln x)} + B\sin(\sqrt{\mu}\ln x) \rightarrow G(1,\xi) = 0 \rightarrow A = 0$

$\underline{x>\xi} \rightarrow G = C\cos(\sqrt{\mu}\ln x) + D\sin(\sqrt{\mu}\ln x)$

$\hookrightarrow G'(e^{\pi},\xi) = 0 \rightarrow D = C\tan(\sqrt{\mu}\,\pi)$



$$G(\xi^-,\xi) = G(\xi^+,\xi)$$

$$B\sin(\sqrt{\mu}\ln\xi) = C\left[\cos(\sqrt{\mu}\ln\xi) + \tan(\sqrt{\mu}\pi)\sin(\sqrt{\mu}\ln\xi)\right]$$

$$G(x,\xi) = C\Big\{\left[\cot(\sqrt{\mu}\ln\xi) + \tan(\sqrt{\mu}\ln\xi)\right]\sin(\sqrt{\mu}\ln x)\,\theta(\xi - x) +$$

$$+\left[\cos(\sqrt{\mu}\ln x) + \tan(\sqrt{\mu}\pi)\sin(\sqrt{\mu}\ln x)\right]\theta(x-\xi)$$

Put $G(x,\xi)$ onto the eq., or alternatively use the formula to get

$$C = -\frac{\sin(\sqrt{\mu}\ln\xi)}{\sqrt{\mu}\,\xi}$$

$$G(x,\xi) = -\frac{1}{\sqrt{\mu}\,\xi\cos(\sqrt{\mu}\pi)}\Big\{\cos[\sqrt{\mu}(\ln\xi - \pi)]\sin(\sqrt{\mu}\ln x)\,\theta(\xi-x) +$$

$$+\cos[\sqrt{\mu}(\ln x - \pi)]\sin(\sqrt{\mu}\ln\xi)\,\theta(x-\xi)\Big\}$$

$\mu = 0$ (compute the limit):

$$G(x,\xi) = -\frac{1}{\xi}\left[\ln x\,\theta(\xi - x) + \ln\xi\,\theta(x-\xi)\right]$$

④ $y'' + y' + \frac{1}{4}y + \lambda y = 0 \quad \oplus \quad y(0) = 0,\ y'(1) + \frac{1}{2}y(1) = 0$

$\hookrightarrow \rho(x) = e^x$

$\lambda = 0 \rightarrow$ No sol.

$$\lambda \neq 0 \rightarrow y = e^{-x/2}\left(Ae^{i\sqrt{\lambda}x} + Be^{-i\sqrt{\lambda}x}\right)$$

$$\text{B.C.} \longrightarrow \lambda_n = \left(n+\frac{1}{2}\right)^2\pi^2 \quad \text{(all positive)}$$

$$y_n = \sqrt{2}\,e^{-x/2}\sin\left[\left(n+\tfrac{1}{2}\right)\pi x\right]$$

$$G(x,\xi) = 2e^{\xi/2}e^{x/2}\sum_{n=0}^{\infty}\frac{\sin\left[\left(n+\frac{1}{2}\right)\pi x\right]\sin\left[\left(n+\frac{1}{2}\right)\pi\xi\right]}{\mu - \left(n+\frac{1}{2}\right)^2\pi^2}$$

$$y'' + y' + \frac{1}{4}y = e^{-x/2}$$

$$y(x) = \int_0^1 G(x,\xi,\mu=0)\, e^{-\xi/2}\, d\xi = -2\int_0^1 e^{-x/2}\left(\sum_{n=0}^{\infty} \frac{\sin[(n+\frac{1}{2})\pi x]\,\sin[(n+\frac{1}{2})\pi \xi]}{(n+\frac{1}{2})^2\pi^2}\right) d\xi$$

$$= \frac{-2e^{-x/2}}{\pi^2} \sum_{n=0}^{\infty} \sin\left[(n+\tfrac{1}{2})\pi x\right] \int_0^1 \frac{\sin[(n+\frac{1}{2})\pi\xi]}{(n+\frac{1}{2})^2}\, d\xi =$$

$$= +\frac{2e^{-x/2}}{\pi^3} \sum_{n=0}^{\infty} \sin\left[(n+\tfrac{1}{2})\pi x\right] \left.\frac{\cos[(n+\frac{1}{2})\pi\xi]}{(n+1/2)^3}\right|_0^1 =$$

$$= -2\,\frac{e^{-x/2}}{\pi^3} \sum_{n=0}^{\infty} \frac{\sin[(n+1/2)\pi x]}{(n+1/2)^3}$$

⑤ $y'' = f(x) \oplus y(0) = e^{\mu}\cdot y(\pi)\,,\; y'(0) = e^{-\mu} y'(\pi)$

$G'' = \delta(x-\xi)\,,$ same B.C.

$x<\xi \quad G = Ax + B$ | $G(0) = e^{\mu} G(\pi)$ | $G'(0) = e^{-\mu} G'(\pi)$

$B = e^{\mu}(C\pi + D)$ | $A = Ce^{-\mu}$

$x>\xi \quad G = Cx + D$

$$G(\xi^-,\xi) = G(\xi^+,\xi) \to Ce^{-\mu}\xi + e^{\mu}(C\pi + D) = C\xi + D$$

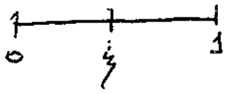
Fix C using the eq. or the formula. $\to C = \dfrac{1}{1-e^{-\mu}}$

$$G(x,\xi) = \frac{e^{\mu}}{e^{\mu}-1}\left[\left(e^{-\mu}x + \xi + \frac{e^{\mu}\pi}{1-e^{\mu}}\right)\theta(\xi - x) + \left(x + e^{-\mu}\xi + \frac{e^{\mu}\pi}{1-e^{\mu}}\right)\theta(x-\xi)\right]$$

$\mu = 0$ is an eigenvalue $\to$ No solution for $G(x,\xi)$.

(9) $y^{(IV)} = f(x) \quad , \; y(0) = y'(0) = y(1) = y'(1) = 0$

$G^{(IV)} = \delta(x-\xi) \oplus$ B. C.



$x < \xi : G = Ax^3 + Bx^2 + \cancel{Cx + D} \rightarrow \boxed{C = D = 0} \rightarrow G = x^2(Ax + B)$

$x > \xi ; G = \tilde{A}(x-1)^3 + \tilde{B}(x-1)^2 + \tilde{C}(x-1) + \tilde{D} \rightarrow \tilde{C} = \tilde{D} = 0$

G''' has a jump $\rightarrow G''$ is cont. $\rightarrow G'$ is cont $\rightarrow G$ is cont.

$G(\xi^-, \xi) = G(\xi^+, \xi)$

$G'(\xi^-, \xi) = G(\xi^+, \xi) \qquad \oplus \; G^{(IV)} = \delta(x-\xi)$

$G''(\xi^-, \xi) = G(\xi^+, \xi)$

$A = -\frac{\xi^3}{4} + \frac{\xi^2}{6} - \frac{1}{6} \qquad \tilde{A} = -\xi^3/3 + \xi^2/2$

$B = \frac{\xi^3}{2} - \xi^2 + \xi/2 \qquad \tilde{B} = -\xi^3/2 + \xi^2/2$

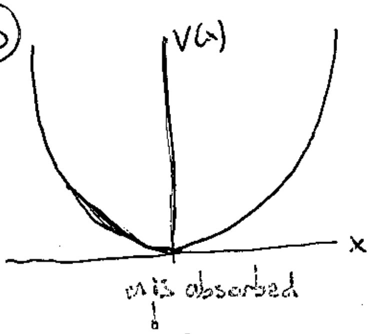
$G = G^+ \theta(\xi - x) + G^- \theta(x - \xi)$

$G' = G^{+\prime} \theta(\xi - x) + G^{-\prime} \theta(x - \xi) + \cancel{(G^- - G^+)\delta(x-\xi)}$

$G'' = \quad " \qquad\qquad " \qquad + \cancel{(G^{-\prime} + G^{+\prime})\delta(x-\xi)}$

$G''' = \quad :$

(10)



$V(x) = \frac{1}{2}K^2x^2$

at $t=0$, $\boxed{x(0) = 0 \;\&\; x(T) = 0}$

$F = F_0^* \sin(\omega t) \quad , \; F_0^* = \text{ctnt.} = F_0 \cdot m$

$\ddot{x} + K^2 x = F_0 \sin(\omega t)$

$$\ddot{G} + K^2 G = \delta(t-\xi)$$

$$t<\xi \rightarrow A\cos(Kt) + B\sin(Kt) = 0 \Rightarrow G(0,\xi) = A = 0$$

$$t>\xi \rightarrow \tilde{C}\cos(Kt) + \tilde{D}\sin(Kt) = 0 = C\sin(Kt+b) \rightarrow G(T,\xi) = 0 = C\sin(KT+b)$$

$$b = -KT$$

$$G(\xi^-,\xi) = G(\xi^+,\xi) \rightarrow B\sin(K\xi) = C\sin[K(\xi-T)] \qquad (1)$$

$$G(t,\xi) = B\sin(Kt)\,\theta(\xi-t) + C\sin[K(t-T)]\,\theta(t-\xi)$$

$$\dot{G}(t,\xi) = +BK\cos(Kt)\,\theta(\xi-t) + CK\cos[K(t-T)]\,\theta(t-\xi) + \underbrace{\left(C\sin[K(t-T)] - B\sin(Kt)\right)}_{=0}\delta(t-\xi)$$

$$\ddot{G}(t,\xi) = \underbrace{-BK^2\sin(Kt)\,\theta(\xi-t) - CK^2\sin[K(t-T)]\,\theta(t-\xi)}_{-K^2 G} +$$

$$+ \left(CK\cos[K(t-T)] - BK\cos(Kt)\right)\delta(t-\xi)$$

$$K\left(C\cos[K(\xi-T)] - B\cos(K\xi)\right) = 1 \qquad (2)$$

$$(1)\ \&\ (2) \Rightarrow B = \frac{\sin[K(\xi-T)]}{K\sin(KT)}, \quad C = \frac{\sin(K\xi)}{K\sin(KT)}$$

$$G(t,\xi) = \frac{1}{K\sin(KT)}\left\{\sin(Kt)\sin(K(\xi-T))\,\theta(\xi-t) + \sin(K\xi)\sin(K(t-T))\,\theta(t-\xi)\right\}$$

$\sin(KT) \neq 0 \rightarrow KT \neq n\pi$ (Fredholm's alternative!)

$$x(t) = \int_0^T G(t,\xi)\, f(\xi)\, d\xi = \int_0^T G(t,\xi) F_0 \sin(\omega\xi)\, d\xi =$$

$$= \frac{F_0}{K\sin(KT)}\int_0^t \sin(K\xi)\sin[K(t-T)]\sin(\omega\xi)\, d\xi + \frac{F_0}{\sin(KT)}\int_t^T \sin(Kt)\sin[K(\xi-T)]\sin(\omega\xi)\, d\xi =$$

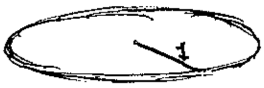
$$= \frac{F_0}{K\sin(KT)}\left[\sin[K(t-T)]\int_0^t \sin(K\xi)\sin(\omega\xi)\, d\xi + \sin(Kt)\int_t^T \sin[K(\xi-T)]\sin(\omega\xi)\, d\xi\right] =$$

$$X(t) = \frac{F_0}{K^2 - \omega^2}\left(\sin(\omega t) - \frac{\sin(\omega T)}{\sin(KT)}\sin(Kt)\right)$$

Resonance $\Rightarrow K = \omega$

$$\lim_{K \to \omega} x(t) = \frac{-F_0}{2K}\left(t\cos(Kt) - T\frac{\cos(KT)}{\sin(KT)}\sin(Kt)\right)$$

④ $u_{tt} = a^2 \nabla^2 u$ POLAR COORDS.



φ $u_{tt} = a^2 \left(\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \varphi^2} \right)$

B.C. $\begin{cases} u(t,1,\varphi) = 0 \\ u \text{ regular everywhere} \end{cases}$

I.C. $\begin{cases} u(0,\rho,\varphi) = f(\rho) \\ u_t(0,\rho,\varphi) = 0 \end{cases}$ → There's no dependence on φ as the I.C. don't deppend on it either.

$u = T(t)R(\rho)\Phi(\varphi)$

$$\frac{\ddot{T}}{T} = a^2 \left[\frac{1}{R} \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dR}{d\rho} \right) + \frac{1}{\rho^2} \underbrace{\frac{1}{\Phi} \frac{d^2\Phi}{d\varphi^2}}_{-m^2} \right] = -a^2 \lambda$$

$\Phi_m = a_m \cos(m\varphi) + b_m \sin(m\varphi)$

$$' \equiv \frac{d}{d\rho} \rightarrow \begin{cases} R'' + \frac{1}{\rho} R' + \left(\lambda - \frac{m^2}{\rho^2} \right) R = 0 \rightarrow \text{Bessel eq.} \\ \ddot{T} + a^2 \lambda T = 0 \end{cases}$$

R regular

$R_m = A_m J_m(\sqrt{\lambda}\rho) + \cancel{B_m N_m(\sqrt{\lambda}\rho)}$ ($B_m = 0$)

$R(1) = 0 \rightarrow J_m(\sqrt{\lambda}) = 0 \rightarrow \lambda_n = \left(\alpha_n^{(m)} \right)^2$

$\ddot{T}_{nm} + a^2 \left(\alpha_n^{(m)} \right)^2 T_{nm} = 0 \rightarrow$ Solution

$$u = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} J_m(\alpha_n^{(m)}\rho) \left[C_{nm} \cos(a\alpha_n^{(m)} t) + D_{nm} \sin(a\alpha_n^{(m)} t) \right] \left[a_m \cos(m\varphi) + b_m \sin(m\varphi) \right]$$

$$u(0,\rho,\varphi) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} J_m(\alpha_n^{(m)}\rho)\, C_{nm} \underbrace{(a_m \cos(m\varphi) + b_m \sin(m\varphi))}_{a_0 \;(=1,\text{ without loss of generality})} = f(\rho)$$

There's no "φ" dependence. $\rightarrow$ $a_m = 0$, $b_m = 0$ $\quad \forall m \geqslant 1$

$\Downarrow$ Gets absorbed into C_{nm}.

$$f(\rho) = \sum_{n=1}^{\infty} J_0(\underbrace{\alpha_n^{(0)}}_{\beta_n}\rho)\, C_{n0} \qquad C_n = \frac{\langle J_0(\beta_n\rho) | f(\rho)\rangle}{\| J_0(\beta_n\rho)\|^2} = \frac{\int_0^1 f(\rho) J_0(\beta_n\rho)\rho\, d\rho}{\frac{1}{2} J_1^2(\beta_1)}$$

$\hookrightarrow$ the vectors of the basis aren't normalized.

$$U_t(0,\rho,\varphi) = 0 = \sum_{n=0}^{\infty} D_{nm} \; \ldots\ldots = 0$$

$\hookrightarrow D_{nm} = 0$

$$\langle J_0(\beta_{n'}\rho) | f(\rho)\rangle = \langle \sum_n J_0(\beta_n\rho) C_n | J_0(\beta_{n'}\rho)\rangle =$$
$$= \delta_{nn'} C_n \langle J_0(\beta_n\rho) | J_0(\beta_{n'}\rho)\rangle$$

Solution:

$$u = \sum_{n=1}^{\infty} J_0(\beta_n\rho)\cos(a\beta_n t)\, \frac{2}{J_1^2(\beta_n)} \left(\int_0^1 f(\rho) J_0(\beta_n\rho)\rho\, d\rho \right)$$

$$f(\rho) = (1-\rho^2)^2 \rightarrow \int_0^1 (1-\rho^2)^2 J_0(\beta_n\rho)\rho\, d\rho \underset{x=\beta_n\rho}{=\!=}$$

$$= \frac{1}{\beta_n^2}\int_0^{\beta_n} \left(1-\frac{x^2}{\beta_n^2}\right)^2 \underbrace{J_0(x)\,x\,dx}_{(xJ_1)'} = \cancel{\frac{1}{\beta_n}\left(1-\frac{x^2}{\beta_n^2}\right) x J_1 \Big|_0^{\beta_n}}_{=0} - \frac{1}{\beta_n^2}\int_0^{\beta_n} \left(1-\frac{x^2}{\beta_n^2}\right)\left(-\frac{4x}{\beta_n^2}\right) x J_1\, dx =$$

$$= \frac{4}{\beta_n^4}\int_0^{\beta_n} \left(1-\frac{x^2}{\beta_n^2}\right) \underbrace{x^2 J_1\, dx}_{(x^2J_2)'} = \cancel{\frac{4}{\beta_n^4}\left(1-\frac{x^2}{\beta_n^2}\right) x^2 J_2 \Big|_0^{\beta_n}}_{=0} - \frac{4}{\beta_n^2}\int_0^{\beta_n}\left(-\frac{2x}{\beta_n^2}\right) x^2 J_2\, dx =$$

$$= \frac{8}{\beta_n^6}\int_0^{\beta_n} \underbrace{x^3 J_2\, dx}_{(x^3J_3)'} = \frac{8}{\beta_n^6} x^3 J_3 \Big|_0^{\beta_n} = \frac{8}{\beta_n^3} J_3(\beta_n)$$

For this particular case, then:

$$u = \sum_{n=1}^{\infty} \frac{16 J_3(\beta_n)}{\beta_n^3 J_1^2(\beta_1)} \cos(a\beta_n t) J_0(\beta_n\rho)$$

(5) Same as (4)

$$u=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} J_m(\alpha_n^{(m)}\rho)\left[C_{nm}\cos(a\alpha_n^{(m)}t)+D_{nm}\sin(a\alpha_n^{(m)}t)\right]\left(a_m\cos(m\varphi)+b_m\sin(m\varphi)\right]$$

I.C. $\begin{cases} u_t(0,\rho,\varphi)=0 \\ u(0,\rho,\varphi)=J_1(\alpha_1^1\rho)\cos\varphi \end{cases}$

$$u_t(0,\rho,\varphi)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} J_m(\alpha_n^{(m)}\rho)(-D_{nm})\,a\,\alpha_n^{(m)}\left(a_m\cos(m\varphi)+b_m\sin(m\varphi)\right)=0\rightarrow D_{nm}=0$$

$$u(0,\rho,\varphi)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} J_m(\alpha_n^{(m)}\rho)\,C_{nm}\left(a_m\cos(m\varphi)+b_m\sin(m\varphi)\right)=\underbrace{J_1(\alpha_1^{(1)}\rho)\cos\varphi}_{\text{one of the terms of the series}}$$

$b_m=0,\ \forall m\ ;\ a_m=0\ \forall m\neq 1$

$$\sum_{n=1}^{\infty} J_1(\alpha_n^{(1)}\rho)\,C_{n1}\,a_1\cos\varphi=J_1(\alpha_1^{(1)}\rho)\cos\varphi$$

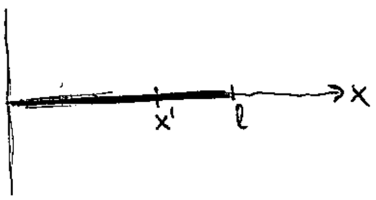
$a_1C_{n1}=0\ \forall n\neq 1$

$$\boxed{a_1C_{11}=1}$$

Solution:

$$u=J_1(\alpha_1^{(1)}\rho)\cos(a\alpha_1^{(1)}t)=\cos\varphi$$

(12)



$$\frac{\partial^2u}{\partial x^2}c^2=\frac{\partial u}{\partial t}$$

$u(0,t)=u(l,t)=0$ B.C.

$u(x,0)=u_0\delta(x-x')$

$u=TX$

$X=\sin(n\frac{\pi}{l}x),\ \lambda_n=\left(\frac{n\pi}{l}\right)^2$

$T_n=C_n e^{-c^2\frac{n^2\pi^2}{l^2}t}$

$$u=\sum_{n=1}^{\infty} C_n e^{-c^2\frac{n^2\pi^2}{l^2}t}\sin(n\frac{\pi}{l}x)$$

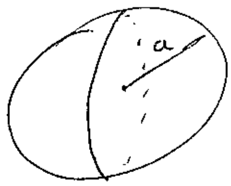
$$u(0,x)=\sum_{n=1}^{\infty} C_n \sin\left(n\frac{\pi}{\ell}x\right)=u_0\,\delta(x-x')$$

$$C_n = u_0\,\frac{\int_0^{\ell}\delta(x-x')\sin\left(n\frac{\pi}{\ell}x\right)dx}{\left\langle \sin\left(n\frac{\pi}{\ell}x\right)\middle|\sin\left(n\frac{\pi}{\ell}x\right)\right\rangle}=\frac{u_0\sin\left(n\frac{\pi}{\ell}x'\right)}{\ell/2}$$

$$u(t,x;x')=\frac{2}{\ell}u_0\sum_{n=1}^{\infty} e^{-c^2\frac{n^2\pi^2}{\ell^2}t}\sin\left(n\frac{\pi}{\ell}x\right)\sin\left(n\frac{\pi}{\ell}x'\right)$$

(9) $u_t = c^2\nabla^2 u$

$u = TRZ\Phi$



Periodicity $\quad \Phi(0)=\Phi(2\pi)$, $\quad \Phi'(0)=\Phi'(2\pi)$

$= -m^2$

$$\frac{\dot T}{T}=\frac{c^2}{r^2}\Bigg[\frac{1}{R}(r^2R')'+\underbrace{\frac{1}{\sin\theta}\frac{1}{Z}\frac{d}{d\theta}\left(\sin\theta\frac{dZ}{d\theta}\right)+\frac{1}{\sin^2\theta}\left(\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}\right)}_{=-\mu}\Bigg]=-c^2\lambda$$

Associated Legendre eq. $\longrightarrow \mu = \ell(\ell+1)$

$P_\ell^m(\cos\theta)$

Radial part of the eq.: $R''+\frac{2}{r}R'+\left(\lambda-\frac{\ell(\ell+1)}{r^2}\right)R=0$ — Not Bessel eq.

Change of vars.: $R=\frac{S}{\sqrt{r}}$

$S''+\frac{1}{r}S'+\left(\lambda-\frac{(\ell+1/2)^2}{r^2}\right)S=0 \quad$ Spherical Bessel eq.

Not regular at $r=0$.

$S = A_\ell J_{\ell+1/2}(\sqrt{\lambda}\,r) + \cancel{B_\ell J_{-(\ell+1/2)}(\sqrt{\lambda}\,r)}$

$R_\ell(r)=\frac{A_\ell}{\sqrt{r}}J_{\ell+1/2}(\sqrt{\lambda}\,r) \xrightarrow[\text{B.C.}]{} \lambda_n$ (not easy)

$Y_\ell^m(\theta,\varphi)$

$$u=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{m=-\ell}^{\ell} A_{\ell m}\,e^{-c^2\lambda_n t}\,P_\ell^m(\cos\theta)\,\frac{J_{\ell+1/2}(\sqrt{\lambda_n}\,r)}{\sqrt{r}}\,(a_m\cos(m\varphi)+b_m\sin(m\varphi))$$

$\to -\ell\leq m\leq \ell$

ALWAYS if we find P_ℓ^m

(10) $u_{xx} = u_{tt} + \sin x$

$u(t,0) = 7(1-t)$ | $u(0,x) = 7$

$u_x(t,\pi) = 0$ | $u_t(0,x) = 0$

$\underline{V = u - 7(1-t)}$ $\rightarrow$ New B.C. $\Rightarrow V(t,0) = V_x(t,\pi) = 0$ ✓

New I.C. $\Rightarrow V(0,x) = 0,\ V_t(0,x) = 7$

$$\left.\begin{array}{l} V_{xx} = u_{xx} \\ V_{tt} = u_{tt} \end{array}\right\} \quad V_{xx} = V_{tt} + \sin x$$

Solve $V_{xx} = V_{tt}$ by separation of vars.

$V = TX \rightarrow \dfrac{X''}{X} = \dfrac{\ddot{T}}{T} = -\lambda$ $\rightarrow$ this won't be the eq. for T anymore

$X'' + \lambda X = 0 \ \oplus\ X(0) = 0,\ X'(\pi) = 0$

$\lambda_n = (n + 1/2)^2$

$X_n = \sin([n + 1/2]x)$

$$V = \sum_{n=0}^{\infty} T_n(t) \sin[(n+1/2)x]$$

$$V_{xx} = -\sum_{n=0}^{\infty} T_n(t)(n+1/2)^2 \sin[(n+1/2)x] =$$

$$= V_{tt} + \sin x = \sum_{n=0}^{\infty} \ddot{T}_n \sin[(n+1/2)x] + \sin x$$

$$\sum_{n=0}^{\infty} \left(\ddot{T}_n + (n+1/2)^2 T_n\right) \sin[(n+1/2)x] = -\sin x = -\sum_{n=0}^{\infty} a_n \sin[(n+1/2)x]$$

$$\ddot{T}_n + (n+1/2)^2 T_n = -a_n \quad (*)$$

$\rightarrow T = 4\pi$

$$\sin x = \sum_{n=0}^{\infty} a_n \sin[(n+1/2)x] \rightarrow a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cdot \sin[(n+1/2)x]\,dx =$$

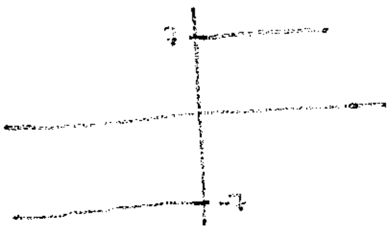
$\hookrightarrow$ Could also be done with $T = 4\pi$

$$= \frac{8}{\pi} \frac{(-1)^{n+1}}{(2n+3)(2n-1)}$$

From (*) we get: $T_n = \dfrac{8(-1)^n}{(2n+3)(2n-1)(n+\frac{1}{2})^2} + A_n \cos((n+1/2)t) + B_n \sin((n+1/2)t)$

$$V(0,x) = \sum_{n=0}^{\infty} T_n(0) \sin[(n+1/2)x] = 0 \rightarrow T_n(0) = 0 \rightarrow A_n = -\frac{8(-1)^n}{(2n+3)(2n-1)(n+\frac{1}{2})^2}$$

$$V_t(0,x) = \sum_{n=0}^{\infty} \dot{T}_n(0) \sin[(n+1/2)x] = \sum_{n=0}^{\infty} B_n (n+1/2) \sin[(n+1/2)x] = 7$$

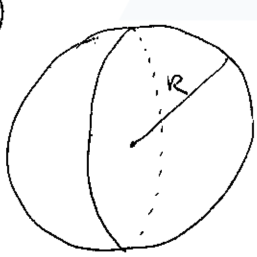


Must expand 7, but it has to be odd, as the expansion only has sines.

$$7 = \sum_{n=0}^{\infty} \underbrace{\frac{14}{\pi} \frac{1}{n+1/2}}_{B_n(n+1/2)} \sin[(n+1/2)x]$$

$$u = V + 7(1-t)$$

(15)



$u_t = \alpha \nabla^2 u$

No φ dependence

Stationary

$u(r,\theta) \Rightarrow \nabla^2 u = 0$ (Laplace eq.)

$$u = \sum_{l=0}^{\infty} \left(a_l r^l + \cancel{\frac{b_l}{r^{l+1}}}\right) P_l(\cos\theta) = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta)$$

↳ Regularity at origin $\rightarrow b_l = 0$

$$\text{B.C.} \Rightarrow u(R,\theta) = \sin^2\theta = \sum_{l=0}^{\infty} a_l R^l P_l(\cos\theta) = \frac{2}{3}(P_0 - P_2)$$

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \left(\frac{2}{3}P_2 + \frac{1}{3}\right) = \frac{2}{3}(P_0 - P_2)$$

$P_0 = 1$

$P_2 = \frac{1}{2}(3\cos^2\theta - 1)$

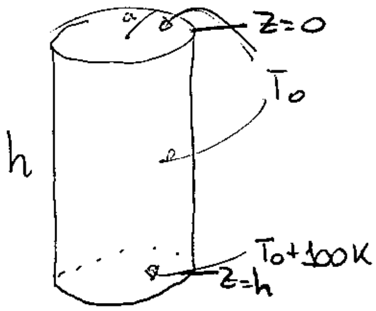
$a_l = 0, \ l \neq 0, 2$

$a_0 = \frac{2}{3}$

$a_2 = -2/3 \, \frac{1}{R^2}$

$$\boxed{u(r,\theta) = \frac{2}{3} - \frac{2}{3}\frac{r^2}{R^2} P_2(\cos\theta)}$$

19



$$u_t = a^2 \nabla^2 u \rightarrow \nabla^2 u = 0$$

$u_t = 0$ — stationary

$u_\varphi = 0$

↳ Due to simetry ⇒ No φ depende (No need to suppose it, anyway)

$u = T_0 + V$

↳ for hom. B.C.

$V = R(\rho) Z(z)$

$V=0$ for $\rho = a$ (*)

$V=0$ on top of cylinder

$V=100$ on bottom "

$$\frac{1}{\rho R}(\rho R')' + \underbrace{\frac{1}{Z}\cdot\frac{d^2Z}{dz^2}}_{K} = 0$$

$$R'' + \frac{1}{\rho}R' + KR \rightarrow \text{Bessel eq. } (m=0) \rightarrow R = A J_0(\sqrt{K}\rho) + \cancel{B N_0(\sqrt{K}\rho)}$$

$B = 0$ r-Regularity

$$(*) \rightarrow \sqrt{K_n}\, a = \alpha_n^{(0)} = \beta_n \rightarrow K_n = \frac{\beta_n^2}{a^2}$$

$$\frac{d^2 Z_n}{dz^2} - \frac{\beta_n^2}{a^2} Z_n = 0 \rightarrow Z_n = C_n e^{\frac{\beta_n}{a}z} + D_n e^{-\frac{\beta_n}{a}z}$$

$$V = \sum_{n=1}^{\infty} J_0\left(\frac{\beta_n}{a}\rho\right)\left(C_n e^{\frac{\beta_n}{a}z} + D_n e^{-\frac{\beta_n}{a}z}\right)$$

$$V(\rho,0) = 0 = \sum_{n=1}^{\infty} J_0\left(\frac{\beta_n}{a}\rho\right)(C_n + D_n) \rightarrow C_n + D_n = 0 \rightarrow -C_n = D_n$$

$$V = \sum_{n=1}^{\infty} J_0\left(\frac{\beta_n}{a}\rho\right) 2C_n \sinh\left(\frac{\beta_n}{a}z\right)$$

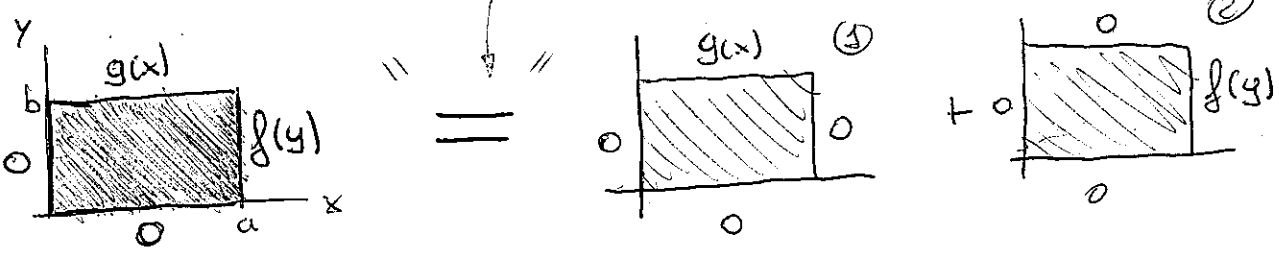
$$V(\rho,-h) = 100 = -\sum_{n=1}^{\infty} J_0\left(\frac{\beta_n}{a}\rho\right) 2C_n \sinh\left(\frac{\beta_n}{a}h\right)$$

$$-2C_n \sinh\left(\frac{\beta_n}{a}h\right) = \frac{\langle 100 | J_0(\frac{\beta_n}{a}\rho)\rangle}{\frac{1}{2}J_1^2(\beta_n)} = \ldots = \frac{200}{\beta_n J_1(\beta_n)}$$

$$u = T_0 - \sum_{n=1}^{\infty} J_0\left(\frac{\beta_n}{a}h\right)\frac{200}{\beta_n J_1(\beta_n)}\cdot\frac{\sinh\left(\frac{\beta_n}{a}z\right)}{\sinh\left(\frac{\beta_n}{a}h\right)}$$

11) $u_{xx} + u_{yy} = 0$

because of the linearity of the Laplace eq.



① $u = XY$ $\quad \dot{} \equiv \frac{d}{dy}$

$$\frac{X''}{X} = -\frac{\ddot{Y}}{Y} = -\lambda \Rightarrow X_n = \sin\left(n\frac{\pi}{a}x\right)$$

$$\hookrightarrow Y_n = A_n \cosh\left(n\frac{\pi}{a}y\right) + B_n \sinh\left(n\frac{\pi}{a}y\right)$$

$$Y_n(0) = 0 \rightarrow A_n = 0$$

$$u^{①} = \sum_{n=1}^{\infty} B_n \sinh\left(n\frac{\pi}{a}y\right) \sin\left(n\frac{\pi}{a}x\right)$$

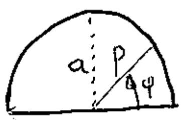
$$u^{①}(x,b) = g(x) = \sum_{n=1}^{\infty} B_n \sinh\left(n\frac{\pi}{a}b\right) \sin\left(n\frac{\pi}{a}x\right)$$

$$B_n \sinh\left(n\frac{\pi}{a}b\right) = \frac{2}{a}\int_0^a g(x) \sin\left(n\frac{\pi}{a}x\right) dx$$

② → "Same"

---

20) $$u_{tt} = c^2\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}\right]$$



As usual :

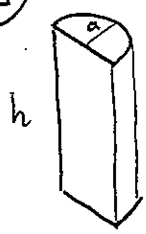
$$u = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} J_m\left(\frac{\alpha_n^{(m)}}{a}\rho\right)\left(C_{nm}\cos\left(c\frac{\alpha_n^{(m)}}{a}t\right) + D_{nm}\sin\left(c\frac{\alpha_n^{(m)}}{a}t\right)\right)\left(a_m\cos(m\varphi) + b_m\sin(m\varphi)\right)$$

$u(\varphi=0) = 0 \rightarrow a_m = 0 \;\; \forall m \longrightarrow m=0$ is not possible anymore.
$\hookrightarrow u=0$

$u(\varphi=\pi) = 0 \rightarrow$ Already due to

$$u=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \cdots \sin(m\varphi) \Longrightarrow \text{MINIMUM: } \cancel{c\,\frac{\alpha_1}{a}}$$

(21)



$$u_{tt}=c^2\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2u}{\partial\varphi^2}+\frac{\partial^2u}{\partial z^2}\right]$$

$$u=R\Phi Z$$

$$\frac{\ddot{T}}{T}=c^2\left[\frac{1}{\rho}\frac{1}{R}(\rho R')'+\frac{1}{\rho^2}\overbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}}^{-m^2}+\overbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}^{\varkappa}\right]=-c^2\lambda$$

$$R''+\frac{1}{\rho}R'+\left[(\varkappa+\lambda)-\frac{m^2}{\rho^2}\right]R=0 \rightarrow R=J_m\left(\sqrt{\varkappa+\lambda}\,\rho\right)$$

$$R(a)=0 \rightarrow J_m\left(\sqrt{\varkappa+\lambda}\,a\right)=0 \rightarrow \sqrt{\varkappa+\lambda}=\alpha_n^{(m)}/a$$

$$R_{mn}=J_m\left(\alpha_n^{(m)}\frac{\rho}{a}\right)$$

$$\frac{d^2Z}{dz^2}=\varkappa Z \rightarrow Z=Ae^{\sqrt{\varkappa}z}+Be^{-\sqrt{\varkappa}z} \qquad \begin{array}{l} Z(0)=0 \\ Z(h)=0 \end{array}$$

↳ must be 0 at

$$\varkappa<0 \rightarrow Z=A\cos(\sqrt{-\varkappa}\,z)+B\sin(\sqrt{-\varkappa}\,z)$$

$$\varkappa_k=-\frac{k^2\pi^2}{h^2} \rightarrow \sin\left(k\frac{\pi}{h}z\right)$$

$$\text{Sol.: } c\sqrt{\left(\frac{\alpha_n^{(m)}}{a}\right)^2+\frac{\pi^2}{h^2}}$$

(23)

$$u_t=a^2\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)\right)+b^2$$

$$u=T_{(t)}X_{(\rho)}$$

$$\frac{\dot{T}}{T}=a^2\frac{1}{\rho X}(\rho X')'=-a^2\lambda$$

$$X''+\frac{1}{\rho}X'+\lambda X=0 \rightarrow X=J_0(\sqrt{\lambda}\rho)$$

$$T\left[J_0(\sqrt{\lambda}R)+R\sqrt{\lambda}\,J_0'(\sqrt{\lambda}R)\right]=0 \rightarrow$$ Eigenvalues are $\sqrt{\lambda_n}R=\gamma_n$ where γ_n are $J_0(\gamma_n)+\gamma_n J_0'(\gamma_n)=0$.

$$u=\sum_{n=1}^{\infty} T_n(t)\, J_0\left(\frac{\gamma_n}{R}\rho\right) \rightarrow \dot{T}_n + a^2\frac{\gamma_n^2}{R^2}T_n = a_n \leftarrow b^2 = \sum_{n=1}^{\infty} a_n J_0\left(\frac{\gamma_n}{R}\rho\right)$$

Sol.:

$$u=\frac{2R^2b^2}{a^2}\sum_{n=1}^{\infty}\frac{1}{\gamma_n^3(1+\gamma_n^2)\,J_1(\gamma_1)}\left(1-e^{-a^2\frac{\gamma_n^2}{R^2}t}\right) J_0\left(\frac{\gamma_n}{R}\rho\right)$$

22 $\phi_t=\phi_{xx}+2\phi_x$, $t\geq 0$, $0\leq x\leq\pi$

B.C.: $\phi(t,0)=2t$, $\phi(t,\pi)=2(t+1)$

I.C.: $\phi(0,x)=\frac{2x}{\pi}$

Homogeneize the B.C.:

$$V=\phi-2t-\frac{2x}{\pi} \overset{\text{B.C.}}{\underset{\text{I.C.}}{\Longrightarrow}} \begin{cases} V(t,0)=V(t,\pi)=0 \\ V(0,x)=0 \end{cases}$$

$\Downarrow$ New PDE

$$V_t=V_{xx}+2V_x+\frac{4}{\pi}-2$$

$$V=TX \rightarrow \frac{\dot{T}}{T}=\frac{X''+2X'}{X}=-\lambda$$

$$X''+2X'+\lambda X=0 \quad \oplus \quad X(0)=X(\pi)=0$$

$$(\dots)\rightarrow \lambda_n=1+n^2,\ X_n=e^{-x}\sin(nx),\ n=1,2,\dots$$

$$V=\sum_{n=1}^{\infty} T_n e^{-x}\sin(nx)$$

$$V_t=\sum_{n=1}^{\infty}\dot{T}_n e^{-x}\sin(nx)=V_{xx}+2V_x+\frac{4}{\pi}-2=$$

$$=\sum_{n=1}^{\infty}T_n\underbrace{(X_n''+2X_n')}_{-\lambda_n X_n}+\frac{4}{\pi}-2=-\sum_{n=1}^{\infty}T_n(1+n^2)e^{-x}\sin(nx)+\frac{4}{\pi}-2$$

$$\sum_{n=1}^{\infty}\left(\dot{T}_n+(1+n^2)T_n\right)e^{-x}\sin(nx)=\frac{4}{\pi}-2$$

weight given by $X''+2X'+\lambda X=0$

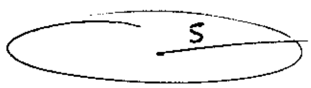
$$\frac{4}{\pi}-2=\sum_{n=1}^{\infty} b_n e^{-x}\sin(nx)\rightarrow b_n=\frac{\langle X_n|\frac{4}{\pi}-2\rangle}{\langle X_n|X_n\rangle}=\frac{\int_0^{\pi}(\frac{4}{\pi}-2)e^{-x}\sin(nx)e^{2x}dx}{\int_0^{\pi}(e^{-x}\sin(nx))^2e^{2x}dx}=$$

$$=(\dots)=\frac{2}{\pi}\left(\frac{4}{\pi}-2\right)n\,\frac{[1-e^{\pi}(-1)^n]}{1+n^2}$$

$$\dot{T}_n+(1+n^2)T_n=b_n\rightarrow T_n=\frac{b_n}{(1+n^2)}+C_n e^{-(1+n^2)t}$$

$$\sum_{n=1}^{\infty}T_n(0)e^{-x}\sin(nx)\overset{\text{I.C.}}{=}0\rightarrow \underset{0}{\underset{\parallel}{T_n(0)}}=\frac{b_n}{1+n^2}+C_n,\forall n\rightarrow C_n=\frac{-b_n}{1+n^2}$$

(26) $u_{tt}=c^2\left(u_{\rho\rho}+\frac{1}{\rho}u_\rho\right)$ $\begin{cases} u_t(0,\rho)=(5-\rho)10 \\ u(0,\rho)=0\end{cases}$ $\begin{cases}\text{Regularity}\\ \text{Edges are fixed.}\end{cases}$



$u=TR$

$$R''+\frac{1}{\rho}R'+\lambda R=0\rightarrow R=J_0(\sqrt{\lambda}\rho)$$

$$J_0(\sqrt{\lambda}5)=0\rightarrow\sqrt{\lambda_n}5=\alpha_n^{(0)}=\beta_n\Rightarrow\lambda_n=\frac{\beta_n^2}{25}\rightarrow J_0\left(\frac{\beta_n}{5}\rho\right)$$

$$\ddot{T}_n+c^2\frac{\beta_n^2}{25}T_n=0$$

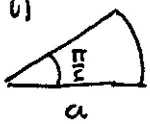
$$u=\sum_{n=1}^{\infty}\left[A_n\cos\left(c\frac{\beta_n}{5}t\right)+B_n\sin\left(c\frac{\beta_n}{5}t\right)\right]J_0\left(\frac{\beta_n}{5}\rho\right)$$

$$u(0,\rho)=0\rightarrow A_n=0$$

$$u_t(0,\rho)=(5-\rho)10\longrightarrow B_n$$

$$u=\sum_{n=1}^{\infty}\frac{500\cdot\mu}{\beta_n^2\,c\,J(\beta_n)}\sin\left(\frac{\beta_n}{5}ct\right)J_0\left(\beta_n\frac{\rho}{5}\right)$$

$$\mu=\frac{\int_0^{\beta_n}J_0(x)dx}{\beta_n^2\,J_1(\beta_n)}$$

27 i) 

$$u_{tt} = c^2\left(u_{\rho\rho} + \frac{1}{\rho}u_\rho + \frac{u_{\varphi\varphi}}{\rho^2}\right)$$

S.V. $\rightarrow u = TR\phi$

$$\phi'' + n^2\phi = 0 \rightarrow \Phi_m = a_m\cos(m\varphi) + b_m\sin(m\varphi)$$

$$R'' + \frac{1}{\rho}R' + R\left(\lambda - \frac{m^2}{\rho^2}\right) = 0 \rightarrow R = J_m(\sqrt{\lambda}\rho)$$

B.C. $\rightarrow J_m(\sqrt{\lambda}a) = 0 \rightarrow \lambda_n = \left(\frac{\alpha_n^{(m)}}{a}\right)^2$

$$\ddot{T} + c^2\left(\frac{\alpha_n^{(m)}}{a}\right)^2 T = 0$$

$$u = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[A_{nm}\cos\left(c\frac{\alpha_n^{(m)}}{a}t\right) + B_{nm}\sin\left(c\frac{\alpha_n^{(m)}}{a}t\right)\right]\cdot$$

$$\cdot J_{nm}\left(\frac{\alpha_n^{(m)}}{a}\rho\right)\left[a_m\cos(m\varphi) + b_m\sin(m\varphi)\right]$$

$u(\varphi=0) = 0 \rightarrow a_m = 0$ ($\Rightarrow$ m starts at $m=1$).

$$u(\varphi = \pi/6) = 0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{nm}\cos(\ ) + B_{nm}\sin(\ )\right]J_m\left(\frac{\alpha_n^{(m)}}{a}\rho\right)b_m\sin\left(m\frac{\pi}{6}\right) = 0$$

$m\frac{\pi}{6} = k\pi \rightarrow m = 6k$, $k = 1, 2, 3, \dots$

$$u = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left[\cos\left(c\frac{\alpha_n^{(6k)}}{a}t\right) + \sin(\ )\right]J_{6k}\left(\frac{\alpha_n^{(6k)}}{a}\rho\right)\sin(6k\varphi)$$

$k=1, n=1 \rightarrow \omega = c\frac{\alpha_1^{(6)}}{a}$

ii) $\cancel{u_t} = c^2\left(u_{\rho\rho} + \frac{1}{\rho}u_\rho + \frac{u_{\varphi\varphi}}{\rho^2}\right) \rightarrow$ stationary $\Rightarrow \underline{u_t = 0}$

S.V. $\rightarrow u = R\Phi$

$\Phi_m = a_m\cos(m\varphi) + b_m\sin(m\varphi)$

Regularity

$m \neq 0 \rightarrow R'' + \frac{1}{\rho}R' - \frac{m^2}{\rho^2}R = 0 \rightarrow$ Cauchy-Euler $\rightarrow R_m = A_m\rho^m + \cancel{B_m\rho^{-m}}$

$m = 0 \rightarrow R_0 = A\log\rho + B \rightarrow A = 0$ (regularity)

$$u = \sum_{m=0}^{\infty} \rho^m \left(C_m \cos(m\varphi) + D_m \sin(m\varphi) \right)$$

$$u(\varphi=0) = 0 \rightarrow C_m = 0 \quad \forall m \geq 0$$

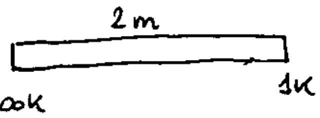
$$u(\varphi = \pi/6) = 0 \rightarrow \sum_{m=1}^{\infty} D_m \rho^m \sin(m\,\pi/6) = 0 \rightarrow m\frac{\pi}{6} = K\pi \rightarrow m = 6K, \; K = 1, 2, \dots$$

$$u = \sum_{K=1}^{\infty} E_K \sin(6K\varphi)\, \rho^{6K}$$

$$u(\rho = a) = \varphi = \sum_{K=1}^{\infty} E_K \sin(6K\varphi)\, a^{6K} \rightarrow E_K a^{6K} = \frac{12}{\pi}\int_0^{\pi/6} \varphi \sin(6K\varphi)\, d\varphi = \frac{(-1)^{K+1}}{3K}$$

$$u = \sum_{K=1}^{\infty} \frac{(-1)^{K+1}}{3K}\left(\frac{\rho}{a}\right)^{6K} \sin(6K\varphi)$$

(28)



$$u_t = \alpha u_{xx} + Q\sin\left(\frac{\pi x}{2}\right)$$

B.C. non-hom. $\rightarrow$ $u = V + 500 - 200x \Rightarrow \begin{cases} V(x=0) = 0 \\ V(x=2) = 0 \end{cases}$ New B.C.

$$\underbrace{V_t = \alpha V_{xx}} + Q\sin\left(\frac{\pi x}{2}\right)$$

$$V = T X$$

$$\left.\begin{aligned} X'' + \lambda X = 0 \\ X(0) = X(2) = 0 \end{aligned}\right\} \sqrt{\lambda_n} = \frac{n\pi}{2}, \quad X_n = \sin\left(n\frac{\pi}{2}x\right)$$

$$V = \sum_{n=1}^{\infty} T_n(t)\sin\left(n\frac{\pi}{2}x\right) \xrightarrow{\text{to PDE}} \sum_{n=1}^{\infty} \dot{T}_n \sin\left(n\frac{\pi}{2}x\right) = -\alpha \sum_{n=1} T_n \frac{n^2\pi^2}{4}\sin\left(n\frac{\pi}{2}x\right) + Q\sin\left(\frac{\pi x}{2}\right)$$

$$n \neq 1 \rightarrow \dot{T}_n + \alpha\frac{n^2\pi^2}{4} T_n = 0 \rightarrow T_n = A_n e^{-\alpha\frac{n^2\pi^2}{4}t}$$

$$n = 1 \rightarrow \dot{T}_1 + \alpha\frac{\pi^2}{4} T_1 = Q \rightarrow T_1 = \frac{4Q}{\alpha\pi^2} + A_1 e^{-\alpha\frac{\pi^2}{4}t}$$

$$u = 500 - 200x + \sum_{n=1}^{\infty} A_n e^{-\alpha^2\frac{n^2\pi^2}{4}t}\sin\left(n\frac{\pi}{2}x\right) + \frac{4Q}{\alpha\pi^2}\sin\left(\frac{\pi}{2}x\right)$$

(24) $u_t = c^2 u_{xx}$

$\left.\begin{array}{l} u(0,t)=1 \\ u(L,t)=2e^{-\mu t} \end{array}\right\}$ B.C. (non-hom)

$u(0,x) = 3x+1 \rightarrow$ I.C.

$V = u - g(x)e^{-\mu t} + f(x) \rightarrow$ Hom. B.C. on $\underline{\underline{V}}$ $\left\{\begin{array}{l} g(0)=0,\ g(L)=2 \\ f(0)=-1,\ f(L)=0 \end{array}\right. \rightarrow \begin{array}{l} g(x)=2\frac{x}{L} \\ f(x)=\frac{x-L}{L} \end{array}$

$V = u - \frac{2x}{L}e^{-\mu t} + \frac{x}{L} - 1$

NEW PDE :

$$\underbrace{V_t - c^2 V_{xx}}_{\text{S.V.}} = 2\mu \frac{x}{L} e^{-\mu t}$$

$V = XT \rightarrow X'' + \lambda X = 0 \rightarrow \lambda_n = \frac{n^2\pi^2}{L^2} \rightarrow X_n = \sin\left(n\frac{\pi}{L}x\right)$

$$V = \sum_{n=1}^{\infty} T_n \sin\left(n\frac{\pi}{L}x\right)$$

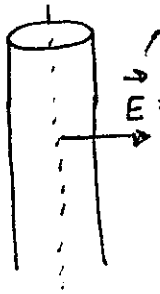
expand

$$V_t - V_{xx} = \sum_{n=1}^{\infty}\left(\dot{T}_n + c^2 n^2 \frac{\pi^2}{L^2} T_n\right)\sin\left(n\frac{\pi}{L}x\right) = 2\mu\frac{x}{L}e^{-\mu t} = \frac{4\mu}{\pi}e^{-\mu t}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\left(n\frac{\pi}{L}x\right)$$

$$\dot{T}_n + c^2\frac{n^2\pi^2}{L^2}T_n = \frac{4\mu}{\pi}e^{-\mu t}\frac{(-1)^{n-1}}{n}, \quad \forall n = 1,2,\dots$$

$$T_n = \underset{\text{known}}{A e^{-\mu t}} + \underset{\text{use I.C.}}{C_n e^{-c^2\frac{n^2\pi^2}{L^2}t}}$$

(34)

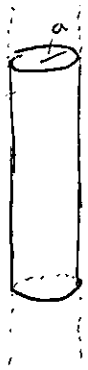


ctant. $\rightarrow \phi = \phi(\rho)$ (linear) $\rightarrow$ outside

$\vec{E} = -\nabla\phi$

$\nabla^2\phi = 0$ $\rightarrow$ inside

(30)



$$u_t = \alpha^2 \nabla^2 u$$

$\rightarrow v = u - T_2$

$T(t,a) = T_2 \oplus$ Regularity at axis

$T(0,\rho) = T_1$

$\rightarrow$ z and φ are ommitted because of symetry

$$u_t = \alpha^2 \left(u_{\rho\rho} + \frac{2}{\rho} u_\rho\right)$$

For a finite cilinder, we cannot ommit the z dependence!

$J_0\left(\alpha_n^{(m)} \frac{\rho}{a}\right) \sin\left(k \frac{\pi z}{h}\right)$

(25) $\phi = XY$

$$x^2 \frac{X'' + \frac{1}{x} X'}{X} = -\frac{\ddot{Y}}{Y} = \lambda^2$$

$$\ddot{Y} + \lambda^2 Y = 0 \rightarrow \begin{cases} A\cos(\lambda y) + B\sin(\lambda y) & \lambda^2 > 0 \\ Ay + B & \lambda^2 = 0 \\ A\cosh(|\lambda| y) + B\sinh(|\lambda| y) & \lambda^2 < 0 \end{cases}$$

$$X'' + \frac{X'}{x} - \frac{\lambda^2}{x^2} X = 0 \quad \begin{cases} X = Cx^\lambda + Dx^{-\lambda} & \lambda \neq 0 \\ Y = C + D\ln x & \lambda = 0 \end{cases}$$

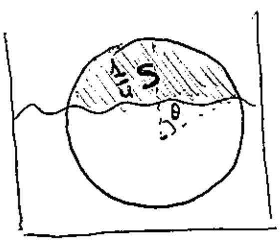
$$X'(a) = 0 \longrightarrow C = Da^{-2\lambda} \rightarrow D = 0$$

$$\left[\left(\frac{x}{a}\right)^\lambda + \left(\frac{x}{a}\right)^{-\lambda}\right]\left(A\cos(\lambda y) + B\sin(\lambda y)\right] + (\bar{A}y + B)C$$

Using B.C.:

$$\phi = \overset{\text{data}}{A} \cos\left(x + \frac{a^2}{x}\right) y + \bar{A}y + \bar{B}$$

(29)



$$\Delta T = 0$$

$$T = \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell(\cos\theta)$$

Spherical cap: $S = 2\pi r^2 (1 - \cos\theta) \longrightarrow \frac{1}{4}$(Total area) $\rightarrow \cos\theta = \frac{1}{2} \rightarrow \theta = \frac{\pi}{3}$

$$T(R,\theta) = \begin{cases} T_0 & 0 \leq \theta \leq \pi/3 \\ T_1 & \text{otherwise} \end{cases}$$

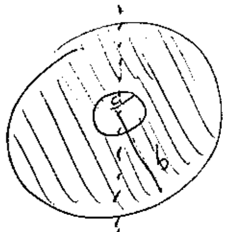
$x = \cos\theta$

$$\frac{2l+1}{2}\left(T_0\int_{1/2}^{1} P_l(x)\,dx + T_1\int_{-1}^{1/2} P_l(x)\,dx\right) = a_l$$

(31) $\nabla^2 u = 0$

$u(a,\theta) = T_a$ ← Only P_0 and P_1 are involved

$u(b,\theta) = T_b(1-\cos\theta)$



$$u = \sum_{l=0}^{\infty}\left(a_l r^l + \frac{b_l}{r^{l+1}}\right) P_l(\cos\theta)$$

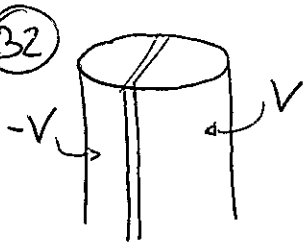
$$u(a,\theta) = \sum_{l=0}^{\infty}\left(a_l a^l + \frac{b_l}{a^{l+1}}\right) P_l(\cos\theta) = T_a$$

$$u(b,\theta) = \sum_{l=0}^{\infty}\left(a_l b^l + \frac{b_l}{b^{l+1}}\right) P_l(\cos\theta) = T_b(1-\cos\theta)$$

$$a_l, b_l = 0 \ \forall l \geq 2 \rightarrow a_0, b_0, a_1, b_1$$

$$u = \frac{1}{a-b}\left[aT_1 - bT_2 + \frac{ab}{r}(T_b - T_a)\right] + \frac{b^2 T_b\cos\theta}{a^3-b^3}\left(r - \frac{a^3}{r^2}\right)$$

(32)



Cylindrical symmetry

$\nabla^2 U = 0$

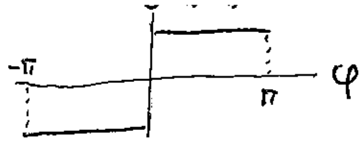
$U = R(\rho)\,\Phi(\varphi)$

$\Phi_m = A_m\cos(m\varphi) + B_m\sin(m\varphi)$

$$R'' + \frac{1}{\rho}R' - \frac{m^2}{\rho^2}R = 0 \rightarrow R = C_1\rho^m + \overset{C_2=0}{\cancel{C_2\rho^{-m}}}$$

$$U = \sum_{m=0}^{\infty}\rho^m\left(A_m\cos(m\varphi) + B_m\sin(m\varphi)\right)$$

$$U_{(\rho>a)} = \begin{cases} V & 0<\varphi<\pi \\ -V & -\pi<\varphi<0 \end{cases}$$



$$U(\rho=a) = \sum_{m=0}^{\infty} a^m \left(A_m \cos(m\varphi) + B_m \sin(m\varphi)\right) = \begin{cases} V & 0<\varphi<\pi \\ -V & -\pi<\varphi<0 \end{cases}$$

FOURIER SERIES PROBLEM

$$A_m = 0, \quad B_{2k} = 0, \quad B_{2k+1} = \frac{4V}{\pi(2k+1)} \frac{1}{a^{2k+1}}$$

(3) $u_{xx}+3u_{xy}-4u_{yy}-u_x+u_y=0$

$A=1,\ B=3,\ C=-4$

$B^2-4AC=25>0 \rightarrow$ Hyperbolic

$A\,dy^2-B\,dx\,dy+C\,dx^2=0$

$(dy+dx)(dy-4dx)=0 \longrightarrow$ CHARACTERISTICS: $\begin{cases} y+x=C_1 \\ y-4x=C_2 \end{cases}$

CHANGE OF VARS. $\begin{cases} \xi=x+y \\ \eta=y-4x \end{cases} \Rightarrow \dfrac{\partial}{\partial x}=\dfrac{\partial\xi}{\partial x}\dfrac{\partial}{\partial\xi}+\dfrac{\partial\eta}{\partial x}\dfrac{\partial}{\partial\eta}=\dfrac{\partial}{\partial\xi}-4\dfrac{\partial}{\partial\eta}$

$$\frac{\partial}{\partial y}=\frac{\partial\xi}{\partial y}\frac{\partial}{\partial\xi}+\frac{\partial\eta}{\partial y}\frac{\partial}{\partial\eta}=\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}$$

$$\frac{\partial^2}{\partial x^2}=\frac{\partial}{\partial x}\cdot\frac{\partial}{\partial x}=\left(\frac{\partial}{\partial\xi}-4\frac{\partial}{\partial\eta}\right)\left(\frac{\partial}{\partial\xi}-4\frac{\partial}{\partial\eta}\right)=$$

$$=\frac{\partial^2}{\partial\xi^2}-8\frac{\partial^2}{\partial\xi\partial\eta}+16\frac{\partial^2}{\partial\eta^2}$$

New form of PDE:

$\boxed{u_{\xi\eta}=\dfrac{u_\eta}{5}}$: $u_\eta=q \longrightarrow q_\xi=\dfrac{q}{5}\longrightarrow q=f(\eta)e^{\xi/5}$

$u_\eta=f(\eta)e^{\xi/5} \rightarrow \boxed{u=F(\eta)e^{\xi/5}+g(\xi)}$

$$\boxed{u=F(y-4x)\,e^{\frac{x+y}{5}}+g(x+y)}$$

Boundary conds.:

$\begin{cases} u|_{y=4x}=5x+e^x=F(0)\,e^x+g(5x) \\ u|_{y=-x}=1=F(-5x)+g(0)\longrightarrow F=1-g(0)=\text{ctant.} \end{cases}$

$5x+\cancel{e^x}=(\cancel{1}-g(0))e^x+g(5x)\rightarrow g(5x)=5x+g(0)e^x \rightarrow$

$\rightarrow g(\otimes) = \otimes + g(0)\, e^{\otimes/5}$

$$u = x + y + e^{\frac{x+y}{5}}$$

Sol. is unique because we're giving B.C., not I.C.!!

④ $y u_{yy} - x\, u_{xy} + u_y = x^2$

$A = 0,\ B = -x,\ C = y$

$B^2 - 4AC = x^2 > 0 \Rightarrow$ Hyperbolic $(x \neq 0)$

$+x\,dx\,dy + y\,dx^2 = 0 \begin{cases} dx = 0 \rightarrow x = C_1 \\ y\,dx + x\,dy = 0 \rightarrow xy = C_2 \end{cases}$

$\left\{ \begin{matrix} \xi = x \\ \eta = xy \end{matrix} \right\} \Rightarrow \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + y \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\eta}{\xi} \frac{\partial}{\partial \eta}$

$y = \eta/\xi$

$\frac{\partial}{\partial y} = x \frac{\partial}{\partial \eta} = \xi \frac{\partial}{\partial \eta}$

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial \xi} + \frac{\eta}{\xi} \frac{\partial}{\partial \eta} \right)\left(\frac{\partial}{\partial \xi} + \frac{\eta}{\xi} \frac{\partial}{\partial \eta} \right) =$$

$$= \frac{\partial^2}{\partial \xi^2} - \frac{\eta}{\xi^2} \frac{\partial}{\partial \eta} + \frac{\eta}{\xi} \frac{\partial^2}{\partial \xi \partial \eta} + \frac{\eta}{\xi} \frac{\partial^2}{\partial \eta \partial \xi} + \frac{\eta}{\xi} \frac{1}{\xi} \frac{\partial}{\partial \eta} + \left(\frac{\eta}{\xi} \right)^2 \frac{\partial^2}{\partial y^2}$$

$\partial_{xy} = \partial_\eta + \xi\, \partial_{\xi\eta} + \eta\, \partial_{\eta\eta}, \quad \partial_{yy} = \xi^2 \partial_{\eta\eta}$

New PDE:

$-\xi^2 u_{\xi\eta} = x^2 = \xi^2 \quad (\xi \neq 0) \rightarrow u_{\xi\eta} = -1$

$u_\xi = -\eta + F(\xi) \rightarrow u = -\eta\xi + f(\xi) + \Phi(\eta)$

$u = -x^2 y + f(x) + \phi(xy)$

⑤ a) Hyperbolic

b) Elliptic if $x \neq 1$; Parabolic if $x = 1$

c) Hyperbolic

Charact. $\rightarrow \begin{matrix} x = C_1 \\ y = C_2\, e^{x^2/2} \end{matrix}$

d) Parabolic

e) $u = 1 \rightarrow$ Sol. (singular solution) $\rightarrow$ Parabolic for $u \neq 1$

f) Got the second order eq. $\rightarrow$ Parabolic

MORE EXERCISES

① $\underbrace{(x+y)\left(x\partial_x - y\partial_y\right)}_{D} z = (x-y)z$

$Dz = (x-y)z$

$\frac{dy}{dx} = \frac{Dy}{Dx} = \frac{-y}{x} \Rightarrow xy = \eta$ ($D\eta = 0$)

η = ctant. are characteristics.

Choose $\xi \to \xi = x+y$ $\longrightarrow \begin{cases} D\xi = (x+y)(x-y) \\ D\eta = 0 \end{cases} \Rightarrow D = \overbrace{(x+y)(x-y)}^{\text{on } \xi,\eta} \frac{\partial}{\partial \xi}$

$\{x,y\} \longrightarrow \{\xi, \eta\}$.

PDE: $\underbrace{(x+y)}_{\xi}\cancel{(x-y)}\frac{\partial z}{\partial \xi} = \cancel{(x-y)}z \rightarrow \xi \frac{\partial z}{\partial \xi} = z$

$z = F(\eta)\xi = (x+y)F(xy) \Rightarrow$ General sol., F arbitrary.

I.C. $\Rightarrow$ $y = x+1$, $z = 1$ $\Rightarrow 1 = (2x+1)F(x(x+1)) \rightarrow F(\underbrace{x(x+1)}_{A}) = \frac{1}{2x+1}$

$\hookrightarrow x^2 + x = A \rightarrow x = \frac{-1 \pm \sqrt{1+4A}}{2}$

$F(A) = \frac{1}{\pm\sqrt{1+4A}}$

$z = (x+y)\frac{1}{(+)\sqrt{1+4xy}}$

For the I.C. to be satisfied.

② $\underbrace{2xy(\partial_x - \partial_y)}_{D} u = (x-y)u$

$\frac{dy}{dx} = \frac{Dy}{Dx} = -1 \rightarrow x+y = \eta$

We choose $\xi = xy$

$\{x,y\} \rightarrow \{\eta, \xi\}$

POE: $2xy\,\cancel{(y-x)}\,\frac{\partial u}{\partial \xi} = \underset{(-)}{\cancel{(x-y)}}\,u \rightarrow 2\xi\,\frac{\partial u}{\partial \xi} = -u \rightarrow u = \frac{f(\eta)}{\sqrt{\xi}} = \frac{f(x+y)}{\sqrt{xy}}$

I.C.: $x=t,\ y=t,\ u=t^2$

$$t^2 = \frac{f(2t)}{t} \rightarrow f(2t) = t^3 \rightarrow f(A) = \frac{A^3}{8} \rightarrow u = \frac{(x+y)^3}{8\sqrt{xy}}$$

I.C.': $\underbrace{x=t,\ y=-t}_{\text{CHARACTERISTIC}},\ u=t^2 \rightarrow$ NO SOL.

③ $\underbrace{(2y\,\partial_x - x\,\partial_y)}_{D} u = 2xy(2y^2-x^2)$

$$\frac{dy}{dx} = \frac{Dy}{Dx} = -\frac{x}{2y} \rightarrow 2y^2+x^2=\eta \rightarrow \text{Choose } \xi:\ \xi = 2y^2-x^2$$

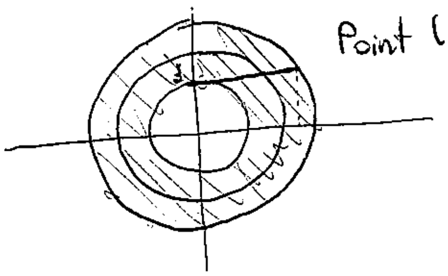
$$\text{PDE} \rightarrow \frac{\partial u}{\partial \xi} = -\xi/4 \rightarrow u = -\xi^2/8 + F(\eta) = -\frac{1}{8}(2y^2-x^2)^2 + f(2y^2+8)$$

I.C.: $u(x,1) = x^2,\ x\in[0,3]$

$$x^2 = -\frac{1}{8}(2-x^2)^2 + f(2+x^2),\ x\in[0,3] \qquad f(A) = A-2+\frac{1}{8}(4-A)^2,\ A\in[2,11]$$

$$f(x^2+2) = x^2+2-2+\frac{1}{8}(4-(2+x^2))^2,\quad x^2+2\in[2,11]$$

$$u = -\frac{1}{8}(2y^2-x^2)^2 + 2y^2+x^2-2+\frac{1}{8}(4-2y^2-x^2)^2 = x^2y^2$$



Point $(2,2)$ is not included. $\rightarrow$ Not possible!

⑤ $y f_{xx} + \underbrace{(y-x)}_{B} f_{xy} \underbrace{-x}_{C} f_{yy} + a \frac{x-y}{x+y}(f_x + f_y) = (x^2+y^2)(x+y)^2$

$\underset{A}{y}$

$B^2 - 4AC = (y+x)^2 > 0 \; , \; x \neq 0, y \neq 0 \Rightarrow$ Hyperbolic

Characteristics : $A\,dy^2 - B\,dx\,dy + C\,dx^2 = 0$

$$y\,dy^2 - (y-x)\,dx\,dy - x\,dx^2 = 0 \quad \begin{cases} \eta = y - x \\ \xi = x^2 + y^2 \end{cases}$$

$\underline{a=1} \rightarrow -2\frac{\partial f}{\partial \xi \partial \eta} = \xi \rightarrow f = -\frac{1}{4}\eta\xi^2 + F(\xi) + g(\eta) =$

$$= F(x^2+y^2) + g(y-x) - \frac{1}{4}(x^2+y^2)^2(y-x)$$

<u>I.C.</u>

$f|_{y=0} = 0 \longrightarrow 0 = F(x^2) + g(-x) + \frac{x^4}{4}$

c will disappear when f g is written on the PDE.

$\frac{\partial f}{\partial y}\Big|_{y=0} \longrightarrow 0 = 0 + g'(-x) - \frac{x^4}{4} \rightarrow g = \frac{x^5}{20} + c$

$F(x^2) - \frac{x^5}{20} + c + \frac{x^4}{4} = 0 \rightarrow F(x^2) = \frac{1}{4}\left(x^4 - \frac{x^5}{5}\right) = \frac{x^4}{4}\left(1 - \frac{x}{5}\right)$

$y = x$ won't give a sol, as it's one of the characteristics

⑥. Wave eq. with speed c.

$y(x) = \begin{cases} \sin\left(\frac{\pi x}{a}\right), & -a \leq x \leq a \\ 0, & |x| > a \end{cases} \quad , \quad y_t(x,0) = 0$

$$y(x) = \sin\left(\frac{\pi x}{a}\right)\left\{\theta(x+a) - \theta(x-a)\right\}$$

D'Alembert solution:

$$y(t,x) = \frac{1}{2}\sin\left[\frac{\pi}{a}(x-ct)\right]\left\{\theta(x-ct+a) - \theta(x-ct-a)\right\} + \frac{1}{2}\sin\left[\frac{\pi}{a}(x+ct)\right]\left\{\theta(x+ct+a) - \theta(x+ct-a)\right\}$$

and $\equiv \cap$
or $\equiv \cup$

⊛ 52 cards

a) Prob. of drawing 2 even cards (2, 4, 6, 8, 10) given that the card is not replaced.

$$P(E_1) = \frac{20}{52} \qquad P(E_1 \cap E_2) = P(E_1) \cdot P(E_2 | E_1) = \frac{20}{52} \cdot \frac{19}{51}$$

$$P(E_2 | E_1) = \frac{19}{51}$$

b) Prob. of 1^{st} card being even and second card not.

$$P(E_1) = \frac{20}{52} \qquad P(E_1 \cap \cancel{E_2}) = P(E_1) \cdot P(\cancel{E_2} | E_1) = \frac{20}{52} \cdot \frac{32}{51}$$

$$P(\cancel{E_2} | E_1) = \frac{32}{51}$$

c) Prob. of both cards not being even.

$$P(\cancel{E_1}) = \frac{32}{52} \qquad P(\cancel{E_1} \cap \cancel{E_2}) = \frac{32}{52} \cdot \frac{31}{51}$$

$$P(\cancel{E_2} | \cancel{E_1}) = \frac{31}{51}$$

⊛ P(+)? → RILEY

$$P(+ | ILL) = 99,99\,\%$$

$$P(+ | NOT\ ILL) = 0,02\,\%$$

$$P(ILL) = 10^{-5} = \frac{1}{10000}$$

$$P(ILL | +) = P(+ | ILL) \cdot \frac{P(ILL)}{P(+)} = 0,9999 \cdot \frac{10^{-5}}{0,9999 \cdot 10^{-5} + 0,0002 \cdot (1 - 10^{-5})} = \frac{1}{3}$$

$$P(A|B) = P(B|A) \frac{P(A)}{P(B)} \qquad \qquad 1 - P(ILL)$$

$$P(+) = P(+ | ILL) \cdot P(ILL) + P(+ | NOT\ ILL) \cdot P(NOT\ ILL)$$

⊛ In a bar from Salamanca, a customer is suporting the Athletic. What is the prob. of he being from Bilbao?

B: The person lives within 30 Km around Bilbao.

A: He supports the Athletic.

DATA: $P(B) = \dfrac{26}{1000000}$, $P(A|B) = \dfrac{92}{100}$, $P(A|\overline{B}) = \dfrac{1}{1000}$

$P(B|A)$? BAYES THEOREM $\longrightarrow$ P([illegible]) $\cdot \dfrac{P(A)}{P(B)}$

$$P(A) = P(A|B)\,P(B) + P(A|\overline{B})\,P(\overline{B}) =$$

$$= \frac{92}{100} \cdot \frac{26}{1000000} + \frac{1}{1000} \cdot \frac{999974}{1000000} =$$

$$P(B|A) = P(A|B)\,\frac{P(B)}{P(A)}$$

⊛ 4 broken pendrives, one of which has your assignment. There's a probability P of retrieving the assignment if it is in the correct pendrive.

$R \equiv$ recovery of the assignment

$A_i \equiv$ assignment is in pendrive i

$R_i \equiv R \cap A_i$

$P(R_i)$?

$P(R|A_i) = P$

$$P(R_i) = P(R \cap A_i) = P(R|A_i) \cdot P(A_i) = P \cdot \frac{1}{4} = \frac{P}{4}$$

$$P(R_1) = \frac{P}{4}$$

$$P(R_2|\overline{R_1}) = P(\overline{R_1}|R_2) \cdot \frac{P(R_2)}{P(\overline{R_1})} = 1 \cdot \frac{P/4}{1 - P/4} = \frac{P}{4-P}$$

3^{rd} pendrive:

$$P(R_3|\bar{R_1}\cap\bar{R_2}) = P(\bar{R_1}\cap\bar{R_2}|R_3)\cdot\frac{P(R_3)}{P(\bar{R_1}\cap\bar{R_2})} = 1\cdot\frac{P/4}{P(\bar{R_1}\cap\bar{R_2})}$$

$$P(\bar{R_1}\cap\bar{R_2}) = \underbrace{P(\bar{R_1}|\bar{R_2})}_{\hookrightarrow 1-P(R_1|\bar{R_2})}\cdot P(\bar{R_1}) = \left(1-\frac{P}{4-P}\right)(1-P/4) = \frac{1}{4}(4-2P)$$

$$P(R_3|\bar{R_1}\cap\bar{R_2}) = \frac{P/4}{\frac{1}{4}(4-2P)} = \frac{P}{4-2P}$$

4^{th} pendrive:

$$P(R_4|\bar{R_1}\cap\bar{R_2}\cap\bar{R_3})$$

⊛ Particle physics theory:

- Fermions $\rightarrow m_F$
- Scalars $\rightarrow m_s$

$$\begin{cases} \cdot\ \rho(m_F)\,dm_F = A\sqrt{\underbrace{\gamma^2}_{\text{DATA}}-m_F^2}\;dm_F \quad , m_F\in[0,\gamma] \\ \cdot\ m_s^2 = m_F^2 - m_F - 2 \quad (*) \end{cases}$$

a) FIND A

b) FIND THE PROB. DISTRIBUTION FOR m_s

a) $$\int_0^\gamma \rho(m_F)\,dm_F = 1 \Rightarrow \int_0^\gamma A\sqrt{\gamma^2-m_F^2}\;dm_F = 1 \Rightarrow A = \frac{4}{\pi\gamma^2}$$

b) $$\rho(m_F)\,dm_F = \frac{4}{\pi\gamma^2}\sqrt{\gamma^2-m_F^2}\;dm_F$$

$$P_y(y)\,dy = P_x(x_1(y))\left|\frac{dx}{dy}\right|_{x_1(y)} + P_x(x_2(y))\left|\frac{dx}{dy}\right|_{x_2(y)}$$

(*) $$m_F^2 - m_F - (2+m_s^2) = 0 \longrightarrow m_F = \frac{1\pm\sqrt{9+4m_s^2}}{2}$$

$$P_{m_s}(m_s)\,dm_s = (\dots) = \left[\frac{4}{\pi\gamma^2}\sqrt{\gamma^2-\left(\frac{1+\sqrt{9+4m_s^2}}{2}\right)^2}\cdot\frac{m_s}{2\sqrt{9+4m_s^2}} + \frac{4}{\pi\gamma^2}\sqrt{\gamma^2-\left(\frac{1-\dots}{\dots}\right)^2}\cdot\frac{m_s}{2\sqrt{9+4m_s^2}}\right]dm_s$$

② 52 cards

Draw two cards, without replacing the first one.

a) A: First card even

B: Second card even

$$P(A) = \frac{n}{N} = \frac{20}{52} = \frac{5}{13}$$

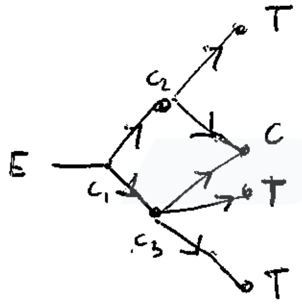
$$P(B|A) = \frac{n-1}{N-1} = \frac{19}{51}$$

$$P(A \cap B) = P(A) \cdot P(B|A) = \frac{5}{13} \cdot \frac{19}{51} = \frac{95}{663}$$

b) $P = \frac{20}{52} \cdot \left(\frac{32}{51}\right) = \frac{--}{-}$

c) $P(2 \text{ not even}) = P(\bar{A}) \cdot P(\bar{B}|\bar{A}) = \frac{32}{52} \cdot \frac{31}{51}$

③



a) $P(C) = P(C \cap C_2) + P(C \cap C_3) =$

$= P(C_2) \cdot P(C|C_2) + P(C_3) \cdot P(C|C_3) =$

$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} = \frac{7}{12}$

⑭ $P(A \cap B) = P(A) + P(B) - P(A \cup B) \geqslant P(A) + P(B) - 1$

$P(A \cup B) = 1 \iff A \cup B = S$

⑦ $N = \frac{N!}{(N-2)!} = N^2 - N$

$$P(A) = \frac{2N}{N^2 - N} = \frac{2}{N-1}$$

$n_A = 2N$

⑧ $n=5$

$P_s = \frac{1}{3}, \quad P_f = \frac{2}{3}$
$\downarrow$
fail

$$\left(p(K) = \frac{n!}{(n-K)!\,K!} p^K (1-p)^{n-K} \right)$$

$$\langle K \rangle = \sum_i k_i P(k_i) = np = 5 \cdot \frac{1}{3} = 5/3$$

$$\langle x \rangle = \langle K \rangle \cdot 1 - 0{,}5(5 - \langle K \rangle) = 0$$

b) $\underline{N|}$ $\quad S = a \cdot K - b(N-K)$

a) $10 = aN - b \cdot 0 \rightarrow a = \frac{10}{N}$

b) $\langle S \rangle = \frac{10}{N} \langle K \rangle - b(N - \langle K \rangle) = 0 = \frac{10}{N} \cdot \frac{N}{3} - b \frac{2N}{3} \rightarrow b = \frac{5}{N}$

$\langle K \rangle = N \cdot p = \frac{N}{3}$

$S = g_K = g_c \cdot K - g_{cN} \cdot (N-K)$

Score : $0 - \frac{5}{N} \cdot N = -5$

$$\langle S^2 \rangle = \sum_{K=0}^{N} \frac{N!}{(N-K)!\,K!} \left(\frac{1}{3}\right)^K \left(\frac{2}{3}\right)^{N-K} (g_K)^2 = \sum_K P_K (g_K)^2$$

$$\sigma = \sqrt{\langle S^2 \rangle - \underset{0}{\langle S \rangle^2}} = \sqrt{\langle S^2 \rangle}$$

(CONT...)

$$\begin{cases} \langle K \rangle = np \\ \sigma^2 = \langle K^2 \rangle - \langle K \rangle^2 \\ \quad = np(1-p) \end{cases}$$

⑨ $\lambda_{obs} = 100$

$\sigma = \sqrt{100} = 10$

$\alpha = \beta = 0{,}05 \quad (1 - \alpha - \beta = 0{,}9)$

$$\underline{\alpha = P(k \leq K_{obs} | \lambda_+)} = \sum_{K=0}^{100} \frac{e^{-\lambda_+} \lambda_+^K}{K!} = 0{,}05$$

GAUSSIAN APROXIMATION $\longrightarrow \int_{-\infty}^{100+1/2} \frac{1}{\sqrt{2\pi}\,\sqrt{\lambda_+}} \exp\left[-\frac{(\lambda-\lambda_+)^2}{2\lambda_+}\right] = \Phi\left(\frac{100+\frac{1}{2}-\lambda_+}{\sqrt{\lambda_+}}\right) = 0{,}05$

$\langle K \rangle = \lambda_+$, $\langle (K - \langle K \rangle)^2 \rangle$ ... λ_+

$$\frac{100 + 1/2 - \lambda_+}{\sqrt{\lambda_+}} = \Phi(0{,}05) \approx -1{,}64 \xrightarrow{\lambda_+ = u^2} (\ldots) \rightarrow \lambda_+ = \ldots$$

b)

$P_1 = P(K \leq 1 | \lambda_+) = \alpha$

$P_2 = P(K \geq 1 | \lambda_-) = \alpha \cdot \beta$

$$\begin{cases} P_1 = \sum_{K=0}^{1} \frac{\lambda_+^K e^{-\lambda_+}}{K!} = e^{-\lambda_+} + \lambda_+ e^{-\lambda_+} = \alpha \\ P_2 = \sum_{K=1}^{\infty} \frac{\lambda_-^K e^{-\lambda_-}}{K!} = 1 - \sum_{K=0}^{1} \frac{\lambda_-^K e^{\lambda_-}}{K!} = \beta \end{cases}$$

(10) $\langle K \rangle = \lambda = 16$

$$P(K) = \sum_{K=12}^{20} \frac{16^K e^{-16}}{K!} = 0{,}741$$

$$P(12 \leq K \leq 20) = \int_{12-1/2}^{20+1/2} \frac{1}{4\sqrt{2\pi}} \exp\left[-\frac{(K-16)^2}{32}\right] = \Phi\left(\frac{20+1/2-16}{4}\right) - \Phi\left(\frac{12-1/2-16}{4}\right) = 0{,}682$$

(6) $X \equiv$ result of JON $\qquad X \equiv r_1 + r_2 \in \{2, \dots, 8\} \to p_X(x)$

$Y \equiv$ guess made by RUTH

a) $P_Y(y) = \frac{1}{7} \quad \forall y$

b) $P_Y(y) = P_X(y)$

c) $P_Y(y) = \begin{cases} y = y_0 & P_Y(y) = 1 \\ 0 & \text{otherwise} \end{cases}$

If Ruth's guess is correct, $\overset{\to \text{gains}}{g} = X^2$

otherwise, $g = -X$

$$g(x,y) = \begin{cases} X^2 & x = y \\ -X & x \neq y \end{cases}$$

$P(x,y) = P_X(x) \cdot P_Y(y)$

$$\langle g \rangle = \sum_{x,y} P(x,y)\, g(x,y) = \sum_{x=y} P(x,x) \underbrace{g(x,x)}_{x^2} + \sum_{x \neq y} P(x,y) \underbrace{g(x,y)}_{-x}$$

a) $p(x,x) = P_X(x) \cdot \frac{1}{7}$

$$\langle g \rangle = \frac{1}{7} \sum_x P_X(x) \cdot X^2 - \frac{1}{7} \sum_{x \neq y} P_X(x) X = \frac{1}{7} \langle X^2 \rangle - \frac{6}{7} \overbrace{\sum_x P_X(x) X}^{\langle X \rangle}$$

$$\sum_{x \neq y} = \sum_x \underbrace{\sum_{y \neq x}}_{=6}$$

(5) $A \equiv 1^{st}$ detector $\rightarrow N_{12}+N_1$ times

$B \equiv 2^{nd}$ detector $\rightarrow N_{12}+N_2$ times

$C \equiv$ Detected in both $\rightarrow N_{12}$ times

$D \equiv$ No detection $\rightarrow N-(N_{12}+N_1+N_2)$ times

a) $P_r(A) = \frac{N_1+N_{12}}{N}$

$P_r(B) = \frac{N_2+N_{12}}{N}$

b) $P_r(A \cap B) = P_r(C) = N_{12}$

c) $P(\bar{A}|\bar{B}) = \frac{P(\bar{A}\cap\bar{B})}{P(\bar{B})}$

$P(\bar{A}|\bar{B}) = P(\bar{A})$

$P(\bar{A}\cap\bar{B}) = P(D) = P(\bar{A})P(\bar{B})$

$(\ldots)$

d) $P(A \cup B) = P(A) + P(B) - \underset{C}{P(A\cap B)} = \frac{N_1+N_2+N_{12}}{N_{12}+N_1+N_2+\frac{N_1N_2}{N_{12}}}$

(8b CONT)

$$(g_k)^2 = \sum_k P_k(g_k)^2 = \sum_k P_k\left(g_c^2k^2 - 2g_c\,g_{NC}(N-k)k + g_{NC}^2(N-k)^2\right) =$$

$$= g_c^2\langle k^2\rangle - 2g_c\,g_{NC}\langle (N-c)k\rangle + g_{NC}^2\langle (N-k)^2\rangle$$

$$\langle (N-c)k\rangle \leftrightarrow (N\langle k\rangle - \langle k^2\rangle)$$

(12) $\begin{cases} X_1 = 12 \pm 2 \rightarrow \sigma_1 \\ X_2 = 9 \pm 3 \rightarrow \sigma_2 \end{cases}$

$\bar{X} = \frac{x_1+x_2}{2} = 10{,}5$

~~$\sigma^2 = \sigma_1^2 + \sigma_2^2 \rightarrow \sigma = \sqrt{13} = 3{,}6$~~

$X = 10{,}5 \pm 3{,}6$

$P_r(\mu_- \le \mu \le \mu_+) = 0{,}9 = 1-\alpha-\beta \xrightarrow{\alpha=\beta} \alpha = 0{,}05$

$\mu_+ = \bar{X} + \sigma_{\bar{x}}\,\phi^{-1}(\alpha) = 12{,}37$

$\mu_- = \bar{X} - \phi^{-1}(1-\alpha)\sigma_{\bar{x}} = 7{,}86$

⊛ $X = \{x_i\} = \{20.0,\ 19.7,\ 20.6,\ 18.5,\ 21.2,\ 20.8,\ 20.7\} \rightarrow x \sim \mu, \sigma$

$\hookrightarrow N = 7$

a) Calculate the sample mean. Estime the error on $\bar{x}$, knowing $\sigma = 0{,}8$

b) Calculate $\hat{\sigma}_{\bar{x}}$, knowing that $\mu = 20{,}0$.

c) " " , if we don't know the mean of the distrib. (μ).

d) Calculate $\sigma_{\bar{x}}$, without knowing μ or σ.

a) $\bar{X} = \frac{1}{N}\sum_{i=1}^{7} x_i = 20{,}13 \equiv \hat{\mu}$

$f(x_1, \ldots, x_N) = \bar{X} \rightarrow \frac{\partial f}{\partial x_i} = \frac{1}{N}$

$\sigma_{\bar{x}}$ $\nearrow$ C.L.T. $\rightarrow \frac{\sigma}{\sqrt{N}}$

$\sigma_{\bar{x}}$ $\searrow$ $f(x_1, \ldots, x_n) \xrightarrow[\text{from each other}]{x_i \text{ indep.}} \sigma_f^2 = \sum_{i=1}^{N}\left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_i^2 = \frac{1}{N^2}\sum_i \sigma_i^2 = \frac{\sigma^2}{N}$

$\sigma_{\bar{x}} = \frac{0{,}8}{\sqrt{7}} = 0{,}32$

b) $\hat{\sigma}_x = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2 \longrightarrow \hat{\sigma}_x = 0{,}89$

c) $\hat{\sigma}_x^{\,2} = \frac{1}{N-1}\sum_{i=1}^{N}(x_1 - \bar{x}) \longrightarrow \hat{\sigma}_x = 0{,}93 \rightarrow$ Each measurement

d) $\sigma_{\bar{x}} = \frac{0{,}93}{\sqrt{N}} = \frac{0{,}93}{\sqrt{7}} \longrightarrow \hat{\sigma}_x = 0{,}38 \rightarrow$ Population.

⊛ H (cm) X: $\{194, 168, 177, 180, 171, 190, 151, 169, 172, 182\}$

W (kg) Y: $\{75, 53, 72, 80, 75, 75, 57, 67, 46, 68\}$

$\bar{X} = \frac{1}{N}\sum_i x_i = 175{,}7$; $\bar{Y} = 66{,}8$

SAMPLE STATISTICS $S_x^2 = \frac{1}{N}\sum_i (x_i - \bar{x})^2$; $S_y^2 = \frac{1}{N}\sum_i (y_i - \bar{y})^2$

$$\mathrm{Cov}[x,y] = V_{xy} = \frac{1}{N}\sum_i (x_i-\bar{x})(y_i-\bar{y}) = \frac{1}{N}\sum_i x_i y_i - \frac{1}{N^2}\left(\sum_i x_i\right)\left(\sum_i y_i\right) = 66,39$$

$$r_{xy} \equiv \frac{V_{xy}}{S_x S_y} = 0,54$$

BIASED BIASED

$$\hat{\mu}_x = \bar{x} \; ; \; \hat{\mu}_y = \bar{y} \; ; \; \hat{\sigma}_x = \sqrt{\frac{N}{N-1}}\, S_x = 12,2 \; ; \; \hat{\sigma}_y = \sqrt{\frac{N}{N-1}}\, S_y = 11,2$$

$$\hat{\mathrm{Cov}}[x,y] = \frac{1}{N-1}\sum_i (x_i-\bar{x})(y_i-\bar{y}) = \frac{N}{N-1}\,\widetilde{\mathrm{Cov}}[x,y]$$

$$\hat{\mathrm{CORR}}[x,y] = \frac{N}{N-1}\, r_{xy}$$

13) $\{x_i\}$

$$a_i = a_{i-1} + \frac{x_i - a_{i-1}}{i} \stackrel{?}{=} \sum_{k=1}^{i} \frac{x_k}{i}$$

For $i=1$:

$$a_1 = 0 + \frac{x_1 - 0}{1} = x_1 = \frac{1}{1}\sum_{k=1}^{1} x_k$$

Apply induction: Suppose it works for $i-1$:

$$a_{i-1} = \sum_{k=1}^{i-1} \frac{x_k}{i-1}$$

$$\frac{1}{i-1}\sum_{k=1}^{i-1} x_k + \frac{x_i - \sum_{k=1}^{i-1}\frac{x_k}{i-1}}{i} = \frac{1}{i-1}\sum_{k=1}^{i-1} x_k + \frac{(i-1)x_i - \sum_{k=1}^{i-1} x_k}{i(i-1)} =$$

$$= \frac{i\sum_{k=1}^{i-1} x_k + (i-1)x_i - \sum_{k=1}^{i-1} x_k}{i(i-1)} = \frac{\cancel{(i-1)}\sum_{k=1}^{i-1} x_k + \cancel{(i-1)}x_i}{i\cancel{(i-1)}} = \frac{1}{i}\sum_{k=1}^{i} x_i$$

b) Error of mean calc. with second algorithm: $\hat{\sigma}_{\bar{x}_i} = \frac{\sigma_x}{\sqrt{i}}$

$$a_i = a_{i-1} + \frac{x_i - a_{i-1}}{i} = a_{i-1} - \frac{1}{i}a_{i-1} + \frac{x_i}{i} = \left(1 - \frac{1}{i}\right)a_{i-1} + \frac{x_i}{i} = \frac{i-1}{i}a_{i-1} + \frac{x_i}{i}$$

$$\sigma_f^2 = \sum_k \left(\frac{\partial f}{\partial x_k}\right)^2 \sigma_k^2 = \left(\frac{i-1}{i}\right)^2 \sigma_{a_{i-1}}^2 + \frac{\sigma^2}{i^2} \rightarrow \sigma_{a_i} = \frac{\sigma}{\sqrt{i}}$$

(12) $\bar{x} = \frac{1}{N}\sum_i x_i$ $x_i \sim \mu_i, \sigma_i \longrightarrow \sigma_{\bar{x}}^2 = \frac{\sigma^2}{N}$

Another way...

$$\bar{x} = \frac{\sum_i \frac{x_i}{\sigma_i^2}}{\sum_i \frac{1}{\sigma_i^2}} \rightarrow \text{Weighted mean} \Rightarrow \sigma_{\bar{x}}^2 = \frac{1}{\sum_i \frac{1}{\sigma_i^2}}$$

(18) T : Tom goes free, $\bar{T}$, complies

D : Dick " "

H : Harry " "

t : Governor <u>NAMES</u> Tom

d : " " Dick

h : " " Harry

$P(T) = P(D) = P(H) = 2/3$

$$P(T|d) = P(d|T)\frac{P(T)}{P(d)} = \frac{1}{2}\cdot\frac{2/3}{P(d)} = 2/3 \ \dots$$

$$P(d) = P(d|\bar{T})P(\bar{T}) + P(d|\bar{H})P(\bar{H}) + P(d|\bar{D})P(\bar{D}) =$$
$$= \frac{1}{2}\cdot 1/3 + 1\cdot 1/3 + 0\cdot 1/3 = 1/2$$

(19) $2\text{-}\ell_1 \rightarrow I_1 = 13\ ph/s$

$1\text{-}\ell_2 \rightarrow I_2 = 11\ ph/s$

$$\Delta t = 6{,}5\ s \rightarrow \begin{cases} \langle N_1\rangle = I_1\cdot\Delta t = 84{,}5\ ph \\ \langle N_2\rangle = I_2\cdot\Delta t = 71{,}5\ ph \end{cases}$$

POISSON DISTRIB $\rightarrow$ APPROX. GAUSSIAN $\rightarrow$ $\sigma_1^2 = \mu_1 = 13\ ph/s$, $\sigma_2^2 = \mu_2 = 11\ ph/s$

• (---)

• $\ell_1 \rightarrow \mu_1 = \sigma_1^2 = 13\Delta t$ $P(\mu_- \le \mu \le \mu_+) = 0{,}95 = 1-2\alpha$

$\ell_2 \rightarrow \mu_2 = \sigma_2^2 = 11\Delta t$

$\int_{-\infty}^{\mu_{obs}}$