Stability Analysis of the Particle Dynamics in Particle Swarm Optimizer

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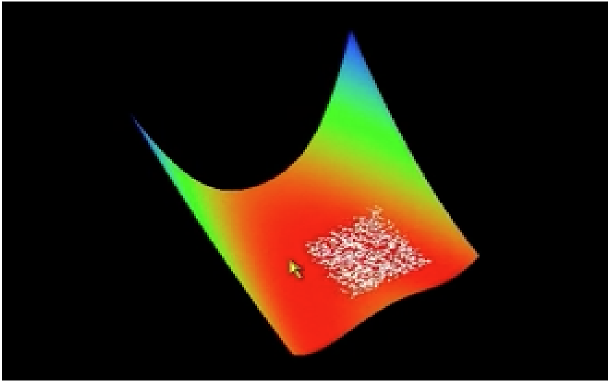
Introduction

How works PSO?



Introduction

- Previous stability analysis over deterministic versions (all parameters are non-random)[1, 3].
- They provide a stability analysis of the **stochastic** particle dynamics.
- Analysis can be carried out on the **1-D case** without loss of generality because each dimension is updated independently from the others (linked via objective function).
- Represent the particle dynamics as a nonlinear feedback controlled system ¿?

Pseudocode PSO

```
Initialize particles (positions and velocities)
Do
    For each particle
        Calculate fitness value
        If the fitness value is better than the best fitness value (pbest) in history
            Set current value as the new pbest.
    end
    Choose the particle with the best fitness value of all the particles as the gbest
    For each particle
        Calculate particle velocity according to equation (1)
        Update particle position according to equation (2)
    end
While maximum iterations or minimum error criteria is not attained.
```

Particle dynamics in 1D

$$v_{t+1} = wv_t + \alpha_t^{(l)} \left(p^{(l)} - x_t \right) + \alpha_t^{(g)} \left(p^{(g)} - x_t \right) \quad (1)$$

$$x_{t+1} = x_t + v_{t+1} \quad (2)$$

where

- v_t is the **particle velocity** and x_t is the particle position at the t-th iteration,
- $p^{(l)}$ is the particle's **best position** thus far,
- $p^{(g)}$ is the **best solution** among all particles,
- w is the **inertia factor** (not in original definition), and
- $\alpha_t^{(l)} \sim \mathcal{U}[0, c_1]$, and $\alpha_t^{(g)} \sim \mathcal{U}[0, c_2]$, are random parameters where c_1 and c_2 are constants (***acceleration coefficients***).

Particle dynamics in 1D

The following statements can be derived from the particle dynamics of (1).

1. The system dynamics are stochastic and of order two.
2. The system does not have an equilibrium point if $p^{(g)} \neq p^{(l)}$.
3. If $p^{(g)} = p^{(l)} = p$ is time invariant, there is a unique equilibrium point at $v_* = 0$, $x_* = p$.

An equilibrium point thus exists only for the best particle whose local best solution is the same as that of the global best solution.

Particle dynamics in 1D

The particle dynamics associated with the best particle or attraction point:

$$\begin{aligned} v_{t+1} &= w v_t + \alpha_t \left(p - x_t\right) \\ x_{t+1} &= x_t + v_{t+1} \end{aligned}$$

where

- $\alpha_t = \alpha_t^{(l)} + \alpha_t^{(g)}$, no uniform distribution but $0 < \alpha_t < (c_1 + c_2)$
- $p = \frac{\alpha_t^{(l)} p^{(l)} + \alpha_t^{(g)} p^{(g)}}{\alpha_t}$, is time varying if $p^{(g)} \neq p^{(l)}$ and if $\alpha_t^{(l)}$ and $\alpha_t^{(g)}$ are random

State-space form

$$y_{t+1} = Ay_t + Bp$$

$$y_t = \begin{pmatrix} x_t \\ v_t \end{pmatrix}, \qquad A = \begin{pmatrix} 1-\alpha_t & w \\ \alpha_t & w \end{pmatrix}, \qquad B = \begin{pmatrix} \alpha_t \\ \alpha_t \end{pmatrix},$$

In the context of the dynamic system theory [1]:

- y_t is the *particle state* made up of its current position and velocity,
- A is the *dynamic matrix* (or *state matrix*) whose properties determine the time behavior of the particle
- p is the *external input* used to drive the particle towards a specified position, and
- B is the *input matrix* that gives the effect of the external input on the particle state.

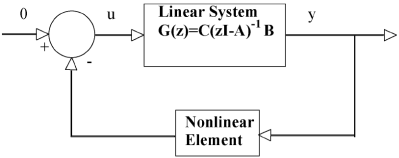
Particle dynamics in 1D

By treating the random variable α_t as a constant
—> deterministic particle dynamics,
—> a simple time-invariant linear second-order dynamic model.

Standard results from dynamic system theory say that the time behavior of the particle depends on the eigenvalues (both have magnitude less than 1) of the dynamic matrix A.
The conditions for convergence are given by [1]:
$w < 1$ and $0 < \alpha_t < (c_1 + c_2) < 2\,(w + 1)$

Characteristics of the particle dynamics

The stability analysis of the particle dynamics can be mapped to the problem of absolute stability of nonlinear feedback systems, known as Lure's stability problem.
The stochastic particle dynamics are thus represented as a feedback controlled dynamic system ¿?:



Feedback control system representation of particle dynamics.

Characteristics of the particle dynamics

The equations governing the dynamics in this new representation, under the conditions of p being time invariant:

$$\begin{aligned} \xi_{t+1} &= A\xi_t + Bu_t \\ \xi_t &= \begin{pmatrix} x_t - p \\ v_t \end{pmatrix} \\ y_t &= C\xi_t \\ u_t &= -\alpha_t y_t \end{aligned} \qquad \begin{aligned} A &= \begin{pmatrix} 1 & w \\ 0 & w \end{pmatrix} \\ B &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ C &= \begin{pmatrix} 1 & 0 \end{pmatrix} \end{aligned}$$

where u_t is the control input signal, and C is the output matrix.

Characteristics of the particle dynamics

Definition (Equilibrium [30]): ξ_* is an equilibrium point of a dynamical system in the state-space form $\xi_{t+1} = f_t(\xi_t)$ if it satisfies $\xi_* = f_t(\xi_*)$ for every $t \geq 0$.

Remark: For the PSO, the dynamical systems with feedback can be rewritten in the following state-space representation:

$$\xi_{t+1} = (A - \alpha_t BC)\xi_t \quad (17)$$

$$(A - \alpha_t BC) = \begin{pmatrix} 1-\alpha_t & w \\ -\alpha_t & w \end{pmatrix}. \quad (18)$$

If $w \neq 0$, then $(A - \alpha_t BC)$ is nonsingular, hence, the only solution that satisfies $\xi_* = (A - \alpha_t BC)\xi_*$ is $\xi_* = 0$. Hence, the particle dynamics specified in (13)–(15) have a unique equilibrium point at the origin in the ξ state space.

Characteristics of the particle dynamics

Definition (*Controllability* *[32]):* A system is completely controllable if the system state $x(t_f)$ at time t_f can be forced to take on any desired value by applying a control input $u(t)$ over a period of time from t_0 until t_f. Suppose n, m, and l are given integers, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{l\times n}$, $D \in \Re^{l\times m}$ and $x_{t+1} = Ax_t + Bu_t$, and $y_t = Cx_t + Du_t$ represents the dynamics of the linear systems. Then, the pair (A, B) is said to be controllable if Rank$[B\ AB \ldots A^{n-1}B] = n$.

(...) The linear part of the PSO system is controllable
The linear plant pair $\{A, B\}$ is controllable

Characteristics of the particle dynamics

Definiton (Observability [32]): A system is completely observable if any initial state vector $x(t_0)$ can be reconstructed by examining the system output $y(t)$ over some period of time from t_0 until t_f. Suppose n, m, and l are given integers $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{l\times n}$, $D \in \Re^{l\times m}$, and $x_{t+1} = Ax_t + Bu_t$, $y_t = Cx_t + Du_t$ represents the dynamics of the linear systems. Then, the pair (C, A) is said to be observable if Rank$[C\ CA \ldots CA^{n-1}]^T = n$.

(...) The linear part of the PSO system is observable.
The linear plant pair $\{A, C\}$ is observable.

Stability Analysis

The passivity idea and the Lyapunov stability idea are combined to analyze the Lure stability problem.

Passive system

Definition [28]: The linear plant has a stable matrix A, if its eigenvalues lie strictly inside the unit circle in the Z plane or equivalently $|\lambda_i\{A\}| < 1$ for all i. Here, $\lambda_i\{\cdot\}$ represents the ith eigenvalues of A.

Definition [28]: A dynamical system is said to be *passive* if there is a nonnegative scalar function $V(\xi)$ with $V(0) = 0$ which satisfies

$$V(\xi_{t+1}) - V(\xi_t) \leq y_t u_t. \qquad (26)$$

Lyapunov stability

Theorem (Lyapunov Stability [28]): Let $\xi = 0$ be an equilibrium point of the system. The equilibrium point is asymptotically stable if there is a nonnegative scalar function $V(\xi)$ with $V(0) = 0$ which satisfies

$$V(\xi_{t+1}) - V(\xi_t) < 0. \tag{27}$$

Remark: Lyapunov stability analysis is based on the idea that if the total energy in the system continually decreases, then the system will asymptotically reach the zero energy state associated with an equilibrium point of the system.

A system is said to be asymptotically stable if all the states approach zero with time.

Lure stability problem

For linear systems, the passivity property can be related to a condition in the frequency domain known as positive real transfer functions.

Definition [28]: The transfer function $H(z)$ of a dynamical system is said to be positive real if and only if the system is stable and

$$\Re\left\{H(e^{j\theta})\right\} > 0$$

for every $\theta \in [0, 2\pi)$, where $\Re\{\cdot\}$ indicates the real part of its argument and $j = \sqrt{-1}$ is the imaginary number.

Discrete-time positive real lemma

An important result that is necessary for the stability analysis is the discrete-time positive real lemma which links the concepts of positive real transfer functions and the existence of a Lyapunov function.

Lemma (Discrete-Time Positive Real Lemma [33], [34]): Let $H(z) = C(zI - A)^{-1}B + D$ be a transfer function, where A is a stable matrix or a semistable matrix with a simple pole on $|z| = 1$, $\{A, B\}$ is controllable, and $\{A, C\}$ is observable. Then, $H(z)$ is strictly positive real if and only if there exist a symmetric positive definite matrix P, matrices W and L, and a positive constant ε such that [33], [34]

$$A^T PA - P = -L^T L \tag{29}$$

$$B^T PA = C - W^T L \tag{30}$$

$$D + D^T - B^T PB = W^T W. \tag{31}$$

Stability analysis

Theorem (Main Result): Let the particle dynamics be represented by (20)–(22) and satisfy (5) with an equilibrium point at the origin. Then, the origin is asymptotically stable if $|w| < 1$, $w \neq 0$, and

$$K < \left(\frac{2\left(1 - 2|w| + w^2\right)}{1 + w} \right).$$

Proof: Consider the Lyapunov function

$$V(\xi) = \xi_t^T P \xi_t \qquad (32)$$

where P is a symmetric positive definite matrix.

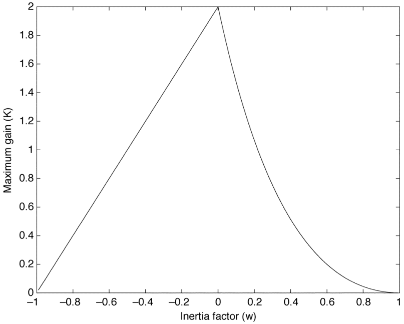
(...)

Stability analysis

- The equilibrium point at the origin represents the particle position reaching the minimum location with zero velocity.
- Lyapunov stability results give only sufficient conditions and, hence, can be very conservative.
- Violation of the stability conditions do not imply instability, rather that stability cannot be guaranteed.

Stability analysis

The maximum gain that gives sufficient guarantees for the stability of particle dynamics decreases with the increase in inertia factor when it is positive. This is in contrast to the previous results derived under nonrandom constant gain assumptions where the maximum gain increased with the inertia factor.



Illustrative examples

The stability analysis given in this paper can be interpreted in the frequency domain and time domain.

Nyquist Plot and Circle Criterion

The circle criterion when applied to the stability of particle dynamics simply states that the Nyquist plot of the linear plant in the feedback system representation should lie to the right side of the point $-(1/\kappa) + j0$ in the Z plane .

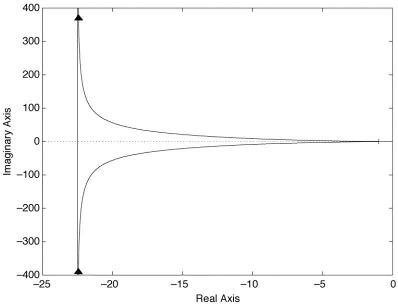


Fig. 3. Discrete-time Nyquist plot for inertia factor = 0.8 and limit value for its real part.

Lyapunov Function and Particle Trajectories

Ideally, the choice for w is for it to lie in the region $0 < w < 1$ (as in previous analysis).

They will determine a candidate positive definite matrix P in the Lyapunov function for the chosen inertia factor w. Consider the system with $w = 0.8$ then the system state matrix is

$$A = \begin{pmatrix} 1 & 0.8 \\ 0 & 0.8 \end{pmatrix}$$

For this case, stability requires $K < 0.044$. A choice of $K = 0.04$ that satisfies this condition but is close to the limit is made for the analysis of this particle.

By solving for P from (29)–(31), the solutions are given by

$$P_1 = \begin{pmatrix} 0.008 & 0.032 \\ 0.032 & 0.4372 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0.008 & 0.032 \\ 0.032 & 0.2108 \end{pmatrix}$$

Lyapunov Function and Particle Trajectories

All simulations are carried out based on (1) and (2) and with initial conditions of $x = 1$ and $v = 0$.

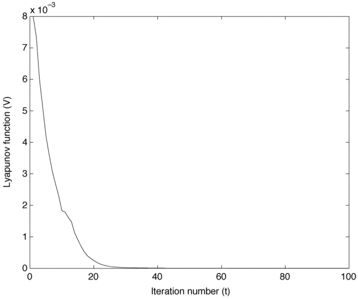


Fig. 5. Lyapunov function with $K = 0.04$ and $w = 0.8$.

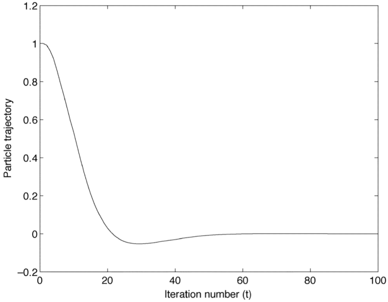


Fig. 7. Particle trajectories with $K = 0.04$ and $w = 0.8$.

Lyapunov Function and Particle Trajectories

The behavior of the particle under conditions that do not guarantee stability

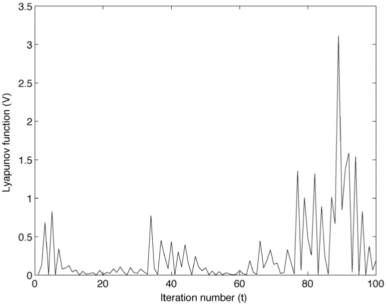


Fig. 9. Lyapunov function with $K = 2.5$ and $w = 0.8$.

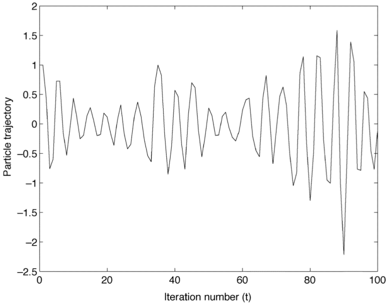


Fig. 10. Particle trajectories with $K = 2.5$ and $w = 0.8$.

Lyapunov Function and Particle Trajectories

The particles **escape from the search region**, not monotonically, but at various times. Movement of particles outside the relevant search region is undesirable.

To investigate the relationship of the number of times in a simulation, the particles exceed some search region defined by a threshold, for specific w values and varying K ; 1000 Monte Carlo simulations for each design choice were carried out. The relevant search region was defined as

$$S = \{x : |x| < \delta\}$$

where δ is a threshold.

Lyapunov Function and Particle Trajectories

Instability count: the number of simulations in which the particle escaped the region at some time during the particle motion.

MATLAB CODE FOR MONTE CARLO SIMULATION

```
T=1000;                          % Simulation time interval
w=0.9;                           % Inertia factor
K=3.5;                           % Maximum gain
instabilitycount=0;
threshold=100;
for S=1:1000                     % Number of Monte Carlo trials
  alpha1=K*rand(1,T);
  alpha2=K*rand(1,T);
  alpha=0.5*alpha1+0.5*alpha2;
  x(1)=1; v(1)=0;                % initial parameter
  for t=2:T
  v(t)=w*v(t-1)-alpha(t)*x(t-1);
  x(t)=x(t-1)+v(t);
  end
  if max(abs(x(:))) ≥threshold
  instabilitycount=instabilitycount+1;
  end
end
```

TABLE I
THRESHOLD AND INSTABILITY COUNT FOR 1000 MONTE CARLO RUNS

Threshold	w=0.8 and K=3.5	w=0.9 and K=2.5	w=0.95 and K=2
10	93	240	817
100	14	75	609
1000	1	18	374
10000	0	2	215

Conclusions

- They have provided a different approach to the stability analysis of PSO with stochastic parameters.
- The passivity theorem and Lyapunov stability methods were applied to the particle dynamics in determining sufficient conditions for asymptotic stability and, hence, convergence to the equilibrium point.
- The results are conservative (are based on the Lyapunov function approach), and, hence, violation of these conditions do not imply instability.
- The results can be used to infer **qualitative design guidelines**.

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